DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SOLUTIONS OF DYNAMIC EQUATIONS OF SECOND ORDER ON TIME SCALES

by
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June, 2023
İZMİR

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We have read the thesis entitled "**SOLUTIONS OF DYNAMIC EQUATIONS OF SECOND ORDER ON TIME SCALES**" completed by **GİZEM MOLO** under supervision of **PROF. DR. BAŞAK KARPUZ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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SOLUTIONS OF DYNAMIC EQUATIONS OF SECOND ORDER ON TIME SCALES

ABSTRACT

First of all, we will give the main definitions, theorems and examples on time scale calculus in this thesis.

Secondly, we will present the solutions and properties of first-order dynamic equations. At the same time, we will examine the solutions under specific time scales with applications.

Finally, we will introduce new solution techniques for second-order dynamic equations with constant coefficients. We will apply the technique to general form of second-order dynamic equations, and then we will get a familiar form (as in the case of differential equations) of the solution after making some arrangements considering possible cases of the roots of the so-called characteristic parabola. Further, we will give new formulas for the solutions when the characteristic equation has complex roots.

Keywords: Time scale, dynamic equation, second-order.

ZAMAN SKALASINDA İKİNCİ MERTEBE DİNAMİK DENKLEMLERİN ÇÖZÜMLERİ

ÖZ

İlk olarak, bu tezde zaman skalası hesabı hakkındaki ana tanımları, teoremleri ve örnekleri vereceğiz.

İkinci olarak, birinci mertebeden dinamik denklemlerin çözümlerini ve özelliklerini sunacağız. Aynı zamanda, çözümleri uygulamalar ile özel zaman skalaları altında inceleyeceğiz.

Son olarak, sabit katsayılı ikinci dereceden dinamik denklemler için yeni çözüm teknikleri tanıtacağız. Tekniği ikinci mertebeden dinamik denklemlerin genel formuna uygulayacağız ve sonra karakteristik parabolün köklerinin olası durumlarını göz önünde bulundurarak bazı düzenlemeler yaptıktan sonra çözümün tanıdık bir formunu (diferansiyel denklemlerde olduğu gibi) elde edeceğiz. Ayrıca, karakteristik denklemin karmaşık köklere sahip olduğu durumdaki çözümler için yeni formüller vereceğiz.

Anahtar kelimeler: Zaman skalası, dinamik denklem, ikinci mertebe

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LIST OF SYMBOLS

$\mathbb{T}$	: The time scale
$\mathbb{R}$	: Set of the real numbers
$\mathbb{N}$	: Set of the natural numbers
$\mathbb{Z}$	: Set of the integers
$\mathbb{C}$	: Set of the complex numbers
$\mathbb{C}_h$	: Hilger's complex plane
$\mathbb{P}_{a,b}$	: $\cup_{\ell=-\infty}^{\infty}[\ell(a+b), \ell(a+b)+a]_{\mathbb{R}}$
$\mathbb{C}_{\mu(\mathbb{T})}$	: $\{z \in \mathbb{C}: \ z \in \mathbb{C}_{\mu(t)}$ for all $t \in \mathbb{T}\}$
C_{rd}	: Set of the rd-continuous functions
$[a, b]_{\mathbb{T}}$	: The intersection of the usual interval $[a, b]$ and $\mathbb{T}$
$\sigma(t)$	: Forward jump operator
$\rho(t)$	: Backward jump operator
$\mu(t)$	: Graininess function
$f^{\sigma}(t)$	: $f(\sigma(t))$
$f^{\Delta}(t)$	: Delta derivative of f at t
$\int_s^t f(\eta)\Delta\eta$	: The Cauchy integral of function f
$\int f(\eta)\Delta\eta$	: Indefinite integral of function f
$\mathrm{Re}_h(z)$	: Hilger's real part of z
$\mathrm{Im}_h(z)$	: Hilger's imaginary part of z
$\mathrm{i}_h(\theta)$	: Hilger's purely imaginary number
$\oplus_h$	: Circle plus addition in $\mathbb{C}_h$
$\ominus_h$	: Circle minus substraction in $\mathbb{C}_h$
$\mathrm{e}_p(\cdot, s)$	: Generalized exponential function
$\mathrm{m}_p(t, s)$	: Generalized monomial operator
$y_{\mathsf{h}}(t)$	: The homogeneous solution
$y_{\mathsf{p}}(t)$	: The particular solution
$\overline{z}$	: Complex conjugate of z

CHAPTER 1
TIME SCALE

Definition 1 (Time Scale)**.** *A time scale is an arbitrary nonempty closed (with respect to the usual topology on $\mathbb{R}$) subset of the real numbers.*

Throughout the paper, we will denote a time scale by the symbol $\mathbb{T}$. First, we will define the forward jump operator, backward jump operator and the graininess function on $\mathbb{T}$. By the time scale interval $[a, b]_{\mathbb{T}}$ we mean the real interval $[a, b]$ intersected with $\mathbb{T}$, i.e., $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$.

Now, we define the basic operators on time scales.

Definition 2 ((Bohner & Peterson, 2001, Definition 1.1))**.** *Let $\mathbb{T}$ be a time scale. The* forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ *is defined by* $\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}$*, the* backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ *by* $\rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}$ *and the* graininess function $\mu : \mathbb{T} \to [0, \infty)$ *is defined by* $\mu(t) := \sigma(t) - t$ *for* $t \in \mathbb{T}$*. Here, it is assumed that* $\inf \emptyset := \sup \mathbb{T}$ *and* $\sup \emptyset := \inf \mathbb{T}$*. We define the set* $\mathbb{T}^{\kappa}$ *as follows. If* $\mathbb{T}$ *has a left-scattered maximum* M*, then we define* $\mathbb{T}^{\kappa} = \mathbb{T} \backslash \{M\}$*. Otherwise, we have* $\mathbb{T}^{\kappa} = \mathbb{T}$*.*

The points on time scales are classified by the following definition.

Definition 3. *Let $\mathbb{T}$ be a time scale. A point $t \in \mathbb{T}$ is called right-scattered if $\sigma(t) > t$, while it is called left-scattered if $\rho(t) < t$, and it is called right-dense if $\sigma(t) = t$, while it is called left-dense if $\rho(t) = t$. Points that are right-scattered and left-scattered at the same time are called isolated. Points that are right-dense and left-dense at the same time are called dense.*

Example 1. *(i) Let $\mathbb{T} = \mathbb{R}$. Then, for $t \in \mathbb{R}$, we have*

$$\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\} = \inf(t, \infty) = t$$

and

$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\} = \sup(-\infty, t) = t.$$

The graininess function μ of $\mathbb{R}$ is

$$\mu(t) = \sigma(t) - t = t - t \equiv 0 \quad \textit{for all } t \in \mathbb{R}.$$

(ii) Let $\mathbb{T} = h\mathbb{Z}$ for $h > 0$. Then, for $t \in h\mathbb{Z}$, we have

$$\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\} = \inf\{t + h, t + 2h, \cdots\} = t + h,$$

and similarly

$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\} = \sup\{\cdots, t - 2h, t - h\} = t - h,$$

and consequently

$$\mu(t) = \sigma(t) - t = (t + h) - t \equiv h.$$

Note that $\lim_{h \to 0^+} h\mathbb{Z} = \mathbb{R}$.

(iii) Let $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^n : \ n \in \mathbb{Z}\} \cup \{0\}$ for $q > 1$. Then, for $t \in \overline{q^{\mathbb{Z}}}$, we have

$$\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\} = \inf\{qt, q^2 t, \cdots\} = qt,$$

and similarly

$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\} = \sup(\{\cdots, t/q^2, t/q\} \cup \{0\}) = t/q,$$

and consequently

$$\mu(t) = \sigma(t) - t = qt - t = (q - 1)t.$$

Note that $\lim_{q \to 1} \overline{q^{\mathbb{Z}}} = [0, \infty)_{\mathbb{R}}$.

(iv) Let $a, b > 0$ and $\mathbb{T} = \mathbb{P}_{a,b} = \cup_{\ell=-\infty}^{\infty} [\ell(a + b), \ell(a + b) + a]_{\mathbb{R}}$. Then, for $t \in \mathbb{P}_{a,b}$,

we have

$$\sigma(t) := \begin{cases} t, & t \in \cup_{\ell=-\infty}^{\infty}[\ell(a+b), \ell(a+b)+a)_{\mathbb{R}} \\ t+b, & t \in \cup_{\ell=-\infty}^{\infty}\{\ell(a+b)+a\} \end{cases}$$

and

$$\rho(t) := \begin{cases} t, & t \in \cup_{\ell=-\infty}^{\infty}(\ell(a+b), \ell(a+b)+a]_{\mathbb{R}} \\ t-b, & t \in \cup_{\ell=-\infty}^{\infty}\{\ell(a+b)\}. \end{cases}$$

For this time scale,

$$\mu(t) := \begin{cases} 0, & t \in \cup_{\ell=-\infty}^{\infty}[\ell(a+b), \ell(a+b)+a)_{\mathbb{R}} \\ b, & t \in \cup_{\ell=-\infty}^{\infty}\{\ell(a+b)+a\} \end{cases} \quad \text{for } t \in \mathbb{P}_{a,b}.$$

Note that $\lim_{a\to 0^+} \mathbb{P}_{a,b} = b\mathbb{Z}$ *and* $\lim_{b\to 0^+} \mathbb{P}_{a,b} = \mathbb{R}$.

One of the most important notions on time scale is the Δ-derivative, which is given below.

Definition 4 ((Bohner & Peterson, 2001, Definition 1.10))**.** *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then we define the number $f^{\Delta}(t)$ to be the number (provided it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood $U_t \subset \mathbb{T}$ of t such that*

$$\big|[f^{\sigma}(t) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\big| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U_t,$$

where $f^{\sigma} := f \circ \sigma$. We say that f is Δ-differentiable *(only differentiable in short) at t if it has a derivative at t. If f is differentiable at each point in $\mathbb{T}^{\kappa}$ we say that f is differentiable on $\mathbb{T}$.*

Some properties related with Δ-derivative are listed as follows.

Theorem 1 ((Bohner & Peterson, 2001, Theorem 1.16))**.** *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then, we have the following:*

(i) *If f is differentiable function at t, then f is continuous at t.*

(ii) *If f is continuous function at t and $\mu(t) > 0$, then f is differentiable at t with*

$$f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\mu(t)}.$$

(iii) *If f is differentiable function at t and $\mu(t) = 0$, then*

$$\lim_{\substack{s \in \mathbb{T} \\ s \to t}} \frac{f(t) - f(s)}{t - s}.$$

Remark 1. *Clearly, $f^{\sigma}(t) = f(t) + f^{\Delta}(t)\mu(t)$ for all $t \in \mathbb{T}^{\kappa}$. This formula is also known as the "simple useful formula".*

Example 2. *(i) Let $\mathbb{T} = \mathbb{R}$ (see Ross (1980), Brand (1966)). We will show that*

$$f^{\Delta}(t) = f'(t)$$

for $t \in \mathbb{T}$. Note that any $t \in \mathbb{T}$ is a right-dense point. We know that

$$f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

from limit definition of the derivative. Hence, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(s)}{t - s} - f'(t) \right| \leq \varepsilon \quad \textit{for all } s \in U_t \backslash \{t\}, \quad \textit{where } U_t := (t - \delta, t + \delta)_{\mathbb{R}},$$

which yields

$$\big| [f(t) - f(s)] - f'(t)[t - s] \big| \leq \varepsilon |t - s| \quad \textit{for all } s \in U_t$$

by Definition 4, $f^{\Delta}(t) = f'(t)$ since $\sigma(t) = t$.

(ii) *Let $\mathbb{T} = h\mathbb{Z}$ for $h > 0$ (see Brand (1966), Kelley & Peterson (2001)). We will show that*

$$f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h} \quad \textit{for } t \in \mathbb{T}.$$

Clearly, any $t \in h\mathbb{Z}$ is a right-scattered point. For any $\varepsilon > 0$, we can let $U_t := (t - \frac{h}{2}, t + \frac{h}{2})_{h\mathbb{Z}}$ with $\delta = \frac{h}{2}$. Note that $U_t = \{t\}$. Then, for any $s \in U_t$, we have

$$\begin{aligned}
&\left|[f^\sigma(t) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \\
&= \left|[f^\sigma(t) - f(t)] - f^\Delta(t)[\sigma(t) - t]\right| \\
&= \left|f(t+h) - f(t) - \frac{f(t+h) - f(t)}{h}[(t+h) - t]\right| \\
&= 0 \\
&\leq \varepsilon|\sigma(t) - s|.
\end{aligned}$$

(iii) Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ for $q > 1$ (see Kac & Cheung (2002)). We will show that

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t} \quad \text{for } t > 0$$

and

$$f^\Delta(0) = f'(0).$$

Let $t > 0$, i.e., t is a right-scattered point. For any $\varepsilon > 0$, we can let $U_t := (t - \frac{q-1}{2q}t, t + \frac{q-1}{2q}t)_{\mathbb{T}}$ with $\delta = \frac{q-1}{2q}t$. Note that $U_t = \{t\}$. Then, for any $s \in U_t$, we have

$$\begin{aligned}
\left|[f^\sigma(t) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| &= \left|[f^\sigma(t) - f(t)] - f^\Delta(t)[\sigma(t) - t]\right| \\
&= \left|f(qt) - f(t) - \frac{f(qt) - f(t)}{(q-1)t}[qt - t]\right| \\
&= 0 \\
&\leq \varepsilon|\sigma(t) - s|.
\end{aligned}$$

Since $t = 0$ is a right-dense point, we have

$$f^\Delta(0) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(0).$$

(iv) $\mathbb{T} = \mathbb{P}_{a,b}$ *for* $a, b > 0$.

$$f^{\Delta}(t) = \begin{cases} f'(t), & t \in \cup_{k=-\infty}^{\infty}[k(a+b), k(a+b)+a)_{\mathbb{R}} \\ \frac{f(t+b)-f(t)}{b}, & t \in \cup_{k=-\infty}^{\infty}\{k(a+b)+a\}. \end{cases}$$

(a) *Let* $t \in [k(a+b), k(a+b)+a)_{\mathbb{R}}$ *for some* $k \in \mathbb{Z}$, *i.e.,* t *is a right-dense point. We know that*

$$f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

In this case, for any $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\left| \frac{f(t) - f(s)}{t - s} - f'(t) \right| \leq \varepsilon \quad \text{for all } s \in U_t \backslash \{t\}, \quad \text{where } U_t := (t-\delta, t+\delta)_{\mathbb{T}}.$$

Without loss of generality, we may let $\delta < \min\{a, b\}$. *It should be noted here that if* $t = k(a+b)$, *then* $U_t = (t-\delta, t+\delta)_{\mathbb{T}} = [t, t+\delta)_{\mathbb{R}}$, *i.e., the limit is only right-sided in usual sense. This yields*

$$\big|[f(t) - f(s)] - f'(t)[t - s]\big| \leq \varepsilon |t - s| \quad \text{for all } s \in U_t$$

by Definition 4, $f^{\Delta}(t) = f'(t)$ *since* $\sigma(t) = t$.

(b) *Let* $t = k(a+b) + a$ *for some* $k \in \mathbb{Z}$, *i.e.,* t *is a right-scattered point. By the continuity of* f *(see Theorem 1 (i)), we can write*

$$\lim_{s \to t} \frac{f(t+b) - f(s)}{(t+b) - s} = \frac{f(t+b) - f(t)}{b},$$

i.e., for every ε, *there exists* $\delta > 0$ *such that*

$$\left| \frac{f(t+b) - f(s)}{(t+b) - s} - \frac{f(t+b) - f(t)}{b} \right| < \varepsilon \quad \text{for all } t \in U_t := (t-\delta, t+\delta)_{\mathbb{T}}.$$

Without loss of generality, we may let $\delta < \min\{a, b\}$, *which yields* $U_t := (t-\delta, t+\delta)_{\mathbb{T}} = (t-\delta, t]_{\mathbb{R}}$. *Hence, for any* $s \in U_t$, *we have*

$$\big|[f^{\sigma}(t) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\big|$$

$$= \left| f(t+b) - f(s) - \frac{f(t+b)-f(t)}{b}[(t+b)-s] \right|$$
$$\leq \varepsilon |(t+b)-s|$$
$$= \varepsilon |\sigma(t) - s|.$$

Example 3. *(i) Let $f(t) \equiv 1$ for $t \in \mathbb{T}$, where $c \in \mathbb{R}$. We will show that $f^{\Delta}(t) \equiv 0$ for $t \in \mathbb{T}^{\kappa}$. This is obvious because for $t \in \mathbb{T}^{\kappa}$ and for any $\varepsilon > 0$,*

$$|[f^{\sigma}(t) - f(s)] - 0 \cdot (\sigma(t) - s)| = |1-1| = 0 \leq \varepsilon |\sigma(t) - s|$$

holds for all $s \in (t-1, t+1)_{\mathbb{T}}$.

(ii) Let $f(t) = t$ for $t \in \mathbb{T}$. We will show that $f^{\Delta}(t) \equiv 1$ for any $t \in \mathbb{T}^{\kappa}$. Then, for any $\varepsilon > 0$,

$$|[f^{\sigma}(t) - f(s)] - 1 \cdot (\sigma(t) - s)| = |[\sigma(t) - s] - (\sigma(t) - s)| = 0 \leq |\sigma(t) - s|$$

holds for all $s \in \mathbb{T}$.

(iii) Let $f(t) = t^2$ for $t \in \mathbb{T}$. We will show that $f^{\Delta}(t) = \sigma(t) + t$ for any $t \in \mathbb{T}^{\kappa}$. This follows since for any $\varepsilon > 0$,

$$\begin{aligned}
& \left|\left[\left(\sigma(t)\right)^2 - s^2\right] - f^{\Delta}(t)\left(\sigma(t) - s\right)\right| \leq \varepsilon |\sigma(t) - s| \\
\iff & \quad |\sigma(t) - s|\left|\left(\sigma(t) + s\right) - f^{\Delta}(t)\right| \leq \varepsilon |\sigma(t) - s| \\
\iff & |\sigma(t) - s|\left|\left(\sigma(t) + s\right) - \left(\sigma(t) + t\right)\right| \leq \varepsilon |\sigma(t) - s| \\
\iff & \qquad |\sigma(t) - s||t - s| \leq \varepsilon |\sigma(t) - s|
\end{aligned}$$

for all $s \in (t-\varepsilon, t+\varepsilon)_{\mathbb{T}}$. Note also that $f^{\Delta}(t) = 2t + \mu(t)$ for $t \in \mathbb{T}^{\kappa}$.

(iv) Let $f(t) = \mathrm{e}^t$ for $t \in \mathbb{T}$.

(a) First, let t be right-scattered, then $\sigma(t) > t$ and $\mu(t) > 0$. By Theorem 1 (ii), we get

$$f^{\Delta}(t) = \frac{\mathrm{e}^{\sigma(t)} - \mathrm{e}^t}{\sigma(t) - t} = \frac{\mathrm{e}^{\sigma(t)} - \mathrm{e}^t}{\mu(t)}.$$

(b) Next, let t be right-dense, then $\sigma(t) = t$ and $\mu(t) = 0$. Since f is differentiable in the usual sense, for any given $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\left|\frac{\mathrm{e}^t - \mathrm{e}^s}{t-s} - \mathrm{e}^t\right| < \varepsilon \quad \text{for all } s \in \mathbb{R} \text{ such that } 0 < |t-s| < \delta.$$

Setting $U_t := (t-\delta, t+\delta)_{\mathbb{T}}$ yields

$$\left|[\mathrm{e}^t - \mathrm{e}^s] - \mathrm{e}^t[t-s]\right| \leq \varepsilon|t-s| \quad \text{for all } s \in U_t,$$

which shows that $f^{\Delta}(t) = \mathrm{e}^t$.

Combining the cases above, we obtain

$$f^{\Delta}(t) = \begin{cases} \mathrm{e}^t, & \mu(t) = 0 \\ \frac{\mathrm{e}^{\sigma(t)} - \mathrm{e}^t}{\mu(t)}, & \mu(t) > 0 \end{cases} \quad \text{for any } t \in \mathbb{T}^{\kappa}.$$

Below, we give some properties of Δ-derivative.

Lemma 1 ((Bohner & Peterson, 2001, Theorem 1.20))**.** *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable functions and let $t \in \mathbb{T}^{\kappa}$. Then, we have the following:*

(i)

$$(f \pm g)^{\Delta}(t) = f^{\Delta}(t) \pm g^{\Delta}(t).$$

(ii) For any constant $a_1 \in \mathbb{R}$,

$$(a_1 f)^{\Delta}(t) = a_1 f^{\Delta}(t).$$

(iii)

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)g^{\sigma}(t) + f(t)g^{\Delta}(t) \\ &= f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t)\end{aligned}$$

$$= f^{\Delta}(t)g(t) + f(t)g^{\Delta}(t) + f^{\Delta}g^{\Delta}(t)\mu(t).$$

(iv) If $f(t)f^{\sigma}(t) \neq 0$, then

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f^{\sigma}(t)}.$$

(v) If $g(t)g^{\sigma}(t) \neq 0$, then

$$\begin{aligned}\left(\frac{f}{g}\right)^{\Delta}(t) &= \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)} \\ &= \frac{f^{\Delta}(t)g^{\sigma}(t) - f^{\sigma}(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)}.\end{aligned}$$

The following is a generalization of the chain rule for time scales.

Definition 5 ((Bohner & Peterson, 2001, Theorem 1.90))**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and assume $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable and*

$$(f \circ g)^{\Delta}(t) = \left\{\int_0^1 f'\big(g(t) + \lambda g^{\Delta}(t)\mu(t)\big)\mathrm{d}\lambda\right\} g^{\Delta}(t),$$

where $\cdot'$ is the usual derivative.

Now, we illustrate the chain rule on some important time scales.

Example 4. *(i) Let $\mathbb{T} = \mathbb{R}$, and f, g be any two continuously differentiable functions, then*

$$(f \circ g)'(t) = \left\{\int_0^1 f'\big(g(t)\big)\mathrm{d}\lambda\right\} g'(t) = f'\big(g(t)\big)g'(t).$$

(ii) Let $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, $f(x) = \mathrm{e}^x$ and $g(t) = t$. We know that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable. Then, we have $f'(x) = \mathrm{e}^x$, $g^{\Delta}(t) = 1$ and $\mu(t) \equiv h$. From Definition 5, we have $f \circ g : \mathbb{T} \to \mathbb{R}$

is Δ-differentiable and

$$\begin{aligned}(f\circ g)^{\Delta}(t) &= \left\{\int_0^1 f'\big(g(t)+\lambda g^{\Delta}(t)\mu(t)\big)\mathrm{d}\lambda\right\}g^{\Delta}(t)\\ &= \int_0^1 \mathrm{e}^{t+\lambda h}\mathrm{d}\lambda = \mathrm{e}^t\int_0^1 \mathrm{e}^{\lambda h}\mathrm{d}\lambda\\ &= \mathrm{e}^t\left[\frac{\mathrm{e}^{\lambda h}}{h}\right]_{\lambda=0}^{\lambda=1} = \frac{\mathrm{e}^{t+h}-\mathrm{e}^t}{h}.\end{aligned}$$

(iii) Let $\mathbb{T}=\overline{q^{\mathbb{Z}}}=\{q^n:\ n\in\mathbb{Z}\}\cup\{0\}$ for $q>1$, $f(x)=\sin(x)$ and $g(t)=t^2$. We have that $f:\mathbb{R}\to\mathbb{R}$ is continuous differentiable and $g:\mathbb{T}\to\mathbb{R}$ is Δ-differentiable. From Example 1 (iii), we know that $\sigma(t)=qt$ and $\mu(t)=(q-1)t$. Then, $f'(x)=\cos(x)$ and $g^{\Delta}(t)=\sigma(t)+t=qt+t=(q+1)t$. From Definition 5, we have $f\circ g:\mathbb{T}\to\mathbb{R}$ is Δ-differentiable and

$$\begin{aligned}(f\circ g)^{\Delta}(t) &= \left\{\int_0^1 \cos\big(t^2+\lambda(q-1)t(q+1)t\big)\mathrm{d}\lambda\right\}(q+1)t\\ &= (q+1)t\left\{\int_0^1 \cos\big(t^2+\lambda t^2(q^2-1)\big)\mathrm{d}\lambda\right\}\\ &= (q+1)t\frac{1}{t^2(q^2-1)}\left[\sin\big(t^2+\lambda t^2(q^2-1)\big)\right]_{\lambda=0}^{\lambda=1}\\ &= \frac{1}{t(q-1)}\left[\sin\big(t^2+t^2(q^2-1)\big)-\sin(t^2)\right]\\ &= \frac{1}{t(q-1)}\left[\sin\big(t^2q^2\big)-\sin(t^2)\right]\\ &= \frac{2}{t(q-1)}\sin\left(\frac{(q^2-1)t^2}{2}\right)\cos\left(\frac{(q^2+1)t^2}{2}\right).\end{aligned}$$

To be able to solve dynamic equations on time scale, the notion of Δ-integral must be introduced after the definition of the Δ-derivative.

Definition 6 ((Bohner & Peterson, 2001, Definition 1.71)). *Assume that $f:\mathbb{T}\to\mathbb{R}$ and $F:\mathbb{T}\to\mathbb{R}$ satisfy $F^{\Delta}=f$ on $\mathbb{T}^{\kappa}$. We define the* indefinite integral *of function f by*

$$\int f(\eta)\Delta\eta = F(t)+c \quad \textit{for } t\in\mathbb{T},$$

where $c \in \mathbb{R}$ is an arbitrary constant. The Cauchy integral *of f on $[s,t)_{\mathbb{T}}$ is defined by*

$$\int_s^t f(\eta)\Delta\eta = F(\eta)\Big|_{\eta=s}^{\eta=t} := F(t) - F(s).$$

Definition 7 ((Bohner & Peterson, 2001, Definition 1.58))**.** *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points for $t \in \mathbb{T}$ and its left-sided limits exist (finite) at left-dense points for $t \in \mathbb{T}$. The set of rd-continuous functions denoted by* $\mathrm{C_{rd}}$.

Theorem 2 ((Bohner & Peterson, 2001, Theorem 1.74))**.** *Every function in the set* $\mathrm{C_{rd}}$ *has an antiderivative. In particular, if $s \in \mathbb{T}$, then $F \in \mathrm{C_{rd}}$ defined by*

$$F(t) = \int_s^t f(\eta)\Delta\eta \quad \textit{for all } t \in \mathbb{T}$$

is an antiderivative of f.

Theorem 3 ((Bohner & Peterson, 2001, Theorem 1.75))**.** *If $f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(\eta)\Delta\eta = \mu(t)f(t).$$

Proof. Suppose that $f^{\Delta}(t) = F(t)$, then we have

$$\begin{aligned}\int_t^{\sigma(t)} f(\eta)\Delta\eta &= F(\sigma(t)) - F(t)\\ &= \mu(t)F^{\Delta}(t)\\ &= \mu(t)f(t),\end{aligned}$$

where the second equation holds from by Remark 1. □

Some important properties of Δ-integrals are given below.

Property 1 ((Bohner & Peterson, 2001, Theorem 1.77))**.** *Assume that $a, b, c \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $f, g \in \mathrm{C_{rd}}$. Then, we have the following:*

(i) $\int_a^b [a_1 f(\eta) + a_2 g(\eta)]\Delta\eta = a_1 \int_a^b f(\eta)\Delta\eta + a_2 \int_a^b g(\eta)\Delta\eta.$

(ii) $\int_a^b f(\eta)\Delta\eta = -\int_b^a f(\eta)\Delta\eta.$

(iii) $\int_a^b f(\eta)\Delta\eta = \int_a^c f(\eta)\Delta\eta + \int_c^b f(\eta)\Delta\eta.$

(iv) $\int_a^b f(\eta)g^{\Delta}(\eta)\Delta\eta = (fg)(\eta)\big|_{\eta=a}^{\eta=b} - \int_a^b f^{\Delta}(\eta)g^{\sigma}(\eta)\Delta\eta.$

(v) $\int_a^a f(\eta)\Delta\eta = 0.$

Explicit definitions of Δ-integral on some particular time scales are given below.

Example 5. *Let $s, t \in \mathbb{T}$ with $t \geq s$ and $f \in \mathrm{C}_{\mathrm{rd}}$.*

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_s^t f(\eta)\Delta\eta = \int_s^t f(\eta)\mathrm{d}\eta,$$

where the integral on the right-hand side is the usual Riemann integral.

(ii) If $\mathbb{T} = h\mathbb{Z}$ for $h > 0$. We will show that

$$\int_s^t f(\eta)\Delta\eta = \sum_{j=\frac{s}{h}}^{\frac{t}{h}-1} f(hj)h,$$

where the empty sum is assumed to be 0. Let

$$F(t) := \sum_{j=0}^{\frac{t}{h}-1} f(hj)h \quad \text{for } t \in h\mathbb{Z}.$$

Then, we compute that

$$\begin{aligned} F^{\Delta}(t) &= \frac{F(t+h) - F(t)}{h} \\ &= \frac{1}{h}\left(\sum_{j=0}^{\frac{t+h}{h}-1} f(hj)h - \sum_{j=0}^{\frac{t}{h}-1} f(hj)h\right) \\ &= \frac{1}{h}\left(\sum_{j=0}^{\frac{t}{h}} f(hj)h - \sum_{j=0}^{\frac{t}{h}-1} f(hj)h\right) \\ &= \frac{1}{h}\big(f(h\tfrac{t}{h})h\big) = f(t). \end{aligned}$$

Finally, we have

$$\int_s^t f(\eta)\Delta\eta = F(t) - F(s) = \sum_{j=0}^{\frac{t}{h}-1} f(hj)h - \sum_{j=0}^{\frac{s}{h}-1} f(hj)h$$
$$= \sum_{j=\frac{s}{h}}^{\frac{t}{h}-1} f(hj)h,$$

which completes the proof.

(iii) If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *for* $q > 1$. *We will show that*

$$\int_s^t f(\eta)\Delta\eta = \sum_{j=\log_q(s)}^{\log_q(t)-1} f(q^j)(q-1)q^j \quad \textit{for } s,t \in q^{\mathbb{Z}} \textit{ with } t \geq s.$$

Let

$$F(t) := \sum_{j=1}^{\log_q(t)-1} f(q^j)(q-1)q^j \quad \textit{for } t \in q^{\mathbb{Z}}.$$

Then, we compute

$$\begin{aligned}
F^{\Delta}(t) &= \frac{F(qt) - F(t)}{(q-1)t} \\
&= \frac{1}{(q-1)t}\left(\sum_{j=1}^{\log_q(qt)-1} f(q^j)(q-1)q^j - \sum_{j=1}^{\log_q(t)-1} f(q^j)(q-1)q^j\right) \\
&= \frac{1}{(q-1)t}\left(\sum_{j=1}^{\log_q(t)} f(q^j)(q-1)q^j - \sum_{j=1}^{\log_q(t)-1} f(q^j)(q-1)q^j\right) \\
&= \frac{1}{(q-1)t}\left(f(q^{\log_q(t)})(q-1)q^{\log_q(t)}\right) \\
&= \frac{1}{(q-1)t} f(t)(q-1)t \\
&= f(t).
\end{aligned}$$

Finally, we have

$$\int_s^t f(\eta)\Delta\eta = F(t) - F(s)$$
$$= \sum_{j=1}^{\log_q(t)-1} f(q^j)(q-1)q^j - \sum_{j=1}^{\log_q(s)-1} f(q^j)(q-1)q^j$$

$$= \sum_{j=\log_q(s)}^{\log_q(t)-1} f(q^j)(q-1)q^j,$$

which completes the proof.

(iv) $\mathbb{T} = \mathbb{P}_{a,b} = \cup_{\ell=-\infty}^{\infty}[\ell(a+b), \ell(a+b)+a]_{\mathbb{R}}$ *for* $a, b > 0$. *Let us define*

$$F(t) := \sum_{j=0}^{\lfloor \frac{t}{a+b} \rfloor - 1} \left[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b \right] + \int_{\lfloor \frac{t}{a+b} \rfloor (a+b)}^{t} f(\eta)\mathrm{d}\eta \tag{1.1}$$

for $t \in \mathbb{P}_{a,b}$, *where* $\lfloor \cdot \rfloor$ *denotes the greatest integer function.*

(a) Let $s \in [k(a+b), k(a+b)+a)_{\mathbb{R}}$ *and* $t \in [n(a+b), n(a+b)+a)_{\mathbb{R}}$ *for some* $k, n \in \mathbb{Z}$, *i.e.,* $k = \lfloor \frac{s}{a+b} \rfloor$ *and* $n = \lfloor \frac{t}{a+b} \rfloor$. *We simply compute that* $F^{\Delta} = f$, *which shows*

$$\begin{aligned}
\int_s^t f(\eta)\Delta\eta &= F(t) - F(s) \\
&= \sum_{j=0}^{n-1} \left[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b \right] \\
&\quad + \int_{n(a+b)}^{t} f(\eta)\mathrm{d}\eta \\
&\quad - \left(\sum_{j=0}^{k-1} \left[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b \right] \right. \\
&\qquad \left. + \int_{k(a+b)}^{s} f(\eta)\mathrm{d}\eta \right) \\
&= \int_s^{k(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(k(a+b)+a\big)b \\
&\quad + \sum_{j=k+1}^{n-1} \left[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b \right] \\
&\quad + \int_{n(a+b)}^{t} f(\eta)\mathrm{d}\eta.
\end{aligned}$$

(b) Let $s = k(a+b)+a$ *and* $t \in [n(a+b), n(a+b)+a)_{\mathbb{R}}$ *for some* $k, n \in \mathbb{Z}$.

We simply compute that $F^{\Delta} = f$, which shows

$$\begin{aligned}\int_s^t f(\eta)\Delta\eta &= F(t) - F(s)\\ &= \sum_{j=0}^{n-1}\Big[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b\Big]\\ &\quad + \int_{n(a+b)}^{t} f(\eta)\mathrm{d}\eta\\ &\quad - \sum_{j=0}^{k-1}\Big[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b\Big]\\ &= f(s)b + \sum_{j=k+1}^{n-1}\Big[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b\Big]\\ &\quad + \int_{n(a+b)}^{t} f(\eta)\mathrm{d}\eta.\end{aligned}$$

(c) *Let $s \in [k(a+b), k(a+b)+a)_{\mathbb{R}}$ and $t = n(a+b)+a$ for some $k, n \in \mathbb{Z}$. We compute that*

$$\begin{aligned}F^{\Delta}(t) &= \frac{F(t+b)-F(t)}{b}\\ &= \frac{1}{b}\Bigg\{\Bigg(\sum_{j=0}^{n}\Big[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b\Big]\Bigg)\\ &\qquad - \Bigg(\sum_{j=0}^{n-1}\Big[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b\Big]\\ &\qquad\qquad + \int_{n(a+b)}^{n(a+b)+a} f(\eta)\mathrm{d}\eta\Bigg)\Bigg\}\\ &= \frac{1}{b}\Big(f\big(n(a+b)+a\big)b\Big) = f(t).\end{aligned}$$

Then

$$\begin{aligned}\int_s^t f(\eta)\Delta\eta &= F(t) - F(s)\\ &= \sum_{j=0}^{n-1}\Big[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b\Big]\\ &\quad + \int_{n(a+b)}^{n(a+b)+a} f(\eta)\mathrm{d}\eta\\ &\quad - \Bigg(\sum_{j=0}^{k-1}\Big[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b\Big]\\ &\qquad + \int_{k(a+b)}^{s} f(\eta)\mathrm{d}\eta\Bigg)\\ &= \int_s^{k(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(k(a+b)+a\big)b\\ &\quad + \sum_{j=k+1}^{n-1}\Big[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b\Big]\\ &\quad + \int_{n(a+b)}^{t} f(\eta)\mathrm{d}\eta.\end{aligned}$$

(d) Let $s = k(a+b)+a$ and $t = n(a+b)+a$ for some $k, n \in \mathbb{Z}$. Then, we can easily show that

$$\begin{aligned}\int_s^t f(\eta)\Delta\eta = f(s)b + \sum_{j=k+1}^{n-1}\Big[\int_{j(a+b)}^{j(a+b)+a} f(\eta)\mathrm{d}\eta + f\big(j(a+b)+a\big)b\Big]\\ + \int_{n(a+b)}^{t} f(\eta)\mathrm{d}\eta.\end{aligned}$$

Example 6. *Let $\mathbb{T}$ is an arbitrary time scale, and $a, b \in \mathbb{T}$.*

(i) Let $f(t) \equiv 1$ for $t \in \mathbb{T}$. Since $F(t) = t$ satisfies $F^{\Delta}(t) = 1$, we have

$$\int_s^t f(\eta)\Delta\eta = \int_s^t F^{\Delta}(\eta)\Delta\eta = F(t) - F(s) = t - s.$$

For the next three functions, is not easy to write an explicit definition for the integral on arbitrary time scales, and thus, we will consider the time scales in Example 1 whenever the computation is simple.

(ii) Let $f(t)=t$.

(a) Let $\mathbb{T}=\mathbb{R}$. Clearly, $F(t)=\frac{t^2}{2}$. Then,

$$\int_s^t \eta\Delta\eta = \frac{\eta^2}{2}\bigg|_{\eta=s}^{\eta=t} = \frac{t^2-s^2}{2}.$$

(b) Let $\mathbb{T}=h\mathbb{Z}$. Clearly, $F(t)=\frac{t(t-h)}{2}$ satisfies $F^{\Delta}(t)=t$. Then,

$$\int_s^t \eta\Delta\eta = \frac{\eta(\eta-h)}{2}\bigg|_{\eta=s}^{\eta=t} = \frac{(t-s)(t-s-h)}{2}.$$

(c) Let $\mathbb{T}=\overline{q^{\mathbb{Z}}}=\{q^n:\ n\in\mathbb{Z}\}\cup\{0\}$ for $q>1$. Clearly, $F(t)=\frac{t^2}{q+1}$ satisfies $F^{\Delta}(t)=t$. Then,

$$\int_s^t \eta\Delta\eta = \frac{\eta^2}{q+1}\bigg|_{\eta=s}^{\eta=t} = \frac{t^2-s^2}{q+1}.$$

(d) Let $a,b>0$ and $\mathbb{T}=\mathbb{P}_{a,b}=\cup_{\ell=-\infty}^{\infty}[\ell(a+b),\ell(a+b)+a]_{\mathbb{R}}$. Then,

$$F(t)=\begin{cases}\dfrac{t^2}{2}, & t\in\cup_{\ell=-\infty}^{\infty}[\ell(a+b),\ell(a+b)+a)_{\mathbb{R}}\\ \dfrac{t(t-b)}{2}, & t\in\cup_{\ell=-\infty}^{\infty}\{\ell(a+b)+a\}.\end{cases}$$

It is obvious that $F^{\Delta}(t)=t$. Let $s\in[k(a+b),k(a+b)+a]_{\mathbb{R}}$ and $t\in[n(a+b),n(a+b)+a]_{\mathbb{R}}$ for some $k,n\in\mathbb{Z}$, then we compute that

$$\begin{aligned}\int_s^t \eta\Delta\eta &= \int_s^{k(a+b)+a}\eta\Delta\eta+\int_{k(a+b)+a}^{(k+1)(a+b)}\eta\Delta\eta+\int_{(k+1)(a+b)}^{(k+1)(a+b)+a}\eta\Delta\eta\\ &\quad+\cdots+\int_{n(a+b)}^{t}\eta\Delta\eta\\ &=\frac{\eta^2}{2}\bigg|_s^{k(a+b)+a}+\frac{\eta(\eta-b)}{2}\bigg|_{k(a+b)+a}^{(k+1)(a+b)}+\cdots+\frac{\eta^2}{2}\bigg|_{n(a+b)}^{t}\\ &=\frac{1}{2}\left(\eta^2\bigg|_s^{k(a+b)+a}+\cdots+\eta^2\bigg|_{n(a+b)}^{t}\right)\\ &\quad-\frac{b}{2}\left(\eta\bigg|_{k(a+b)+a}^{(k+1)(a+b)}+\cdots+\eta\bigg|_{(n-1)(a+b)+a}^{n(a+b)}\right)\\ &=\frac{1}{2}\eta^2\bigg|_s^t-\frac{b}{2}\Big(\big[(k+1)(a+b)-\big(k(a+b)+a\big)\big]\end{aligned}$$

$$+ \cdots + \Big[n(a+b) - \big((n-1)(a+b)+a\big)\Big]\Bigg)$$
$$= \frac{t^2-s^2}{2} - b(n-k)\frac{b}{2}$$
$$= \frac{t^2-s^2}{2} - \left\lfloor \frac{t-s}{a+b} \right\rfloor \frac{b^2}{2},$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function.

(iii) Let $f(t) = t^2$.

(a) Let $\mathbb{T} = \mathbb{R}$. Clearly, $F(t) = \frac{t^3}{3}$. Then,

$$\int_s^t \eta^2 \Delta\eta = \frac{\eta^3}{3}\Big|_{\eta=s}^{\eta=t} = \frac{t^3-s^3}{3}.$$

(b) Let $\mathbb{T} = h\mathbb{Z}$. Clearly, $F(t) = \frac{t(t-h)(t-2h)}{6}$ satisfies $F^{\Delta}(t) = t^2$. Then,

$$\int_s^t \eta^2 \Delta\eta = \frac{\eta(\eta-h)(\eta-2h)}{6}\Big|_{\eta=s}^{\eta=t}$$
$$= \frac{t^3-3ht^2+2h^2t}{6} - \frac{s^3-3hs^2+2h^2s}{6}$$
$$= \frac{1}{6}\Big(\big(t^3-s^3\big) - 3h(t^2-s^2) + 2h^2(t-s)\Big).$$

(c) Let $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ for $q > 1$. Clearly, $F(t) = \frac{1}{q^2+q+1}t^3$ satisfies $F^{\Delta}(t) = t^2$. Then,

$$\int_s^t \eta^2 \Delta\eta = \frac{1}{q^2+q+1}\eta^3\Big|_{\eta=s}^{\eta=t} = \frac{t^3-s^3}{q^2+q+1}.$$

(iv) Let $f(t) = \mathrm{e}^t$.

(a) Let $\mathbb{T} = \mathbb{R}$. Clearly, $F(t) = \mathrm{e}^t$. Then,

$$\int_s^t \mathrm{e}^{\eta} \Delta\eta = \mathrm{e}^{\eta}\Big|_{\eta=s}^{\eta=t} = \mathrm{e}^t - \mathrm{e}^s.$$

(b) *Let* $\mathbb{T} = h\mathbb{Z}$*. Clearly,* $F(t) = \frac{h}{\mathrm{e}^h - 1}\mathrm{e}^t$ *satisfies* $F^{\Delta}(t) = \mathrm{e}^t$*. Then,*

$$\int_s^t \mathrm{e}^{\eta}\Delta\eta = \frac{h}{\mathrm{e}^h - 1}\mathrm{e}^t\Big|_{\eta=s}^{\eta=t} = \frac{h(\mathrm{e}^t - \mathrm{e}^s)}{\mathrm{e}^h - 1}.$$

Hilger's complex plane is one of the main tools used in the solutions of autonomous linear dynamic equations when the characteristic equation has some complex roots.

Definition 8 (Hilger's Complex Plane)**.** *Let* $h > 0$*. We define the* Hilger's complex plane *as* $\mathbb{C}_h := \mathbb{C} \setminus \{-\frac{1}{h}\}$*. For* $h = 0$*, we define* $\mathbb{C}_0 := \mathbb{C}$*.*

Now, we introduce Hilger's complex numbers on the Hilger's complex plane.

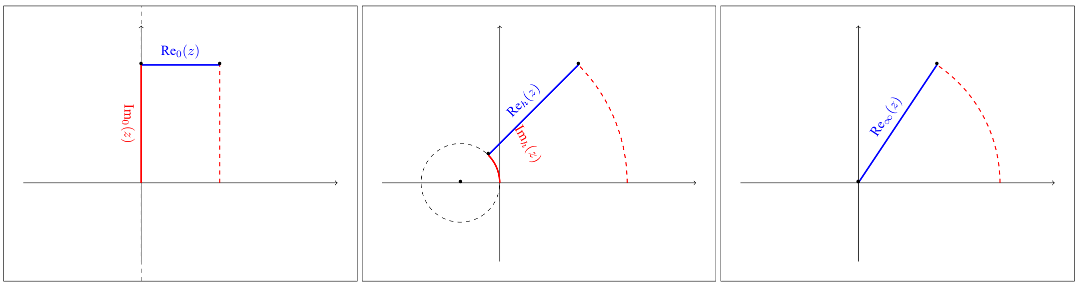
Definition 9 ((Bohner & Peterson, 2001, Definition 2.3))**.** *Let* $h > 0$ *and* $z \in \mathbb{C}_h$*. The* Hilger's real part *of* z *is defined by*

$$\mathrm{Re}_h(z) := \frac{|zh + 1| - 1}{h} \tag{1.2}$$

and the Hilger's imaginary part *of* z *by*

$$\mathrm{Im}_h(z) := \frac{\mathrm{Arg}(zh + 1)}{h}, \tag{1.3}$$

where Arg *is the principal argument function, i.e.,* $-\pi < \mathrm{Arg}(z) \leq \pi$*.*



(a) $h = 0$ (b) $h > 0$ (c) $h = \infty$

Figure 1.1 An illustration of Hilger's complex plane for possible h values.

Definition 10 ((Bohner & Peterson, 2001, Definition 2.4))**.** *Let* $h > 0$ *and* $-\frac{\pi}{h} < \theta \leq \frac{\pi}{h}$*. Hilger's purely imaginary number* $\mathrm{i}_h(\theta)$ *is defined by*

$$\mathrm{i}_h(\theta) := \frac{\mathrm{e}^{\mathrm{i}\theta h} - 1}{h}.$$

An interesting property of Hilger's real part and Hilger's imaginary part reads as follows.

Lemma 2. *Let* $h > 0$*, then*

$$\lim_{h\to 0}\big[\text{Re}_h(z) + \text{i}_h\big(\text{Im}_h(z)\big)\big] = z. \tag{1.4}$$

Proof. To prove (1.4), we will show that

$$\lim_{h\to 0}\big[\text{Re}_h(z) + \text{i}_h\big(\text{Im}_h(z)\big)\big] = \text{Re}(z) + \text{i}\,\text{Im}(z)$$

holds since

$$z = \text{Re}(z) + \text{i}\,\text{Im}(z).$$

More precisely, we will prove that

$$\lim_{h\to 0}\text{Re}_h(z) = \text{Re}(z) \tag{1.5}$$

and

$$\lim_{h\to 0}\text{i}_h\big(\text{Im}_h(z)\big) = \text{i}\,\text{Im}(z). \tag{1.6}$$

First, let us verify (1.5). We compute that

$$\begin{aligned}\text{Re}_h(z) &= \frac{|[\text{Re}(z) + \text{i}\,\text{Im}(z)]h + 1| - 1}{h} \\ &= \frac{\big|\big(\text{Re}(z)h + 1\big) + \text{i}\,\text{Im}(z)h\big| - 1}{h} \\ &= \frac{\sqrt{\big(\text{Re}(z)h + 1\big)^2 + \big(\text{Im}(z)h\big)^2} - 1}{h},\end{aligned}$$

which yields

$$\begin{aligned}&\lim_{h\to 0}\text{Re}_h(z) \\ &= \lim_{h\to 0}\frac{\sqrt{\big(\text{Re}(z)h + 1\big)^2 + \big(\text{Im}(z)h\big)^2} - 1}{h}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\left(\sqrt{\left(\operatorname{Re}(z)h+1\right)^2+\left(\operatorname{Im}(z)h\right)^2}-1\right)\left(\sqrt{\left(\operatorname{Re}(z)h+1\right)^2+\left(\operatorname{Im}(z)h\right)^2}+1\right)}{h\left(\sqrt{\left(\operatorname{Re}(z)h+1\right)^2+\left(\operatorname{Im}(z)h\right)^2}+1\right)}\\
&= \lim_{h\to 0} \frac{2\operatorname{Re}(z)h+\left(\left(\operatorname{Re}(z)\right)^2+\left(\operatorname{Im}(z)\right)^2\right)h^2}{h\left(\sqrt{\left(\operatorname{Re}(z)h+1\right)^2+\left(\operatorname{Im}(z)h\right)^2}+1\right)}\\
&= \lim_{h\to 0} \frac{2\operatorname{Re}(z)+h\left(\left(\operatorname{Re}(z)\right)^2+\left(\operatorname{Im}(z)\right)^2\right)}{\sqrt{\left(\operatorname{Re}(z)h+1\right)^2+\left(\operatorname{Im}(z)h\right)^2}+1}\\
&= \operatorname{Re}(z).
\end{aligned}
$$

This proves (1.5). Next, let us verify (1.6). We compute that

$$
\begin{aligned}
\mathrm{i}_h\left(\frac{\operatorname{Arg}(zh+1)}{h}\right) &= \mathrm{i}_h\left(\frac{1}{h}\arctan\left(\frac{\operatorname{Im}(z)h}{\operatorname{Re}(z)h+1}\right)\right)\\
&= \frac{\mathrm{e}^{\mathrm{i}\arctan\left(\frac{\operatorname{Im}(z)h}{\operatorname{Re}(z)h+1}\right)}-1}{h}
\end{aligned}
$$

and

$$
\lim_{h\to 0}\arctan\left(\frac{\operatorname{Im}(z)h}{\operatorname{Re}(z)h+1}\right)=0,
$$

which yields

$$
\begin{aligned}
\lim_{h\to 0}\mathrm{i}_h\left(\frac{\operatorname{Arg}(zh+1)}{h}\right) &= \lim_{h\to 0}\frac{\mathrm{e}^{\mathrm{i}\arctan\left(\frac{\operatorname{Im}(z)h}{\operatorname{Re}(z)h+1}\right)}-1}{h}\\
&= \lim_{h\to 0}\frac{\mathrm{e}^{\mathrm{i}\arctan\left(\frac{\operatorname{Im}(z)h}{\operatorname{Re}(z)h+1}\right)}-\mathrm{e}^0}{\mathrm{i}\arctan\left(\frac{\operatorname{Im}(z)h}{\operatorname{Re}(z)h+1}\right)-0}\,\frac{\mathrm{i}\arctan\left(\frac{\operatorname{Im}(z)h}{\operatorname{Re}(z)h+1}\right)-0}{h-0}\\
&= \left(\lim_{h\to 0}\frac{\mathrm{e}^{\mathrm{i}\arctan\left(\frac{\operatorname{Im}(z)h}{\operatorname{Re}(z)h+1}\right)}-\mathrm{e}^0}{\mathrm{i}\arctan\left(\frac{\operatorname{Im}(z)h}{\operatorname{Re}(z)h+1}\right)-0}\right)\left(\lim_{h\to 0}\frac{\mathrm{i}\arctan\left(\frac{\operatorname{Im}(z)h}{\operatorname{Re}(z)h+1}\right)-0}{h-0}\right)\\
&= \lim_{h\to 0}\frac{\mathrm{i}\arctan\left(\frac{\operatorname{Im}(z)h}{\operatorname{Re}(z)h+1}\right)}{h}\\
&= \lim_{h\to 0}\frac{\mathrm{i}\operatorname{Im}(z)}{(zh+1)(\overline{z}h+1)}\\
&= \mathrm{i}\operatorname{Im}(z).
\end{aligned}
$$

This proves (1.6). Combining (1.5) and (1.6), we arrive at (1.4). □

Now, we introduce Hilger's binary operations on the Hilger's complex plane.

Definition 11. *For $h \geq 0$ and $z, w \in \mathbb{C}_h$, "circle plus" addition $\oplus_h$ and "circle minus" substraction $\ominus_h$ on $\mathbb{C}_h$ are defined by*

$$z \oplus_h w := z + w + hzw \quad \textit{and} \quad z \ominus_h w := z \oplus_h (\ominus_h w),$$

where $\ominus_h z := -\frac{z}{zh+1}$.

Example 7. *Let $z, w \in \mathbb{C}_h$ for $h \geq 0$. We will compute the expression $z \ominus_h w$.*

$$\begin{aligned} z \ominus w &= z \oplus_h (\ominus_h w) \\ &= z \oplus_h \left(-\frac{w}{wh+1}\right) \\ &= z + \left(-\frac{w}{wh+1}\right) - z\frac{w}{wh+1}h \\ &= \frac{z(wh+1) - w - hzw}{wh+1} \\ &= \frac{z(wh+1-hw) - w}{wh+1} \\ &= \frac{z-w}{wh+1}. \end{aligned}$$

We can see easily that

$$z \ominus_0 w = z - w.$$

Some properties regarding Hilger's real part and Hilger's imaginary part are listed below.

Lemma 3. *Let $h > 0$ and $z \in \mathrm{C}_{\mathrm{rd}}$. Then, the following identities hold.*

(i) $\mathrm{Re}_h(\ominus_h z) = \ominus_\mu \mathrm{Re}_h(z)$.

(ii) $\mathrm{Im}_h(\ominus_h z) = -\mathrm{Im}_h(z)$.

(iii) $\mathrm{Re}_h(\overline{z}) = \mathrm{Re}_h(z)$.

(iv) $\mathrm{Im}_h(\overline{z}) = -\mathrm{Im}_h(z)$.

(v) $\mathrm{i}_h(-\theta) = \ominus_h \mathrm{i}_h(\theta)$ *where* $-\frac{\pi}{h} < \theta \leq \frac{\pi}{h}$.

Proof. (i) Let us take $z \in \mathbb{C}_h$. Using (1.2) and Definition 11, we compute that

$$\begin{aligned}\mathrm{Re}_h(\ominus z) &= \frac{\left|-\frac{z}{zh+1}h+1\right|-1}{h}\\ &= \frac{\left|\frac{1}{zh+1}\right|-1}{h}\\ &= \frac{1-|zh+1|}{h|zh+1|}\\ &= -\frac{\frac{|zh+1|-1}{h}}{\frac{|zh+1|-1}{h}h+1}\\ &= \ominus\,\mathrm{Re}_h(z).\end{aligned}$$

This completes the proof.

(ii) Let us take $z \in \mathbb{C}_h$. Using (1.3) and Definition 11, we compute that

$$\begin{aligned}\mathrm{Im}_h(\ominus z) &= \frac{\mathrm{Arg}(-\frac{z}{zh+1}h+1)}{h} = \frac{\mathrm{Arg}(\frac{1}{zh+1})}{h}\\ &= -\frac{\mathrm{Arg}(zh+1)}{h} = -\,\mathrm{Im}_h(z).\end{aligned}$$

This completes the proof.

(iii) Let us take $\overline{z} \in \mathbb{C}_h$. Using (1.2), we compute that

$$\mathrm{Re}_h(\overline{z}) = \frac{|\overline{z}h+1|-1}{h} = \frac{|zh+1|-1}{h} = \mathrm{Re}_h(z).$$

This completes the proof.

(iv) Let us take $\overline{z} \in \mathbb{C}_h$. Using (1.3), we compute that

$$\mathrm{Im}_h(\overline{z}) = \frac{\mathrm{Arg}(\overline{z}h+1)}{h} = -\frac{\mathrm{Arg}(zh+1)}{h} = -\,\mathrm{Im}_h(z).$$

This completes the proof.

(v) Let $-\frac{\pi}{h} < \theta \leq \frac{\pi}{h}$. Using Definition 10, we compute that

$$\mathrm{i}_h(-\theta) = \frac{\mathrm{e}^{-\mathrm{i}\theta h} - 1}{h} = -\frac{\frac{\mathrm{e}^{\mathrm{i}\theta h}-1}{h}}{\mathrm{e}^{\mathrm{i}\theta h}} = -\frac{\mathrm{i}_h(\theta)}{\mathrm{i}_h(\theta)h + 1} = \ominus_h \mathrm{i}_h(\theta).$$

This completes the proof. □

Additional properties for z and $\overline{z}$ on Hilger's complex plane are presented below.

Property 2. *For $h \in \mathbb{R}^+$ and $z \in \mathbb{C}_h$, then we have the following:*

(i) $z = \mathrm{Re}_h(z) \oplus_h \mathrm{i}_h\big(\mathrm{Im}_h(z)\big)$.

(ii) $\overline{z} = \mathrm{Re}_h(z) \ominus_h \mathrm{i}_h\big(\mathrm{Im}_h(z)\big)$.

(iii) $\ominus_h z = \big(\ominus_h \mathrm{Re}_h(z)\big) \ominus_h \mathrm{i}_h\big(\mathrm{Im}_h(z)\big)$.

Proof. (i) Let us take $z \in \mathbb{C}_h$. Then, we have

$$\begin{aligned}
\mathrm{Re}_h(z) \oplus_h \mathrm{i}_h\big(\mathrm{Im}_h(z)\big) &= \frac{|zh+1|-1}{h} \oplus_h \mathrm{i}_h \frac{\mathrm{Arg}(zh+1)}{h} \\
&= \frac{|zh+1|-1}{h} \oplus_h \frac{\mathrm{e}^{\mathrm{i}\,\mathrm{Arg}(zh+1)} - 1}{h} \\
&= \frac{|zh+1|-1}{h} + \frac{\mathrm{e}^{\mathrm{i}\,\mathrm{Arg}(zh+1)} - 1}{h} \\
&\quad + h\frac{|zh+1|-1}{h}\frac{\mathrm{e}^{\mathrm{i}\,\mathrm{Arg}(zh+1)} - 1}{h} \\
&= \frac{1}{h}\Big\{|zh+1| - 1 + \mathrm{e}^{\mathrm{i}\,\mathrm{Arg}(zh+1)} - 1 \\
&\qquad + \big[|zh+1| - 1\big]\Big[\mathrm{e}^{\mathrm{i}\,\mathrm{Arg}(zh+1)} - 1\Big]\Big\} \\
&= \frac{1}{h}\Big\{|zh+1|\mathrm{e}^{\mathrm{i}\,\mathrm{Arg}(zh+1)} - 1\Big\} \\
&= \frac{(zh+1)-1}{h} \\
&= z,
\end{aligned}$$

where we have used Definition 9 for the first equality and Definition 10 for the second equality.

(ii) Let us take $z \in \mathbb{C}_h$. Then, we have

$$\begin{aligned}\overline{z} &= \mathrm{Re}_h(\overline{z}) \oplus_h \mathrm{i}_h\big(\mathrm{Im}_h(\overline{z})\big) \\ &= \mathrm{Re}_h(z) \oplus_h \mathrm{i}_h\big(-\mathrm{Im}_h(z)\big) \\ &= \mathrm{Re}_h(z) \ominus_h \mathrm{i}_h\big(\mathrm{Im}_h(z)\big),\end{aligned}$$

where we have used Lemma 3 (iii) for the second equality and (ii) for the third equality.

(iii) Let us take $z \in \mathbb{C}_h$. Then, we have

$$\begin{aligned}\ominus_h z &= \mathrm{Re}_h(\ominus_h z) \oplus_h \mathrm{i}_h\big(\mathrm{Im}_h(\ominus_h z)\big) \\ &= \ominus_h \mathrm{Re}_h(z) \oplus_h \mathrm{i}_h(-\mathrm{Im}_h(z) \\ &= (\ominus_h \mathrm{Re}_h(z)) \ominus_h \mathrm{i}_h(\mathrm{Im}_h(z)),\end{aligned}$$

where the second equation holds by Lemma 3 (i) (ii) and the third equation holds by (iv) and (v). □

CHAPTER 2
FIRST-ORDER DYNAMIC EQUATIONS

We start this section the definition of regressivity, which is important in the existence and uniqueness property of solutions to dynamic equations.

Definition 12. *A function* $p : \mathbb{T} \to \mathbb{R}$ *is called regressive provided*

$$p(t)\mu(t) + 1 \neq 0 \quad \textit{for all} \quad t \in \mathbb{T}^{\kappa}.$$

Definition 13 ((Bohner & Peterson, 2001, Theorem 2.32))**.** *The first-order dynamic equation*

$$y^{\Delta} = p(t)y \tag{2.1}$$

is called regressive provided that $p \in \mathrm{C}_{\mathrm{rd}}$ *is regressive.*

Next, we will introduce the generalized exponential function on time scales.

Definition 14 ((Bohner & Peterson, 2001, Definition 2.33))**.** *Suppose that* (2.1) *is regressive, and let* $s \in \mathbb{T}$*. Then, the unique solution of the initial value problem*

$$\begin{cases} y^{\Delta} = p(t)y & \textit{on } \mathbb{T}^{\kappa} \\ y(s) = 1 \end{cases} \tag{2.2}$$

is defined to be $\mathrm{e}_p(\cdot, s)$.

Below, we will give explicit forms of the generalized exponential function on some particular time scales.

Example 8. *Consider the initial value problem*

$$\begin{cases} y^{\Delta} = p(t)y & \textit{on } \mathbb{T}^{\kappa} \\ y(s) = 1, \end{cases} \tag{2.3}$$

where (2.3) *is regressive.*

(i) Let $\mathbb{T} = \mathbb{R}$ *(see Ross (1980), Brand (1966)). We will show that*

$$y(t) = \exp\left\{\int_s^t p(\eta)\mathrm{d}\eta\right\} \quad \textit{for } s, t \in \mathbb{R}. \tag{2.4}$$

To show this we substitute y *into the IVP* (2.3)*. It is obvious that*

$$y(s) = \exp\left\{\int_s^s p(\eta)\mathrm{d}\eta\right\} = 1.$$

Then, we compute that

$$y^{\Delta}(t) = y'(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\int_s^t p(\eta)\mathrm{d}\eta\right)\exp\left\{\int_s^t p(\eta)\mathrm{d}\eta\right\} = p(t)y(t).$$

Hence, (2.4) *is a unique solution of* (2.3) *on* $\mathbb{T} = \mathbb{R}$*.*

(ii) Let $\mathbb{T} = h\mathbb{Z}$ *(see Brand (1966), Kelley & Peterson (2001)). We will show that*

$$y(t) = \prod_{j=\frac{s}{h}}^{\frac{t}{h}-1}[p(jh)h + 1] \quad \textit{for } s, t \in h\mathbb{Z}, \tag{2.5}$$

where the empty product is assumed to be 1*. To show this we substitute* y *into the IVP* (2.3)*. First, it is obvious that*

$$y(s) = \prod_{j=\frac{s}{h}}^{\frac{s}{h}-1}[p(jh)h + 1] = 1.$$

Then, we compute that

$$\begin{aligned}
y^{\Delta}(t) &= \frac{y(t+h) - y(t)}{h} \\
&= \frac{1}{h}\left(\prod_{j=\frac{s}{h}}^{\frac{t+h}{h}-1}[p(jh)h + 1] - \prod_{j=\frac{s}{h}}^{\frac{t}{h}-1}[p(jh)h + 1]\right) \\
&= \frac{1}{h}\left(\prod_{j=\frac{s}{h}}^{\frac{t}{h}}[p(jh)h + 1] - \prod_{j=\frac{s}{h}}^{\frac{t}{h}-1}[p(jh)h + 1]\right) \\
&= \frac{1}{h}\left(\prod_{j=\frac{s}{h}}^{\frac{t}{h}-1}[p(jh)h + 1]\right)\left(p(\tfrac{t}{h}h)h + 1 - 1\right)
\end{aligned}$$

$$= p(t) \prod_{j=\frac{s}{h}}^{\frac{t}{h}-1} [p(jh)h + 1]$$

$$= p(t)y(t).$$

Hence, (2.5) *is a unique solution of* (2.3) *on* $\mathbb{T} = h\mathbb{Z}$*.*

(iii) Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *for* $q > 1$ *(see Kac & Cheung (2002)). We will show that*

$$y(t) = \prod_{j=\log_q(s)}^{\log_q(t)-1} \left[p(q^j)(q-1)q^j + 1\right] \quad \textit{for } s, t \in \mathbb{R}. \tag{2.6}$$

To show this we substitute y *into the IVP* (2.3)*. It is obvious that*

$$y(s) = \prod_{j=\log_q(s)}^{\log_q(s)-1} \left[p(q^j)(q-1)q^j + 1\right] = 1.$$

Then, we compute that

$$\begin{aligned}
y^{\Delta} &= \frac{y(qt) - y(t)}{(q-1)t} \\
&= \frac{1}{(q-1)t}\left(\prod_{j=\log_q(s)}^{\log_q(qt)-1} \left[p(q^j)(q-1)q^j + 1\right] - \prod_{j=\log_q(s)}^{\log_q(t)-1} \left[p(q^j)(q-1)q^j + 1\right]\right) \\
&= \frac{1}{(q-1)t}\left(\prod_{j=\log_q(s)}^{\log_q(t)} \left[p(q^j)(q-1)q^j + 1\right] - \prod_{j=\log_q(s)}^{\log_q(t)-1} \left[p(q^j)(q-1)q^j + 1\right]\right) \\
&= \frac{1}{(q-1)t}\left(\prod_{j=\log_q(s)}^{\log_q(t)-1} \left[p(q^j)(q-1)q^j + 1\right]\right) \\
&\quad \times \left(p(q^{\log_q(t)})(q-1)q^{\log_q(t)} + 1 - 1\right) \\
&= p(t) \prod_{j=\log_q(s)}^{\log_q(t)-1} \left[p(q^j)(q-1)q^j + 1\right] \\
&= p(t)y(t).
\end{aligned}$$

Hence, (2.6) *is a unique solution of* (2.3) *on* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$*.*

(iv) Let $\mathbb{T} = \mathbb{P}_{a,b}$ for $a, b \in \mathbb{R}^+$. We will show that

$$y(t) = \left(\prod_{j=\lfloor \frac{s}{a+b} \rfloor}^{\lfloor \frac{t}{a+b} \rfloor - 1} \Big(p\big(j(a+b)+a\big)b + 1 \Big) \right) \times \exp\left\{ \sum_{j=\lfloor \frac{s}{a+b} \rfloor}^{\lfloor \frac{t}{a+b} \rfloor - 1} \int_{j(a+b)}^{j(a+b)+a} p(\eta)\mathrm{d}\eta + \int_{\lfloor \frac{t}{a+b} \rfloor (a+b)}^{t} p(\eta)\mathrm{d}\eta - \int_{\lfloor \frac{s}{a+b} \rfloor (a+b)}^{s} p(\eta)\mathrm{d}\eta \right\}, \tag{2.7}$$

where the empty sum is assumed to be 0*, while the empty product is assumed to be* 1*. To show this we substitute y into the IVP* (2.3)*. It is obvious that*

$$\begin{aligned} y(s) &= \left(\prod_{j=\lfloor \frac{s}{a+b} \rfloor}^{\lfloor \frac{s}{a+b} \rfloor - 1} \Big(p\big(j(a+b)+a\big)b + 1 \Big) \right) \times \exp\left\{ \sum_{j=\lfloor \frac{s}{a+b} \rfloor}^{\lfloor \frac{s}{a+b} \rfloor - 1} \int_{j(a+b)}^{j(a+b)+a} p(\eta)\mathrm{d}\eta + \int_{\lfloor \frac{s}{a+b} \rfloor (a+b)}^{s} p(\eta)\mathrm{d}\eta - \int_{\lfloor \frac{s}{a+b} \rfloor (a+b)}^{s} p(\eta)\mathrm{d}\eta \right\} \\ &= 1 \times \mathrm{e}^0 = 1. \end{aligned}$$

Let $t \in [k(a+b), k(a+b)+a)_{\mathbb{R}}$ for some $k \in \mathbb{Z}$, then

$$\begin{aligned} y^{\Delta}(t) = y'(t) &= p(t) \left(\prod_{j=\lfloor \frac{s}{a+b} \rfloor}^{k-1} \Big(p\big(j(a+b)+a\big)b + 1 \Big) \right) \times \exp\left\{ \sum_{j=\lfloor \frac{s}{a+b} \rfloor}^{k-1} \int_{j(a+b)}^{j(a+b)+a} p(\eta)\mathrm{d}\eta + \int_{k(a+b)}^{t} p(\eta)\mathrm{d}\eta - \int_{\lfloor \frac{s}{a+b} \rfloor (a+b)}^{s} p(\eta)\mathrm{d}\eta \right\} \\ &= p(t)y(t). \end{aligned}$$

Let $t = k(a+b)+a$ for some $k \in \mathbb{Z}$, then

$$\begin{aligned}
y^{\Delta}(t) &= \frac{y(t+b)-y(t)}{b} \\
&= \frac{1}{b}\Bigg[\Bigg(\prod_{j=\lfloor\frac{s}{a+b}\rfloor}^{k}\Big(p\big(j(a+b)+a\big)b+1\Big)\Bigg) \\
&\qquad \times \exp\Bigg\{\sum_{j=\lfloor\frac{s}{a+b}\rfloor}^{k}\int_{j(a+b)}^{j(a+b)+a} p(\eta)\mathrm{d}\eta - \int_{\lfloor\frac{s}{a+b}\rfloor(a+b)}^{s} p(\eta)\mathrm{d}\eta\Bigg\} \\
&\qquad - \Bigg(\prod_{j=\lfloor\frac{s}{a+b}\rfloor}^{k-1}\Big(p\big(j(a+b)+a\big)b+1\Big)\Bigg) \\
&\qquad\quad \times \exp\Bigg\{\sum_{j=\lfloor\frac{s}{a+b}\rfloor}^{k}\int_{j(a+b)}^{j(a+b)+a} p(\eta)\mathrm{d}\eta - \int_{\lfloor\frac{s}{a+b}\rfloor(a+b)}^{s} p(\eta)\mathrm{d}\eta\Bigg\}\Bigg] \\
&= p(t)y(t).
\end{aligned}$$

Hence, (2.7) *is a unique solution of* (2.3) *on $\mathbb{T} = \mathbb{P}_{a,b}$.*

We now give the some properties of the exponential function.

Theorem 4. *Let p and q are regressive, then for $t, s \in \mathbb{T}$, the following properties are satisfied.*

(i) $\mathrm{e}_0(t,s) = 1$ *and* $\mathrm{e}_p(t,t) = 1$.

(ii) $\mathrm{e}_p(\sigma(t),s) = [p(t)\mu(t)+1]\mathrm{e}_p(t,s)$ *for $t \in \mathbb{T}$ and $s \in \mathbb{T}^{\kappa}$.*

(iii) $\frac{1}{\mathrm{e}_p(t,s)} = \mathrm{e}_{\ominus_\mu p}(t,s)$ *and* $\mathrm{e}_p(t,s) = \frac{1}{\mathrm{e}_p(s,t)}$.

(iv) $\mathrm{e}_p(t,s)\mathrm{e}_q(t,s) = \mathrm{e}_{p\oplus_\mu q}(t,s)$.

(v) $\dfrac{\mathrm{e}_p(t,s)}{\mathrm{e}_q(t,s)} = \mathrm{e}_{p\ominus_\mu q}(t,s)$.

Proof. (i) We know that $y(t) \equiv 1$ is obviously a solution of the IVP $y^{\Delta} = 0 \cdot y$, $y(s) = 1$. By Definition 14, this problem has only one solution. Hence, we have $y(t) = \mathrm{e}_0(t,s) = 1$.

(ii) By Remark 1, we have

$$\begin{aligned} \mathrm{e}_p(\sigma(t), s) &= \mathrm{e}_p(t, s) + \mu(t)\mathrm{e}_p^{\Delta}(t, s) \\ &= \mathrm{e}_p(t, s) + \mu(t)p(t)\mathrm{e}_p(t, s) \\ &= [p(t)\mu(t) + 1]\mathrm{e}_p(t, s). \end{aligned}$$

(iii) We consider the IVP

$$\begin{cases} y^{\Delta} = (\ominus_{\mu} p)(t)y, \\ y(s) = 1. \end{cases} \tag{2.8}$$

We can easily to see that the dynamic equation in (2.8) is regressive. Now, we will show that $y(t) = \frac{1}{\mathrm{e}_p(t,s)}$ satisfy the IVP for all s. By part (i), we have $\frac{1}{\mathrm{e}_p(s,s)} = 1$ and we use the Lemma 1 (v) to obtain

$$\begin{aligned} y^{\Delta}(t) &= \left(\frac{1}{\mathrm{e}_p(\cdot, s)}\right)^{\Delta}(t) \\ &= -\frac{\mathrm{e}_p^{\Delta}(t, s)}{\mathrm{e}_p(t, s)\mathrm{e}_p^{\sigma}(t, s)} \\ &= -\frac{p(t)}{\mathrm{e}_p^{\sigma}(t, s)} \\ &= -\frac{p(t)}{(p(t)\mu(t) + 1)\mathrm{e}_p(t, s)} \\ &= (\ominus_{\mu} p)(t)y(t). \end{aligned}$$

(iv) We consider the IVP

$$\begin{cases} y^{\Delta} = (p \oplus_{\mu} q)(t)y, \\ y(s) = 1. \end{cases} \tag{2.9}$$

We can easily to see that the dynamic equation in (2.9) is regressive. Now, we will show that $y(t) = \mathrm{e}_p(t, s)\mathrm{e}_q(t, s)$ satisfy the IVP. By part (i), we have $\mathrm{e}_p(s, s)\mathrm{e}_q(s, s) = 1$ and we can compute

$$\begin{aligned} y^{\Delta}(t) &= \left(\mathrm{e}_p(\cdot, s)\mathrm{e}_q(\cdot, s)\right)^{\Delta}(t) \\ &= p(t)\mathrm{e}_p(t, s)\mathrm{e}_q(\sigma(t), s) + \mathrm{e}_p(t, s)q(t)\mathrm{e}_q(t, s) \end{aligned}$$

$$
\begin{aligned}
&= [q(t)\mu(t)+1]p(t)\mathrm{e}_p(t,s)\mathrm{e}_q(t,s) + \mathrm{e}_p(t,s)q(t)\mathrm{e}_q(t,s) \\
&= \big([q(t)\mu(t)+1]p(t)+q(t)\big)\mathrm{e}_p(t,s)\mathrm{e}_q(t,s) \\
&= (p \oplus_\mu q)(t)\mathrm{e}_p(t,s)\mathrm{e}_q(t,s) \\
&= (p \oplus_\mu q)(t)y(t).
\end{aligned}
$$

Therefore, we have that $\mathrm{e}_p(t,s)\mathrm{e}_q(t,s) = \mathrm{e}_{p\oplus_\mu q}(t,s)$.

(v) This follows easily using parts (iii) and (iv). □

The time scales exponential function satisfies semigroup property.

Lemma 4. *Let p be regressive, then $\mathrm{e}_p(t,s)\mathrm{e}_p(s,r) = \mathrm{e}_p(t,r)$ for $r,s,t \in \mathbb{T}$ is satisfied.*

Proof. Define $y(t) := \mathrm{e}_p(t,s)\mathrm{e}_p(s,r)$ for $t \in \mathbb{T}$. Then, we have

$$
y^\Delta(t) = p(t)\mathrm{e}_p(t,s)\mathrm{e}_p(s,r) = p(t)y(t) \quad \text{for all } t \in \mathbb{T}^\kappa.
$$

Further, we compute that

$$
y(r) = \mathrm{e}_p(r,s)\mathrm{e}_p(s,r) = \frac{\mathrm{e}_p(r,s)}{\mathrm{e}_p(r,s)} = 1
$$

by Theorem 4 (iii). □

When we will be dealing with second-order dynamic equations, we will be requiring the following function.

Definition 15. *Let $s,t \in \mathbb{T}$, and suppose that p is regressive (i.e., $p(t)\mu(t)+1 \neq 0$ for all $t \in \mathbb{T}$). The* generalized monomial operator *is defined by*

$$
\mathrm{m}_p(t,s) = \int_s^t \frac{1}{p(\eta)\mu(\eta)+1}\Delta\eta.
$$

We define

$$
\mathbb{C}_{\mu(\mathbb{T})} := \bigcap_{h\in\mu(\mathbb{T})} \mathbb{C}_h \tag{2.10}
$$

or equivalently

$$
\begin{aligned}
\mathbb{C}_{\mu(\mathbb{T})} &= \{z \in \mathbb{C}: \ z \in \mathbb{C}_{\mu(t)} \text{ for all } t \in \mathbb{T}\} \\
&= \{z \in \mathbb{C}: \ z\mu(t) + 1 \neq 0 \text{ for all } t \in \mathbb{T}\}.
\end{aligned}
$$

Corollary 1 ((Karpuz, 2017, Corollary 3.2)). *Let $s, t \in \mathbb{T}$ and $z \in \mathbb{C}_{\mu(\mathbb{T})}$. Then by (Karpuz, 2017, Theorem 3.1), we have*

$$
\mathrm{e}_z(t,s) := \exp\bigg\{\int_0^z \mathrm{m}_\lambda(t,s)\mathrm{d}\lambda\bigg\},
$$

where the integral is taken over any regressive curve starting at 0 ending at z.

Example 9. *We will compute $\mathrm{e}_z(t,s)$ for some time scales.*

(i) Let $\mathbb{T} = \mathbb{R}$ and $s, t \in \mathbb{R}$. We know that $\mu(t) \equiv 0$ for $t \in \mathbb{R}$. Then, we have

$$
\begin{aligned}
\mathrm{e}_z(t,s) &= \exp\bigg\{\int_0^z \mathrm{m}_\lambda(t,s)\mathrm{d}\lambda\bigg\} \\
&= \exp\bigg\{\int_0^z \int_s^t \frac{1}{\lambda\mu(\eta)+1}\mathrm{d}\eta\mathrm{d}\lambda\bigg\} \\
&= \exp\bigg\{\int_0^z \int_s^t 1\mathrm{d}\eta\mathrm{d}\lambda\bigg\} \\
&= \exp\bigg\{\int_0^z (t-s)\mathrm{d}\lambda\bigg\} \\
&= \mathrm{e}^{z(t-s)}.
\end{aligned}
$$

(ii) Let $\mathbb{T} = h\mathbb{Z}$ and $s, t \in h\mathbb{Z}$. We know that $\mu(t) \equiv h$ for $t \in h\mathbb{Z}$. Then, we have

$$
\begin{aligned}
\mathrm{e}_z(t,s) &= \exp\bigg\{\int_0^z \mathrm{m}_\lambda(t,s)\mathrm{d}\lambda\bigg\} \\
&= \exp\bigg\{\int_0^z \int_s^t \frac{1}{\lambda h+1}\Delta\eta\mathrm{d}\lambda\bigg\} \\
&= \exp\bigg\{\int_0^z \frac{t-s}{\lambda h+1}\mathrm{d}\lambda\bigg\} \\
&= \exp\bigg\{\frac{t-s}{h}\mathrm{Log}(\lambda h+1)\bigg|_{\lambda=0}^{\lambda=z}\bigg\} \\
&= \exp\bigg\{\frac{t-s}{h}\mathrm{Log}(zh+1)\bigg\}
\end{aligned}
$$

$$= (zh+1)^{\frac{t-s}{h}}.$$

(iii) Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ and $s, t \in \overline{q^{\mathbb{Z}}}$. We know that $\mu(t) = (q-1)t$ for $t \in \overline{q^{\mathbb{Z}}}$. Then, we have

$$\begin{aligned}
\mathrm{e}_z(t,s) &= \exp\left\{\int_0^z \mathrm{m}_\lambda(t,s)\mathrm{d}\lambda\right\} \\
&= \exp\left\{\int_0^z \int_s^t \frac{1}{\lambda(q-1)\eta+1}\Delta\eta\mathrm{d}\lambda\right\} \\
&= \exp\left\{\int_0^z \sum_{j=\log_q(s)}^{\log_q(t)-1} \frac{(q-1)q^j}{\lambda(q-1)q^j+1}\mathrm{d}\lambda\right\} \\
&= \exp\left\{\sum_{j=\log_q(s)}^{\log_q(t)-1} \int_0^z \frac{(q-1)q^j}{\lambda(q-1)q^j+1}\mathrm{d}\lambda\right\} \\
&= \exp\left\{\sum_{j=\log_q(s)}^{\log_q(t)-1} \mathrm{Log}\big(\lambda(q-1)q^j+1\big)\Big|_{\lambda=0}^{\lambda=z}\right\} \\
&= \exp\left\{\sum_{j=\log_q(s)}^{\log_q(t)-1} \mathrm{Log}\big(z(q-1)q^j+1\big)\right\} \\
&= \exp\left\{\mathrm{Log}\left(\prod_{j=\log_q(s)}^{\log_q(t)-1} \big(z(q-1)q^j+1\big)\right)\right\} \\
&= \prod_{j=\log_q(s)}^{\log_q(t)-1} \big(z(q-1)q^j+1\big).
\end{aligned}$$

(iv) Let $\mathbb{T} = \mathbb{P}_{a,b} = \cup_{\ell=-\infty}^{\infty}[\ell(a+b), \ell(a+b)+a]_{\mathbb{R}}$ for $a, b > 0$. In view of (1.1)*, we compute*

$$\begin{aligned}
F(t) = &\sum_{j=0}^{\left\lfloor \frac{t}{a+b}\right\rfloor -1}\left[\int_{j(a+b)}^{j(a+b)+a} \frac{1}{z\mu(\eta)+1}\mathrm{d}\eta + \frac{1}{z\mu\big(j(a+b)+a\big)+1}b\right] \\
&+ \int_{\left\lfloor \frac{t}{a+b}\right\rfloor (a+b)}^{t} \frac{1}{z\mu(\eta)+1}\mathrm{d}\eta \\
= &\sum_{j=0}^{\left\lfloor \frac{t}{a+b}\right\rfloor -1}\left[\int_{j(a+b)}^{j(a+b)+a} 1\mathrm{d}\eta + \frac{1}{zb+1}b\right] + \int_{\left\lfloor \frac{t}{a+b}\right\rfloor (a+b)}^{t} 1\mathrm{d}\eta
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\left\lfloor \frac{t}{a+b} \right\rfloor - 1} \left[a + \frac{b}{zb+1}\right] + \left(t - \left\lfloor \frac{t}{a+b} \right\rfloor (a+b)\right) \\
&= \left\lfloor \frac{t}{a+b} \right\rfloor \left[a + \frac{b}{zb+1}\right] + \left(t - \left\lfloor \frac{t}{a+b} \right\rfloor (a+b)\right) \\
&= t - \frac{zb^2}{zb+1} \left\lfloor \frac{t}{a+b} \right\rfloor .
\end{aligned}
$$

This gives

$$
\begin{aligned}
\mathrm{m}_z(t,s) &= \int_s^t \frac{1}{z\mu(\eta)+1} \Delta\eta \\
&= F(t) - F(s) \\
&= (t-s) - \frac{zb^2}{zb+1} \left(\left\lfloor \frac{t}{a+b} \right\rfloor - \left\lfloor \frac{s}{a+b} \right\rfloor \right).
\end{aligned}
$$

Then, we have

$$
\int_0^z \mathrm{m}_\lambda(t,s) \mathrm{d}\lambda = (t-s)z - \big(bz - \mathrm{Log}(zb+1)\big) \left(\left\lfloor \frac{t}{a+b} \right\rfloor - \left\lfloor \frac{s}{a+b} \right\rfloor \right),
$$

which yields

$$
\begin{aligned}
\mathrm{e}_z(t,s) = &\exp\left\{ (t-s)z - \big(bz - \mathrm{Log}(zb+1)\big) \left(\left\lfloor \frac{t}{a+b} \right\rfloor - \left\lfloor \frac{s}{a+b} \right\rfloor \right) \right\} \\
&= (zb+1)^{\lfloor \frac{t}{a+b} \rfloor - \lfloor \frac{s}{a+b} \rfloor} \exp\left\{ \left((t-s) - b\left(\left\lfloor \frac{t}{a+b} \right\rfloor - \left\lfloor \frac{s}{a+b} \right\rfloor \right) \right) z \right\}.
\end{aligned}
$$

2.1 Homogeneous Equations

The following theorem considers first-order homogeneous linear dynamic equations.

Theorem 5 ((Bohner & Peterson, 2001, Theorem 2.62))**.** *Suppose* (2.1) *is regressive.*

Let $s \in \mathbb{T}$*, and* $y_0 \in \mathbb{R}$*. The unique solution of the initial value problem*

$$\begin{cases} y^{\Delta} - p(t)y = 0 \\ y(s) = y_0 \end{cases}$$

is given by

$$y = y_0 \mathrm{e}_p(t,s) \quad \textit{for } t \in \mathbb{T}.$$

2.2 Non-homogeneous Equations

The following theorem considers first-order non-homogeneous linear dynamic equations.

Theorem 6 ((Bohner & Peterson, 2001, Theorem 2.77))**.** *Suppose* (2.1) *is regressive. Let* $s \in \mathbb{T}$*, and* $y_0 \in \mathbb{R}$*. The unique solution of the initial value problem*

$$\begin{cases} y^{\Delta} - p(t)y = f(t) \\ y(s) = y_0 \end{cases}$$

is given by

$$y = y_0 \mathrm{e}_p(t,s) + \int_s^t \mathrm{e}_p(t, \sigma(\eta)) f(\eta) \Delta\eta \quad \textit{for } t \in \mathbb{T}.$$

Now, we give some examples for Theorem 6 on some well known time scales.

Example 10. *Consider the initial value problem*

$$\begin{cases} y^{\Delta} - p(t)y = f(t) \quad \textit{on } \mathbb{T}^{\kappa} \\ y(s) = y_0, \end{cases} \tag{2.11}$$

where (2.11) *is regressive.*

(i) Let $\mathbb{T}=\mathbb{R}$ (see Ross (1980), Brand (1966)). We will show that

$$y(t)=y_0\exp\left\{\int_s^t p(\eta)\mathrm{d}\eta\right\}+\int_s^t \exp\left\{\int_\eta^t p(\zeta)\mathrm{d}\zeta\right\}f(\eta)\mathrm{d}\eta \quad \textit{for } s,t\in\mathbb{R}. \tag{2.12}$$

To show this, we substitute y into the IVP (2.11). *It is obvious that*

$$\begin{aligned} y(s) &= y_0\exp\left\{\int_s^s p(\eta)\mathrm{d}\eta\right\}+\int_s^s \exp\left\{\int_\eta^s p(\zeta)\mathrm{d}\zeta\right\}f(\eta)\mathrm{d}\eta \\ &= y_0. \end{aligned}$$

Then, we compute that

$$\begin{aligned} y^{\Delta}(t) &= y'(t) \\ &= p(t)y_0\exp\left\{\int_s^t p(\eta)\mathrm{d}\eta\right\}+p(t)\int_s^t \exp\left\{\int_\eta^t p(\zeta)\mathrm{d}\zeta\right\}f(\eta)\mathrm{d}\eta \\ &\quad + f(t) \\ &= p(t)\left(y_0\exp\left\{\int_s^t p(\eta)\mathrm{d}\eta\right\}+\int_s^t \exp\left\{\int_\eta^t p(\zeta)\mathrm{d}\zeta\right\}f(\eta)\mathrm{d}\eta\right) \\ &\quad + f(t) \\ &= p(t)y+f(t). \end{aligned}$$

Hence, (2.12) *is a unique solution of* (2.11) *on $\mathbb{T}=\mathbb{R}$.*

(ii) Let $\mathbb{T}=h\mathbb{Z}$ (see Brand (1966), Kelley & Peterson (2001)). We will show that

$$y(t)=y_0\prod_{j=\frac{s}{h}}^{\frac{t}{h}-1}[p(jh)h+1]+h\sum_{i=\frac{s}{h}}^{\frac{t}{h}-1}\prod_{j=i+1}^{\frac{t}{h}-1}[p(jh)h+1]f(ih), \tag{2.13}$$

where the empty sum is assumed to be 0*, while the empty product is assumed to be* 1*. To show this we substitute y into the IVP* (2.11)*. First, it is obvious that*

$$\begin{aligned} y(s) &= y_0\prod_{j=\frac{s}{h}}^{\frac{s}{h}-1}[p(jh)h+1]+h\sum_{i=\frac{s}{h}}^{\frac{s}{h}-1}\prod_{j=i+1}^{\frac{s}{h}-1}[p(jh)h+1]f(ih) \\ &= y_0. \end{aligned}$$

Then, we compute that

$$
\begin{aligned}
y^{\Delta}(t) &= \frac{y(t+h)-y(t)}{h} \\
&= \frac{1}{h}\Bigg(y_0 \prod_{j=\frac{s}{h}}^{\frac{t}{h}} \big[p(jh)h+1\big] + h \sum_{i=\frac{s}{h}}^{\frac{t}{h}} \prod_{j=i+1}^{\frac{t}{h}} [p(jh)h+1] f(ih) \\
&\qquad - y_0 \prod_{j=\frac{s}{h}}^{\frac{t}{h}-1} \big[p(jh)h+1\big] - h \sum_{i=\frac{s}{h}}^{\frac{t}{h}-1} \prod_{j=i+1}^{\frac{t}{h}-1} [p(jh)h+1] f(ih) \Bigg) \\
&= \frac{1}{h}\Bigg(y_0 p(t) h \prod_{j=\frac{s}{h}}^{\frac{t}{h}-1} \big[p(jh)h+1\big] + h \sum_{i=\frac{s}{h}}^{\frac{t}{h}-1} \prod_{j=i+1}^{\frac{t}{h}} [p(jh)h+1] f(ih) \\
&\qquad + hf(t) - h \sum_{i=\frac{s}{h}}^{\frac{t}{h}-1} \prod_{j=i+1}^{\frac{t}{h}-1} [p(jh)h+1] f(ih) \Bigg) \\
&= y_0 p(t) \prod_{j=\frac{s}{h}}^{\frac{t}{h}-1} \big[p(jh)h+1\big] + [p(t)h+1] \sum_{i=\frac{s}{h}}^{\frac{t}{h}-1} \prod_{j=i+1}^{\frac{t}{h}-1} [p(jh)h+1] f(ih) \\
&\quad + f(t) - \sum_{i=\frac{s}{h}}^{\frac{t}{h}-1} \prod_{j=i+1}^{\frac{t}{h}-1} [p(jh)h+1] f(ih) \\
&= y_0 p(t) \prod_{j=\frac{s}{h}}^{\frac{t}{h}-1} \big[p(jh)h+1\big] + p(t)h \sum_{i=\frac{s}{h}}^{\frac{t}{h}-1} \prod_{j=i+1}^{\frac{t}{h}-1} [p(jh)h+1] f(ih) + f(t) \\
&= p(t)\Bigg(y_0 \prod_{j=\frac{s}{h}}^{\frac{t}{h}-1} \big[p(jh)h+1\big] + h \sum_{i=\frac{s}{h}}^{\frac{t}{h}-1} \prod_{j=i+1}^{\frac{t}{h}-1} [p(jh)h+1] f(ih) \Bigg) + f(t) \\
&= p(t)y + f(t).
\end{aligned}
$$

Hence, (2.13) *is a unique solution of* (2.11) *on* $\mathbb{T} = h\mathbb{Z}$*.*

(iii) Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *for* $q > 1$ *(see Kac & Cheung (2002)). We will show that*

$$
\begin{aligned}
y(t) = y_0 &\prod_{j=\log_q(s)}^{\log_q(t)-1} \big[p(q^j)(q-1)q^j + 1\big] \\
&+ (q-1) \sum_{i=\log_q(s)}^{\log_q(t)-1} \prod_{j=i+1}^{\log_q(t)-1} \big[p(q^j)(q-1)q^j + 1\big] f(q^i)q^i,
\end{aligned}
\tag{2.14}
$$

where the empty sum is assumed to be 0*, while the empty product is assumed to* *be* 1*. To show this we substitute* y *into the IVP* (2.11)*. It is obvious that*

$$\begin{aligned} y(s) &= y_0 \prod_{j=\log_q(s)}^{\log_q(s)-1} \left[p(q^j)(q-1)q^j+1\right] \\ &\quad + (q-1) \sum_{i=\log_q(s)}^{\log_q(s)-1} \prod_{j=i+1}^{\log_q(s)-1} \left[p(q^j)(q-1)q^j+1\right] f(q^i)q^i \\ &= y_0. \end{aligned}$$

Then, we compute that

$$\begin{aligned} y^{\Delta} &= \frac{y(qt)-y(t)}{(q-1)t} \\ &= \frac{1}{(q-1)t}\Bigg(\Bigg[y_0 \prod_{j=\log_q(s)}^{\log_q(t)} \left[p(q^j)(q-1)q^j+1\right] \\ &\qquad + (q-1) \sum_{i=\log_q(s)}^{\log_q(t)} \prod_{j=i+1}^{\log_q(t)} \left[p(q^j)(q-1)q^j+1\right] f(q^i)q^i\Bigg] \\ &\qquad - \Bigg[y_0 \prod_{j=\log_q(s)}^{\log_q(t)-1} \left[p(q^j)(q-1)q^j+1\right] \\ &\qquad + (q-1) \sum_{i=\log_q(s)}^{\log_q(t)-1} \prod_{j=i+1}^{\log_q(t)-1} \left[p(q^j)(q-1)q^j+1\right] f(q^i)q^i\Bigg]\Bigg) \\ &= \frac{1}{(q-1)t}\Bigg(y_0 p(t)(q-1)t \prod_{j=\log_q(s)}^{\log_q(t)-1} \left[p(q^j)(q-1)q^j+1\right] \\ &\qquad + (q-1) \sum_{i=\log_q(s)}^{\log_q(t)-1} \prod_{j=i+1}^{\log_q(t)} \left[p(q^j)(q-1)q^j+1\right] f(q^i)q^i \\ &\qquad + (q-1)f(t)t - (q-1) \\ &\qquad \times \sum_{i=\log_q(s)}^{\log_q(t)-1} \prod_{j=i+1}^{\log_q(t)-1} \left[p(q^j)(q-1)q^j+1\right] f(q^i)q^i\Bigg) \\ &= \frac{1}{(q-1)t}\Bigg(y_0 p(t)(q-1)t \prod_{j=\log_q(s)}^{\log_q(t)-1} \left[p(q^j)(q-1)q^j+1\right] \end{aligned}$$

$$
\begin{aligned}
&\quad + (q-1)[p(t)(q-1)t+1] \\
&\qquad \times \sum_{i=\log_q(s)}^{\log_q(t)-1} \prod_{j=i+1}^{\log_q(t)-1} \big[p(q^j)(q-1)q^j+1\big] f(q^i)q^i \\
&\quad + (q-1)f(t)t \\
&- (q-1)\sum_{i=\log_q(s)}^{\log_q(t)-1} \prod_{j=i+1}^{\log_q(t)-1} \big[p(q^j)(q-1)q^j+1\big] f(q^i)q^i \Bigg) \\
&= \frac{1}{(q-1)t}\Bigg(y_0 p(t)(q-1)t \prod_{j=\log_q(s)}^{\log_q(t)-1} \big[p(q^j)(q-1)q^j+1\big] \\
&\quad + (q-1)^2 p(t)t \sum_{i=\log_q(s)}^{\log_q(t)-1} \prod_{j=i+1}^{\log_q(t)-1} \big[p(q^j)(q-1)q^j+1\big] f(q^i)q^i \\
&\quad + (q-1)f(t)t \Bigg) \\
&= y_0 p(t) \prod_{j=\log_q(s)}^{\log_q(t)-1} \big[p(q^j)(q-1)q^j+1\big] \\
&+ (q-1)p(t) \sum_{i=\log_q(s)}^{\log_q(t)-1} \prod_{j=i+1}^{\log_q(t)-1} \big[p(q^j)(q-1)q^j+1\big] f(q^i)q^i \\
&+ f(t) \\
&= p(t)\Bigg(y_0 \prod_{j=\log_q(s)}^{\log_q(t)-1} \big[p(q^j)(q-1)q^j+1\big] \\
&\quad + (q-1) \sum_{i=\log_q(s)}^{\log_q(t)-1} \prod_{j=i+1}^{\log_q(t)-1} \big[p(q^j)(q-1)q^j+1\big] f(q^i)q^i \Bigg) \\
&+ f(t) \\
&= p(t)y + f(t).
\end{aligned}
$$

Hence, (2.14) *is a unique solution of* (2.11) *on* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$.

CHAPTER 3
SECOND-ORDER DYNAMIC EQUATIONS WITH CONSTANT COEFFICIENTS

This section is concerned with solutions of second-order dynamic equations. We first consider the general equation

$$y^{\Delta\Delta} - p(t)y^{\Delta} + q(t)y = f(t), \tag{3.1}$$

where $p, q, f \in \mathrm{C}_{\mathrm{rd}}$.

Definition 16. *(Bohner & Peterson, 2001, Definition 3.3) We say that equation* (3.1) *is* regressive *provided*

$$q(t)\big(\mu(t)\big)^2 + p(t)\mu(t) + 1 \neq 0 \quad \textit{for all} \quad t \in \mathbb{T}^{\kappa}. \tag{3.2}$$

The notion of Wronskian plays an important role in the definition of solutions of linear dynamic equations, which is introduced below.

Definition 17 ((Bohner & Peterson, 2001, Definition 3.5))**.** *The* Wronskian $W = W(y_1, y_2)$ *of two differentiable functions* y_1 *and* y_2 *is defined by*

$$W(t) := \begin{vmatrix} y_1(t) & y_2(t) \\ y_1^{\Delta}(t) & y_2^{\Delta}(t) \end{vmatrix} \quad \textit{for} \quad t \in \mathbb{T}^{\kappa}.$$

From now on in this section, we will consider second-order linear dynamic equations with constant coefficients, which has the form

$$L_2[y] := y^{\Delta\Delta} - 2a_1 y^{\Delta} + a_2 y = f, \tag{3.3}$$

where $a_1, a_2 \in \mathbb{R}$ and $f \in \mathrm{C}_{\mathrm{rd}}$.

3.1 Homogeneous Equations

Here, we will focus on the second-order homogeneous dynamic equation

$$L_2[y] := y^{\Delta\Delta} - 2a_1 y^{\Delta} + a_2 y = 0, \tag{3.4}$$

which is the so-called associated homogeneous equation with (3.3). Let us suppose that (3.4) has a solution of the form $y = \mathrm{e}_r(\cdot, s)$, where $r \in \mathbb{R}$. Substituting $y = \mathrm{e}_r(\cdot, s)$ into (3.4) yields

$$\begin{aligned} L_2[\mathrm{e}_r(t,s)] &= r^2 \mathrm{e}_r(t,s) - 2a_1 r \mathrm{e}_r(t,s) + a_2 \mathrm{e}_r(t,s) \\ &= (r^2 - 2a_1 r + a_2)\mathrm{e}_r(t,s) \\ &= P_2(r)\mathrm{e}_r(t,s), \end{aligned} \tag{3.5}$$

for $t \in \mathbb{T}$, where the so-called characteristic equation is defined by

$$P_2(\lambda) := \lambda^2 - 2a_1\lambda + a_2 = 0 \quad \text{for } \lambda \in \mathbb{C}. \tag{3.6}$$

Clearly, the characteristic equation, which is a parabola, either has distinct real roots, repeated real roots or complex conjugate roots. Note that if $r \in \mathbb{R}$ is a root of (3.6), then $y = \mathrm{e}_r(\cdot, s)$ is a real solution of (3.4) by (3.5).

3.1.1 Distinct Real Roots

In this case, the roots are given by

$$r_1 := a_1 + \sqrt{a_1^2 - a_2} \quad \text{and} \quad r_2 := a_1 - \sqrt{a_1^2 - a_2}, \tag{3.7}$$

where $a_1^2 > a_2$. Note that (3.4) takes the form

$$L_2[y] := y^{\Delta\Delta} - (r_1 + r_2)y^{\Delta} + r_1 r_2 y = 0. \tag{3.8}$$

It immediately follows from (3.5) that

$$L_2[\mathrm{e}_{r_1}(t,s)] = 0 \quad \text{and} \quad L_2[\mathrm{e}_{r_2}(t,s)] = 0 \quad \text{for } t \in \mathbb{T}.$$

Thus, $\mathrm{e}_{r_1}(\cdot,s)$ and $\mathrm{e}_{r_2}(\cdot,s)$ are two solutions of (3.4).

Let us now show that $\mathrm{e}_{r_1}(\cdot,s)$ and $\mathrm{e}_{r_2}(\cdot,s)$ are linearly independent. Indeed, we have

$$\begin{vmatrix} \mathrm{e}_{r_1}(t,s) & \mathrm{e}_{r_2}(t,s) \\ r_1\mathrm{e}_{r_1}(t,s) & r_2\mathrm{e}_{r_2}(t,s) \end{vmatrix} = (r_2 - r_1)\mathrm{e}_{r_1}(t,s)\mathrm{e}_{r_2}(t,s) = (r_2 - r_1)\mathrm{e}_{r_1 \oplus_\mu r_2}(t,s) \neq 0.$$

Therefore, $\mathrm{e}_{r_1}(\cdot,s)$ and $\mathrm{e}_{r_2}(\cdot,s)$ are two linearly independent solutions of (3.4).

3.1.2 Repeated Real Roots

In this case, the roots are given by

$$r_1 = r_2 = a_1. \tag{3.9}$$

Note also that $a_1^2 = a_2$. Then, (3.4) and (3.6) take the forms

$$L_2[y] := y^{\Delta\Delta} - 2r_1 y^{\Delta} + r_1^2 y = 0 \tag{3.10}$$

and

$$P_2(\lambda) = (\lambda - r_1)^2, \tag{3.11}$$

respectively, and (3.5) becomes

$$L_2[\mathrm{e}_r(t,s)] = (r - r_1)^2 \mathrm{e}_r(t,s).$$

Before proceeding, we compute that

$$\big(\mathrm{m}_r(\cdot,s)\mathrm{e}_r(\cdot,s)\big)^{\Delta}(t) = \frac{1}{1 + r\mu(t)}\mathrm{e}_r(\sigma(t),s) + r\mathrm{m}_r(t,s)\mathrm{e}_r(t,s)$$

$$= \mathrm{e}_r(t,s) + r\mathrm{m}_r(t,s)\mathrm{e}_r(t,s), \tag{3.12}$$

$$\big(\mathrm{m}_r(t,s)\mathrm{e}_r(\cdot,s)\big)^{\Delta^2}(t) = r\mathrm{e}_r(t,s) + r[\mathrm{e}_r(t,s) + r\mathrm{m}_r(t,s)\mathrm{e}_r(t,s)]$$
$$= 2r\mathrm{e}_r(t,s) + r^2\mathrm{m}_r(t,s)\mathrm{e}_r(t,s) \tag{3.13}$$

for $t \in \mathbb{T}$. It follows that

$$\begin{aligned} L_2[\mathrm{m}_r(t,s)\mathrm{e}_r(t,s)] &= \big(2r\mathrm{e}_r(t,s) + r^2\mathrm{m}_r(t,s)\mathrm{e}_r(t,s)\big) \\ &\quad - 2r_1\big(\mathrm{e}_r(t,s) + r\mathrm{m}_r(t,s)\mathrm{e}_r(t,s)\big) + r_1^2\mathrm{m}_r(t,s)\mathrm{e}_r(t,s) \\ &= (2r - 2r_1)\mathrm{e}_r(t,s) + \big(r^2 - 2r_1r + r_1^2\big)\mathrm{m}_r(t,s)\mathrm{e}_r(t,s) \\ &= 2(r - r_1)\mathrm{e}_r(t,s) + (r - r_1)^2\mathrm{m}_r(t,s)\mathrm{e}_r(t,s) \end{aligned}$$

for $t \in \mathbb{T}$, which implies that $r = r_1$ yields a solution of the form $y = \mathrm{m}_{r_1}(\cdot,s)\mathrm{e}_{r_1}(\cdot,s)$. Note also that by using (3.5) and (Karpuz, 2017, Theorem 3.1), we can obtain

$$\begin{aligned} L_2\left[\frac{\partial}{\partial r}\mathrm{e}_r(\cdot,s)\right]_{r=r_1} &= \frac{\partial}{\partial r}L_2[\mathrm{e}_r(\cdot,s)]\bigg|_{r=r_1} = \frac{\partial}{\partial r}[P_2(r)\mathrm{e}_r(\cdot,s)]\bigg|_{r=r_1} \\ &= P_2'(r_1)\mathrm{e}_{r_1}(\cdot,s) + P_2(r_1)\mathrm{m}_{r_1}(\cdot,s)\mathrm{e}_{r_1}(\cdot,s) = 0. \end{aligned}$$

Thus, $\mathrm{e}_{r_1}(\cdot,s)$ and $\mathrm{m}_{r_1}(\cdot,s)\mathrm{e}_{r_1}(\cdot,s)$ are two solutions of (3.4) since $P_2(r_1) = 0$ and $P_2'(r_1) = 0$ by (3.11).

Let us now show that $\mathrm{e}_{r_1}(\cdot,s)$ and $\mathrm{m}_{r_1}(\cdot,s)\mathrm{e}_{r_1}(\cdot,s)$ are linearly independent. Indeed, we have

$$\begin{vmatrix} \mathrm{e}_{r_1}(t,s) & \mathrm{m}_{r_1}(t,s)\mathrm{e}_{r_1}(t,s) \\ r_1\mathrm{e}_{r_1}(t,s) & \mathrm{e}_{r_1}(t,s) + r_1\mathrm{m}_{r_1}(t,s)\mathrm{e}_{r_1}(t,s) \end{vmatrix} = \mathrm{e}_{r_1}(t,s)\mathrm{e}_{r_1}(t,s) = \mathrm{e}_{r_1\oplus_\mu r_1}(t,s) \neq 0.$$

Therefore, $\mathrm{e}_{r_1}(\cdot,s)$ and $\mathrm{m}_{r_1}(\cdot,s)\mathrm{e}_{r_1}(\cdot,s)$ are two linearly independent solutions of (3.4).

3.1.3 Roots in Complex Conjugates

In this case, from (3.7), the complex roots are given by

$$r_1 := a_1 + \mathrm{i}\sqrt{a_2 - a_1^2} \quad \text{and} \quad r_2 := a_1 - \mathrm{i}\sqrt{a_2 - a_1^2}, \tag{3.14}$$

where $a_1^2 < a_2$. Hence, we have

$$\mathrm{Re}(r_1) = a_1 \quad \text{and} \quad \mathrm{Im}(r_1) = \sqrt{a_2 - a_1^2}.$$

Then, (3.4) and (3.6) take the forms

$$L_2[y] = y^{\Delta\Delta} - 2\,\mathrm{Re}(r_1)y^{\Delta} + r_1\overline{r_1}y = 0 \tag{3.15}$$

and

$$P_2(\lambda) = \lambda^2 - 2\,\mathrm{Re}(r_1)\lambda + r_1\overline{r_1},$$

respectively.

Before we proceed, we need to prove some results required in the sequel.

3.1.3.1 Some Definitions and Computations

We start this section by defining “new” trigonometric functions on time scales.

Definition 18. *Let* $s, t \in \mathbb{T}$ *and* $f \in \mathrm{C_{rd}}$. *Then, functions* $\cos_f(t, s)$ *and* $\sin_f(t, s)$ *defined by*

$$\cos_f(t, s) := \cos\left(\int_s^t f(\eta)\Delta\eta\right) \quad \text{and} \quad \sin_f(t, s) := \sin\left(\int_s^t f(\eta)\Delta\eta\right).$$

Let us introduce some properties of the newly defined cosine and sine functions.

Property 3. *Let* $s, t \in \mathbb{T}$ *and* $f \in \mathrm{C_{rd}}$. *Then, the following identities hold.*

(i)
$$\cos_{(-f)}(t,s)=\cos_f(t,s) \quad \textit{and} \quad \sin_{(-f)}(t,s)=-\sin_f(t,s).$$

(ii)
$$\cos_f(t,s)=\cos_f(s,t) \quad \textit{and} \quad \sin_f(t,s)=-\sin_f(s,t).$$

(iii)
$$\cos_f(t,s)=\cos_f(t,r)\cos_f(r,s)-\sin_f(t,r)\sin_f(r,s).$$

(iv)
$$\sin_f(t,s)=\cos_f(t,r)\sin_f(r,s)+\sin_f(t,r)\cos_f(r,s).$$

(v)
$$\begin{aligned}\cos_f(\sigma(t),s) &= \cos_f(\sigma(t),t)\cos_f(t,s)-\sin_f(\sigma(t),t)\sin_f(t,s)\\ &= \cos\big(\mu(t)f(t)\big)\cos_f(t,s)-\sin\big(\mu(t)f(t)\big)\sin_f(t,s).\end{aligned}$$

(vi)
$$\begin{aligned}\sin_f(\sigma(t),s) &= \cos_f(\sigma(t),t)\sin_f(t,s)+\sin_f(\sigma(t),t)\cos_f(t,s)\\ &= \cos\big(\mu(t)f(t)\big)\sin_f(t,s)+\sin(\mu(t)f(t))\cos_f(t,s).\end{aligned}$$

Proof. (i) From Definition 18, we have

$$\cos_{(-f)}(t,s)=\cos\left(-\int_s^t f(\eta)\Delta\eta\right)=\cos\left(\int_s^t f(\eta)\Delta\eta\right)=\cos_f(t,s)$$

and

$$\sin_{(-f)}(t,s)=\sin\left(-\int_s^t f(\eta)\Delta\eta\right)=-\sin\left(\int_s^t f(\eta)\Delta\eta\right)=-\sin_f(t,s).$$

(ii) From Property 1 (ii), we have

$$\begin{aligned}\cos_f(t,s) &= \cos\left(\int_s^t f(\eta)\Delta\eta\right)=\cos\left(-\int_t^s f(\eta)\Delta\eta\right)=\cos\left(\int_t^s f(\eta)\Delta\eta\right)\\ &= \cos_f(s,t)\end{aligned}$$

and

$$\sin_f(t,s) = \sin\left(\int_s^t f(\eta)\Delta\eta\right)=\sin\left(-\int_t^s f(\eta)\Delta\eta\right)=-\sin\left(\int_t^s f(\eta)\Delta\eta\right)$$

$$= -\sin_f(s,t).$$

(iii) From Property 1 (iii), we have

$$\begin{aligned}
\cos_f(t,s) &= \cos\left(\int_s^t f(\eta)\Delta\eta\right) = \cos\left(\int_r^t f(\eta)\Delta\eta + \int_s^r f(\eta)\Delta\eta\right) \\
&= \cos\left(\int_r^t f(\eta)\Delta\eta\right)\cos\left(\int_s^r f(\eta)\Delta\eta\right) \\
&\quad - \sin\left(\int_r^t f(\eta)\Delta\eta\right)\sin\left(\int_s^r f(\eta)\Delta\eta\right) \\
&= \cos_f(t,r)\cos_f(r,s) - \sin_f(t,r)\sin_f(r,s).
\end{aligned}$$

(iv) From Property 1 (iii), we have

$$\begin{aligned}
\sin_f(t,s) &= \sin\left(\int_s^t f(\eta)\Delta\eta\right) = \sin\left(\int_r^t f(\eta)\Delta\eta + \int_s^r f(\eta)\Delta\eta\right) \\
&= \cos\left(\int_r^t f(\eta)\Delta\eta\right)\sin\left(\int_s^r f(\eta)\Delta\eta\right) \\
&\quad + \sin\left(\int_r^t f(\eta)\Delta\eta\right)\cos\left(\int_s^r f(\eta)\Delta\eta\right) \\
&= \cos_f(t,r)\sin_f(r,s) + \sin_f(t,r)\cos_f(r,s).
\end{aligned}$$

(v) From Property 1 (iii) and Theorem 3, we have

$$\begin{aligned}
\cos_f(\sigma(t),s) &= \cos\left(\int_s^{\sigma(t)} f(\eta)\Delta\eta\right) \\
&= \cos\left(\int_s^t f(\eta)\Delta\eta + \int_t^{\sigma(t)} f(\eta)\Delta\eta\right) \\
&= \cos\left(\int_s^t f(\eta)\Delta\eta + \mu(t)f(t)\right) \\
&= \cos\big(\mu(t)f(t)\big)\cos\left(\int_s^t f(\eta)\Delta\eta\right) \\
&\quad - \sin\big(\mu(t)f(t)\big)\sin\left(\int_s^t f(\eta)\Delta\eta\right) \\
&= \cos\big(\mu(t)f(t)\big)\cos_f(t,s) - \sin\big(\mu(t)f(t)\big)\sin_f(t,s).
\end{aligned}$$

(vi) From Property 1 (iii) and Theorem 3, we have

$$\begin{aligned}
\sin_f(\sigma(t), s) &= \sin\left(\int_s^{\sigma(t)} f(\eta)\Delta\eta\right) \\
&= \sin\left(\int_s^t f(\eta)\Delta\eta + \int_t^{\sigma(t)} f(\eta)\Delta\eta\right) \\
&= \sin\left(\int_s^t f(\eta)\Delta\eta + \mu(t)f(t)\right) \\
&= \cos\big(\mu(t)f(t)\big)\sin\left(\int_s^t f(\eta)\Delta\eta\right) \\
&\quad + \sin\big(\mu(t)f(t)\big)\cos\left(\int_s^t f(\eta)\Delta\eta\right) \\
&= \cos\big(\mu(t)f(t)\big)\sin_f(t,s) + \sin\big(\mu(t)f(t)\big)\cos_f(t,s).
\end{aligned}$$

□

For the following results, we need to recall that $\mathbb{C}_{\mu(\mathbb{T})}$ is defined in (2.10). The following result can be regarded as the Euler's formula on time scales.

Lemma 5 ((Karpuz, 2017, Section 7))**.** *Let us take* $s, t \in \mathbb{T}$ *and* $z \in \mathbb{C}_{\mu(\mathbb{T})}$*, then we have*

$$\mathrm{e}_z(t,s) = \mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\big[\cos_{\mathrm{Im}_\mu(z)}(t,s) + \mathrm{i}\sin_{\mathrm{Im}_\mu(z)}(t,s)\big].$$

Some trigonometric relations regarding the usual real and imaginary parts and Hilger's real and Hilger's imaginary parts are given below.

Property 4. *Let* $s, t \in \mathbb{T}$ *and* $z \in \mathbb{C}_{\mu(\mathbb{T})}$*, then we have the followings:*

(i) $\mathrm{Re}\big(\mathrm{e}_z(t,s)\big) = \mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\cos_{\mathrm{Im}_\mu(z)}(t,s)$,

(ii) $\mathrm{Im}\big(\mathrm{e}_z(t,s)\big) = \mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\sin_{\mathrm{Im}_\mu(z)}(t,s)$,

(iii) $\mathrm{Re}\big(\mathrm{e}_{\overline{z}}(t,s)\big) = \mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\cos_{\mathrm{Im}_\mu(z)}(t,s)$,

(iv) $\mathrm{Im}\big(\mathrm{e}_{\overline{z}}(t,s)\big) = -\mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\sin_{\mathrm{Im}_\mu(z)}(t,s)$.

Proof. Let us take $s, t \in \mathbb{T}$ and $z \in \mathbb{C}_{\mu(\mathbb{T})}$. From Definition 18 and Lemma 5, we have

$$\mathrm{e}_z(t,s) = \mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\left[\cos\left(\int_s^t \mathrm{Im}_{\mu(\eta)}(z)\Delta\eta\right) + \mathrm{i}\sin\left(\int_s^t \mathrm{Im}_{\mu(\eta)}(z)\Delta\eta\right)\right].$$

Hence, parts (i) and (ii) are obvious.

From Lemma 3 (iii) and (iv), we have

$$\begin{aligned}\mathrm{e}_{\bar{z}}(t,s) &= \mathrm{e}_{\mathrm{Re}_{\mu}(\bar{z})}(t,s)\left[\cos_{\mathrm{Im}_{\mu}(\bar{z})}(t,s)+\mathrm{i}\sin_{\mathrm{Im}_{\mu}(\bar{z})}(t,s)\right]\\ &= \mathrm{e}_{\mathrm{Re}_{\mu}(z)}(t,s)\left[\cos_{(-\mathrm{Im}_{\mu}(z))}(t,s)+\mathrm{i}\sin_{(-\mathrm{Im}_{\mu}(z))}(t,s)\right]\\ &= \mathrm{e}_{\mathrm{Re}_{\mu}(z)}(t,s)\left[\cos_{\mathrm{Im}_{\mu}(z)}(t,s)-\mathrm{i}\sin_{\mathrm{Im}_{\mu}(z)}(t,s)\right].\end{aligned}$$

Hence, parts (iii) and (iv) are obvious. □

As an immediate consequence of Property 4, we can give the following corollary.

Corollary 2. *Let $s,t\in\mathbb{T}$ and $z\in\mathbb{C}_{\mu(\mathbb{T})}$. Then,*

$$\mathrm{e}_{\overline{z}}(t,s)=\overline{\mathrm{e}_{z}(t,s)}.$$

Proof. From Property 4 (iii) and (iv), we have

$$\begin{aligned}\mathrm{e}_{\overline{z}}(t,s) &= \mathrm{e}_{\mathrm{Re}_{\mu}(z)}(t,s)\left[\cos_{\mathrm{Im}_{\mu}(z)}(t,s)-\mathrm{i}\sin_{\mathrm{Im}_{\mu}(z)}(t,s)\right]\\ &= \overline{\mathrm{e}_{\mathrm{Re}_{\mu}(z)}(t,s)\left[\cos_{\mathrm{Im}_{\mu}(z)}(t,s)+\mathrm{i}\sin_{\mathrm{Im}_{\mu}(z)}(t,s)\right]}\\ &= \overline{\mathrm{e}_{z}(t,s)},\end{aligned}$$

which completes the proof. □

In the sequel, we will be needing the derivative of the newly defined trigonometric functions. To this end, we define the functions sinc : $\mathbb{R}\to\mathbb{R}$ and cosc : $\mathbb{R}\to\mathbb{R}$ by

$$\operatorname{sinc}(x):=\begin{cases}1, & x=0\\ \dfrac{\sin(x)}{x}, & x\neq 0\end{cases}\quad\text{and}\quad \operatorname{cosc}(x):=\begin{cases}0, & x=0\\ \dfrac{\cos(x)-1}{x}, & x\neq 0.\end{cases}$$

The following lemma is important in the computation of the derivatives of $\cos_f$ and $\sin_f$.

Lemma 6. *Let $h \geq 0$ and $z \in \mathbb{C}_h$. Then,*

$$\mathrm{Re}(z) = \mathrm{Re}_h(z)\cos\big(h\,\mathrm{Im}_h(z)\big) + \mathrm{Im}_h(z)\,\mathrm{cosc}\big(h\,\mathrm{Im}_h(z)\big) \tag{3.16}$$

and

$$\mathrm{Im}(z) = \mathrm{Re}_h(z)\sin\big(h\,\mathrm{Im}_h(z)\big) + \mathrm{Im}_h(z)\,\mathrm{sinc}\big(h\,\mathrm{Im}_h(z)\big). \tag{3.17}$$

Proof. For $h = 0$, (3.16) and (3.17) hold trivially. Now, let $h > 0$. First, let us show (3.16). We compute that

$$\begin{aligned}
\mathrm{Re}(z) &= \frac{\big(\mathrm{Re}(z)h+1\big)-1}{h} = \frac{\mathrm{Re}(zh+1)-1}{h} \\
&= \frac{|zh+1|\cos\big(\mathrm{Arg}(zh+1)\big)-1}{h} \\
&= \frac{|zh+1|-1}{h}\cos\big(\mathrm{Arg}(zh+1)\big) + \frac{\cos\big(\mathrm{Arg}(zh+1)\big)-1}{h} \\
&= \frac{|zh+1|-1}{h}\cos\left(h\frac{\mathrm{Arg}(zh+1)}{h}\right) + \frac{\mathrm{Arg}(zh+1)}{h}\,\frac{\cos\big(h\frac{\mathrm{Arg}(zh+1)}{h}\big)-1}{h\frac{\mathrm{Arg}(zh+1)}{h}} \\
&= \mathrm{Re}_h(z)\cos\big(h\,\mathrm{Im}_h(z)\big) + \mathrm{Im}_h(z)\,\mathrm{cosc}\big(h\,\mathrm{Im}_h(z)\big).
\end{aligned}$$

Next, let us show (3.17). We compute that

$$\begin{aligned}
\mathrm{Im}(z) &= \frac{\mathrm{Im}(zh+1)}{h} = \frac{|zh+1|\sin\big(\mathrm{Arg}(zh+1)\big)}{h} \\
&= \frac{|zh+1|-1}{h}\sin\big(\mathrm{Arg}(zh+1)\big) + \frac{\sin\big(\mathrm{Arg}(zh+1)\big)}{h} \\
&= \frac{|zh+1|-1}{h}\sin\left(h\frac{\mathrm{Arg}(zh+1)}{h}\right) + \frac{\mathrm{Arg}(zh+1)}{h}\,\frac{\sin\big(h\frac{\mathrm{Arg}(zh+1)}{h}\big)}{h\frac{\mathrm{Arg}(zh+1)}{h}} \\
&= \mathrm{Re}_h(z)\sin\big(h\,\mathrm{Im}_h(z)\big) + \mathrm{Im}_h(z)\,\mathrm{sinc}\big(h\,\mathrm{Im}_h(z)\big).
\end{aligned}$$

This completes the proof. □

The derivatives of the newly defined trigonometric functions are given below.

Lemma 7. *Let $s, t \in \mathbb{T}$ and $z \in \mathbb{C}_{\mu(\mathbb{T})}$. Then, for $s, t \in \mathbb{T}$, we have*

$$\cos_f^{\Delta_1}(t,s) = -f(t)\sin_f(t,s)\,\mathrm{sinc}\big(\mu(t)f(t)\big) + f(t)\cos_f(t,s)\,\mathrm{cosc}\big(\mu(t)f(t)\big) \tag{3.18}$$

and

$$\sin_f^{\Delta_1}(t,s) = f(t)\cos_f(t,s)\operatorname{sinc}\big(\mu(t)f(t)\big) + f(t)\sin_f(t,s)\operatorname{cosc}\big(\mu(t)f(t)\big). \quad (3.19)$$

Proof. Using Definition 5, we have

$$\begin{aligned}\cos_f^{\Delta_1}(t,s) &= -f(t)\int_0^1 \sin\bigg(\int_s^t f(\eta)\Delta\eta + \lambda\mu(t)f(t)\bigg)\mathrm{d}\lambda \\ &= -f(t)\int_0^1 \big[\sin_f(t,s)\cos\big(\lambda\mu(t)f(t)\big) + \cos_f(t,s)\sin\big(\lambda\mu(t)f(t)\big)\big]\mathrm{d}\lambda \\ &= -f(t)\sin_f(t,s)\int_0^1 \cos\big(\lambda\mu(t)f(t)\big)\mathrm{d}\lambda \\ &\quad - f(t)\cos_f(t,s)\int_0^1 \sin\big(\lambda\mu(t)f(t)\big)\mathrm{d}\lambda.\end{aligned} \quad (3.20)$$

Now, let $\mu(t) = 0$, then (3.20) becomes

$$\cos_f^{\Delta_1}(t,s) = -f(t)\sin_f(t,s).$$

Next, let $\mu(t) > 0$, then (3.20) becomes

$$\cos_f^{\Delta_1}(t,s) = -\frac{1}{\mu(t)}\sin_f(t,s)\sin\big(\mu(t)f(t)\big) + \frac{1}{\mu(t)}\cos_f(t,s)\big[\cos\big(\mu(t)f(t)\big) - 1\big].$$

In both cases, we have (3.18).

Similarly, we compute

$$\begin{aligned}\sin_f^{\Delta_1}(t,s) &= f(t)\int_0^1 \cos\bigg(\int_s^t f(\eta)\Delta\eta + \lambda\mu(t)f(t)\bigg)\mathrm{d}\lambda \\ &= f(t)\int_0^1 \big[\cos_f(t,s)\cos\big(\lambda\mu(t)f(t)\big) - \sin_f(t,s)\sin\big(\lambda\mu(t)f(t)\big)\big]\mathrm{d}\lambda \\ &= f(t)\cos_f(t,s)\int_0^1 \cos\big(\lambda\mu(t)f(t)\big)\mathrm{d}\lambda \\ &\quad - f(t)\sin_f(t,s)\int_0^1 \sin\big(\lambda\mu(t)f(t)\big)\mathrm{d}\lambda \\ &= f(t)\cos_f(t,s)\operatorname{sinc}\big(\mu(t)f(t)\big) + f(t)\sin_f(t,s)\operatorname{cosc}\big(\mu(t)f(t)\big),\end{aligned}$$

which proves (3.19).

This completes the proof. □

The following lemma is also crucial in our proofs.

Lemma 8. *Let* $s, t \in \mathbb{T}$ *and* $z \in \mathbb{C}_{\mu(\mathbb{T})}$*. Then, we have*

$$\begin{aligned}\big(\mathrm{e}_{\mathrm{Re}_\mu(z)}(\cdot,s)\cos_{\mathrm{Im}_\mu(z)}(\cdot,s)\big)^{\Delta}(t) = {}& \mathrm{Re}(z)\mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\cos_{\mathrm{Im}_\mu(z)}(t,s)\\ &- \mathrm{Im}(z)\mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\sin_{\mathrm{Im}_\mu(z)}(t,s)\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\big(\mathrm{e}_{\mathrm{Re}_\mu(z)}(\cdot,s)\sin_{\mathrm{Im}_\mu(z)}(\cdot,s)\big)^{\Delta}(t) = {}& \mathrm{Im}(z)\mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\cos_{\mathrm{Im}_\mu(z)}(t,s)\\ &+ \mathrm{Re}(z)\mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\sin_{\mathrm{Im}_\mu(z)}(t,s).\end{aligned} \tag{3.22}$$

Proof. First, let us show (3.21). We compute that

$$\begin{aligned}&\big(\mathrm{e}_{\mathrm{Re}_\mu(z)}(\cdot,s)\cos_{\mathrm{Im}_\mu(z)}(\cdot,s)\big)^{\Delta}(t)\\
&\quad= \mathrm{Re}_{\mu(t)}(z)\mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\cos_{\mathrm{Im}_\mu(z)}(\sigma(t),s)\\
&\qquad+ \mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\,\mathrm{Im}_{\mu(t)}(z)\big[\cos_{\mathrm{Im}_\mu(z)}(t,s)\,\mathrm{cosc}\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\\
&\qquad\qquad- \sin_{\mathrm{Im}_\mu(z)}(t,s)\,\mathrm{sinc}\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\big]\\
&\quad= \mathrm{Re}_{\mu(t)}(z)\mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\big[\cos_{\mathrm{Im}_\mu(z)}(t,s)\cos\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\\
&\qquad\qquad- \sin_{\mathrm{Im}_\mu(z)}(t,s)\sin\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\big]\\
&\qquad+ \mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\,\mathrm{Im}_{\mu(t)}(z)\big[\cos_{\mathrm{Im}_\mu(z)}(t,s)\,\mathrm{cosc}\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\\
&\qquad\qquad- \sin_{\mathrm{Im}_\mu(z)}(t,s)\,\mathrm{sinc}\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\big]\\
&\quad= \mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\cos_{\mathrm{Im}_\mu(z)}(t,s)\big[\mathrm{Re}_{\mu(t)}(z)\cos\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\\
&\qquad\qquad+ \mathrm{Im}_{\mu(t)}(z)\,\mathrm{cosc}\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\big]\\
&\qquad- \mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\sin_{\mathrm{Im}_\mu(z)}(t,s)\big[\mathrm{Re}_{\mu(t)}(z)\sin\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\\
&\qquad\qquad+ \mathrm{Im}_{\mu(t)}(z)\,\mathrm{sinc}\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\big].\end{aligned} \tag{3.23}$$

By Lemma 6, (3.23) becomes (3.21).

Now let us show (3.22). We compute that

$$\begin{aligned}
&\left(\mathrm{e}_{\mathrm{Re}_\mu(z)}(\cdot,s)\sin_{\mathrm{Im}_\mu(z)}(\cdot,s)\right)^{\Delta}(t)\\
&\quad= \mathrm{Re}_{\mu(t)}(z)\mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\sin_{\mathrm{Im}_\mu(z)}(\sigma(t),s)\\
&\qquad+\mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\,\mathrm{Im}_{\mu(t)}(z)\big[\cos_{\mathrm{Im}_\mu(z)}(t,s)\,\mathrm{sinc}\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\\
&\qquad\qquad+\sin_{\mathrm{Im}_\mu(z)}(t,s)\,\mathrm{cosc}\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\big]\\
&\quad= \mathrm{Re}_{\mu(t)}(z)\mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\big[\cos_{\mathrm{Im}_\mu(z)}(t,s)\sin\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\\
&\qquad\qquad+\sin_{\mathrm{Im}_\mu(z)}(t,s)\cos\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\big]\\
&\qquad+\mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\,\mathrm{Im}_{\mu(t)}(z)\big[\cos_{\mathrm{Im}_\mu(z)}(t,s)\,\mathrm{sinc}\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\\
&\qquad\qquad+\sin_{\mathrm{Im}_\mu(z)}(t,s)\,\mathrm{cosc}\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\big]\\
&\quad= \mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\cos_{\mathrm{Im}_\mu(z)}(t,s)\big[\mathrm{Re}_{\mu(t)}(z)\sin\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\\
&\qquad\qquad+\mathrm{Im}_{\mu(t)}(z)\,\mathrm{sinc}\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\big]\\
&\qquad+\mathrm{e}_{\mathrm{Re}_\mu(z)}(t,s)\sin_{\mathrm{Im}_\mu(z)}(t,s)\big[\mathrm{Re}_{\mu(t)}(z)\cos\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\\
&\qquad\qquad+\mathrm{Im}_{\mu(t)}(z)\,\mathrm{cosc}\big(\mu(t)\,\mathrm{Im}_{\mu(t)}(z)\big)\big].
\end{aligned}\tag{3.24}$$

By Lemma 6, (3.24) becomes (3.22).

This completes the proof. □

3.1.3.2 Solutions Corresponding to the Complex Roots

In this section, we will show that

$$\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\cos_{\mathrm{Im}_\mu(r_1)}(\cdot,s)\quad\text{and}\quad \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\sin_{\mathrm{Im}_\mu(r_1)}(\cdot,s)\tag{3.25}$$

are linearly independent solutions of (3.4) Now, after proving some results we need, we can continue to computations in Section 3.1.3. Using Lemma 8, we compute

$$\begin{aligned}
&\left(\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\cos_{\mathrm{Im}_\mu(r_1)}(\cdot,s)\right)^{\Delta^2}(t)\\
&=\left[\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\big(\mathrm{Re}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(\cdot,s)-\mathrm{Im}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(\cdot,s)\big)\right]^{\Delta}(t)\\
&=\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\,\mathrm{Re}(r_1)\big(\mathrm{Re}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)-\mathrm{Im}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big)
\end{aligned}$$

$$
\begin{aligned}
&- \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\,\mathrm{Im}(r_1)\big(\mathrm{Re}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)+\mathrm{Im}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\big)\\
&= \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\Big\{\Big[\big(\mathrm{Re}(r_1)\big)^2-\big(\mathrm{Im}(r_1)\big)^2\Big]\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\\
&\qquad - 2\,\mathrm{Re}(r_1)\,\mathrm{Im}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\Big\}
\end{aligned}
\tag{3.26}
$$

for $s,t\in\mathbb{T}$. Substituting (3.21), (3.25) and (3.26) into (3.4), we obtain

$$
\begin{aligned}
&L_2\big[\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\big]\\
&= \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\Big\{\Big[\big(\mathrm{Re}(r_1)\big)^2-\big(\mathrm{Im}(r_1)\big)^2\Big]\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\\
&\qquad - 2\,\mathrm{Re}(r_1)\,\mathrm{Im}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\Big\}\\
&\quad - 2a_1\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\big(\mathrm{Re}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)-\mathrm{Im}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big)\\
&\quad + a_2\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\\
&= \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\Big\{\Big[\big(\mathrm{Re}(r_1)\big)^2-\big(\mathrm{Im}(r_1)\big)^2\Big]-2a_1\,\mathrm{Re}(r_1)+a_2\Big\}\\
&\quad - 2\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\{\mathrm{Re}(r_1)-a_1\}\,\mathrm{Im}(r_1)\\
&= \Big\{\underbrace{\Big[a_1^2-\big(a_2-a_1^2\big)\Big]-2a_1^2+a_2}_{0}\Big\}\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\\
&\quad - 2\{\underbrace{a_1-a_1}_{0}\}\sqrt{a_2-a_1^2}\,\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)=0
\end{aligned}
$$

for $s,t\in\mathbb{T}$. Similarly, we can show that

$$
L_2\big[\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big]=0.
$$

Thus, $\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\cos_{\mathrm{Im}_\mu(r_1)}(\cdot,s)$ and $\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\sin_{\mathrm{Im}_\mu(r_1)}(\cdot,s)$ are two solutions of (3.4).

Alternatively, by using (3.5) and (Karpuz, 2017, § 7), we can obtain

$$
\begin{aligned}
L_2\big[\mathrm{Re}\big(\mathrm{e}_{r_{1,2}}(\cdot,s)\big)\big] &= \mathrm{Re}\big(L_2[\mathrm{e}_{r_{1,2}}(\cdot,s)]\big) = \mathrm{Re}\big(P_2(r_{1,2})\mathrm{e}_{r_{1,2}}(\cdot,s)\big)\\
&= P_2(r_{1,2})\,\mathrm{Re}\big(\mathrm{e}_{r_{1,2}}(\cdot,s)\big)=0
\end{aligned}
$$

and

$$L_2\big[\mathrm{Im}\big(\mathrm{e}_{r_{1,2}}(\cdot,s)\big]\big) = \mathrm{Im}\big(L_2[\mathrm{e}_{r_{1,2}}(\cdot,s)]\big) = \mathrm{Im}\big(P_2(r_{1,2})\mathrm{e}_{r_{1,2}}(\cdot,s)\big)$$
$$= P_2(r_{1,2})\,\mathrm{Im}\big(\mathrm{e}_{r_{1,2}}(\cdot,s)\big) = 0.$$

Let us now show that $\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\cos_{\mathrm{Im}_\mu(r_1)}(\cdot,s)$ and $\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\sin_{\mathrm{Im}_\mu(r_1)}(\cdot,s)$ are linearly independent. Indeed, we have

$$\begin{aligned}
&\begin{vmatrix} \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\cos_{\mathrm{Im}_\mu(r_1)}(t,s) & \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\sin_{\mathrm{Im}_\mu(r_1)}(t,s) \\ \big(\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\cos_{\mathrm{Im}_\mu(r_1)}(\cdot,s)\big)^\Delta(t) & \big(\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\sin_{\mathrm{Im}_\mu(r_1)}(\cdot,s)\big)^\Delta(t) \end{vmatrix} \\
&= \mathrm{Im}(r_1)\big(\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\big)^2 + \mathrm{Im}(r_1)\big(\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big)^2 \\
&= \mathrm{Im}(r_1)\big(\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\big)^2\Big[\big(\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\big)^2 + \big(\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big)^2\Big] \\
&= \sqrt{a_2 - a_1^2}\,\mathrm{e}_{\mathrm{Re}_\mu(r_1)\oplus_\mu \mathrm{Re}_\mu(r_1)}(t,s) \neq 0.
\end{aligned}$$

Therefore, $\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\cos_{\mathrm{Im}_\mu(r_1)}(\cdot,s)$ and $\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\cdot,s)\sin_{\mathrm{Im}_\mu(r_1)}(\cdot,s)$ are two linearly independent solutions of (3.4).

3.2 Non-homogeneous Equations

Let $r_1, r_2 \in \mathbb{C}$ be the roots of the characteristic polynomial P_2. Then,

$$r_1 := a_1 + \sqrt{a_1^2 - a_2} \quad \text{and} \quad r_2 := a_1 - \sqrt{a_1^2 - a_2},$$

then

$$r_1 + r_2 = 2a_1 \quad \text{and} \quad r_1 r_2 = a_2.$$

Multiplying (3.3) with $\mathrm{e}_{\ominus_\mu r_1}(\sigma(t),s)$, we find

$$\begin{aligned}
\mathrm{e}_{\ominus_\mu r_1}(\sigma(t),s)f(t) &= \mathrm{e}_{\ominus_\mu r_1}(\sigma(t),s)[y^{\Delta\Delta} - (r_1+r_2)y^\Delta + r_1 r_2 y] \\
&= \mathrm{e}_{\ominus_\mu r_1}(\sigma(t),s)[y^{\Delta\Delta} - r_2 y^\Delta] - r_1 \mathrm{e}_{\ominus_\mu r_1}(\sigma(t),s)[y^\Delta - r_2 y] \\
&= \mathrm{e}_{\ominus_\mu r_1}(\sigma(t),s)[y^\Delta - r_2 y]^\Delta - r_1 \mathrm{e}_{\ominus_\mu r_1}(\sigma(t),s)[y^\Delta - r_2 y]
\end{aligned}$$

$$
\begin{aligned}
&= \left[\mathrm{e}_{\ominus_\mu r_1}(\cdot,s)[y^\Delta - r_2 y]\right]^\Delta(t)\\
&= \left[\mathrm{e}_{\ominus_\mu r_1}(\cdot,s)\mathrm{e}_{r_2}(\sigma(\cdot),s)[\mathrm{e}_{\ominus_\mu r_2}(\cdot,s)y]^\Delta\right]^\Delta(t),
\end{aligned}
$$

which yields upon integrating twice

$$
\begin{aligned}
y(t) &= \mathrm{e}_{r_2}(t,s)\int_s^t \mathrm{e}_{\ominus_\mu r_2}(\sigma(\xi),s)\mathrm{e}_{r_1}(\xi,s)\int_s^\xi \mathrm{e}_{\ominus_\mu r_1}(\sigma(\eta),s)f(\eta)\Delta\eta\Delta\xi\\
&= \mathrm{e}_{r_2}(t,s)\int_s^t \mathrm{e}_{r_2}(s,\sigma(\xi))\mathrm{e}_{r_1}(\xi,s)\int_s^\xi \mathrm{e}_{r_1}(s,\sigma(\eta))f(\eta)\Delta\eta\Delta\xi\\
&= \int_s^t \mathrm{e}_{r_2}(t,\sigma(\xi))\int_s^\xi \mathrm{e}_{r_1}(\xi,\sigma(\eta))f(\eta)\Delta\eta\Delta\xi\\
&= \int_s^t\int_s^\xi \mathrm{e}_{r_2}(t,\sigma(\xi))\mathrm{e}_{r_1}(\xi,\sigma(\eta))f(\eta)\Delta\eta\Delta\xi
\end{aligned}
\tag{3.27}
$$

for $t \in \mathbb{T}$. Applying the change of order formula (see (Karpuz, 2014, Theorem 3.2)) for the integral in (3.27), we obtain

$$
\begin{aligned}
y(t) &= \int_s^t\int_{\sigma(\eta)}^t \mathrm{e}_{r_2}(t,\sigma(\xi))\mathrm{e}_{r_1}(\xi,\sigma(\eta))f(\eta)\Delta\xi\Delta\eta\\
&= \int_s^t f(\eta)\int_{\sigma(\eta)}^t \mathrm{e}_{r_2}(t,\sigma(\xi))\mathrm{e}_{r_1}(\xi,\sigma(\eta))\Delta\xi\Delta\eta
\end{aligned}
\tag{3.28}
$$

for $t \in \mathbb{T}$. Now, we consider three distinct possible cases for the roots of the characteristic parabola (3.6).

3.2.1 Distinct Roots

Let $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then the particular solution in (3.28) becomes

$$
\begin{aligned}
y(t) &= \int_s^t\int_{\sigma(\eta)}^t \mathrm{e}_{r_2}(t,\sigma(\xi))\mathrm{e}_{r_1}(\xi,\sigma(\eta))\Delta\xi f(\eta)\Delta\eta\\
&= \int_s^t \frac{\mathrm{e}_{r_1}(t,\sigma(\eta)) - \mathrm{e}_{r_2}(t,\sigma(\eta))}{r_1 - r_2} f(\eta)\Delta\eta
\end{aligned}
$$

by (Bohner & Peterson, 2001, Theorem 2.39).

3.2.2 Repeated Real Roots

Let $r_1 = r_2$, then the particular solution in (3.28) becomes

$$\begin{aligned} y(t) &= \int_s^t \int_{\sigma(\eta)}^t \frac{\mathrm{e}_{r_1}(t,\sigma(\eta))}{1+r_1\mu(\xi)} \Delta\xi f(\eta)\Delta\eta \\ &= \int_s^t \mathrm{e}_{r_1}(t,\sigma(\eta))\mathrm{m}_{r_1}(t,\sigma(\eta)) f(\eta)\Delta\eta. \end{aligned}$$

3.2.3 Complex Roots

Let $r_1, r_2 \in \mathbb{C}$ and $r_1 = \overline{r_2}$. Considering Subsection 3.2.1, the particular solution in (3.28) becomes

$$\begin{aligned} y(t) &= \int_s^t \frac{\mathrm{e}_{r_1}(t,\sigma(\eta)) - \mathrm{e}_{r_2}(t,\sigma(\eta))}{r_1 - r_2} f(\eta)\Delta\eta \\ &= \int_s^t \frac{\mathrm{e}_{r_1}(t,\sigma(\eta)) - \mathrm{e}_{\overline{r_1}}(t,\sigma(\eta))}{r_1 - \overline{r_1}} f(\eta)\Delta\eta \\ &= \int_s^t \frac{\mathrm{e}_{r_1}(t,\sigma(\eta)) - \overline{\mathrm{e}_{r_1}(t,\sigma(\eta))}}{r_1 - \overline{r_1}} f(\eta)\Delta\eta \\ &= \int_s^t \frac{2\mathrm{i}\,\mathrm{Im}\big(\mathrm{e}_{r_1}(t,\sigma(\eta))\big)}{2\mathrm{i}\,\mathrm{Im}(r_1)} f(\eta)\Delta\eta \\ &= \int_s^t \frac{\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))\sin_{\mathrm{Im}_\mu(r_1)}(t,\sigma(\eta))}{\mathrm{Im}(r_1)} f(\eta)\Delta\eta. \end{aligned}$$

3.3 Conclusions

In this section, we finalize our results with initial value problems for all possible cases of the characteristic roots.

Theorem 7 (Distinct Real Roots)**.** *Suppose that* (3.4) *is regressive and* $a_1^2 > a_2$*. Let* $s \in \mathbb{T}$*, and* $y_0, y_1 \in \mathbb{R}$*. The unique solution of the initial value problem*

$$\begin{cases} y^{\Delta\Delta}(t) - 2a_1 y^{\Delta}(t) + a_2 y(t) = f(t) \\ y(s) = y_0 \quad \textit{and} \quad y^{\Delta}(s) = y_1 \end{cases} \tag{3.29}$$

is given by

$$y(t) = -\frac{r_2 y_0 - y_1}{r_1 - r_2}\mathrm{e}_{r_1}(t,s) + \frac{y_0 r_1 - y_1}{r_1 - r_2}\mathrm{e}_{r_2}(t,s) + \int_s^t \frac{\mathrm{e}_{r_1}(t,\sigma(\eta)) - \mathrm{e}_{r_2}(t,\sigma(\eta))}{r_1 - r_2} f(\eta)\Delta\eta \tag{3.30}$$

for $t \in \mathbb{T}$, where r_1 and r_2 are defined as in (3.7)*, i.e., $r_1 \neq r_2$.*

Proof. We will show that (3.30) satisfies the IVP (3.29). To this end, we define

$$y_{\mathsf{h}}(t) := -\frac{r_2 y_0 - y_1}{r_1 - r_2}\mathrm{e}_{r_1}(t,s) + \frac{y_0 r_1 - y_1}{r_1 - r_2}\mathrm{e}_{r_2}(t,s), \tag{3.31}$$

$$y_{\mathsf{p}}(t) := \int_s^t \frac{\mathrm{e}_{r_1}(t,\sigma(\eta)) - \mathrm{e}_{r_2}(t,\sigma(\eta))}{r_1 - r_2} f(\eta)\Delta\eta. \tag{3.32}$$

First, from (3.31), we compute

$$\begin{aligned} y_{\mathsf{h}}^{\Delta}(t) &= -r_1\frac{r_2 y_0 - y_1}{r_1 - r_2}\mathrm{e}_{r_1}(t,s) + r_2\frac{y_0 r_1 - y_1}{r_1 - r_2}\mathrm{e}_{r_2}(t,s), \\ y_{\mathsf{h}}^{\Delta^2}(t) &= -r_1^2\frac{r_2 y_0 - y_1}{r_1 - r_2}\mathrm{e}_{r_1}(t,s) + r_2^2\frac{y_0 r_1 - y_1}{r_1 - r_2}\mathrm{e}_{r_2}(t,s). \end{aligned} \tag{3.33}$$

We rewrite (3.32) in the form

$$y_{\mathsf{p}}(t) = \frac{\mathrm{e}_{r_1}(t,s)}{r_1 - r_2}\int_s^t \mathrm{e}_{r_1}(s,\sigma(\eta)) f(\eta)\Delta\eta - \frac{\mathrm{e}_{r_2}(t,s)}{r_1 - r_2}\int_s^t \mathrm{e}_{r_2}(s,\sigma(\eta)) f(\eta)\Delta\eta.$$

Next, we compute

$$\begin{aligned} y_{\mathsf{p}}^{\Delta}(t) &= \left(\frac{\mathrm{e}_{r_1}(s,\sigma(t))}{r_1 - r_2}\mathrm{e}_{r_1}(\sigma(t),s) f(t) + \frac{r_1 \mathrm{e}_{r_1}(t,s)}{r_1 - r_2}\int_s^t \mathrm{e}_{r_1}(s,\sigma(\eta)) f(\eta)\Delta\eta\right) \\ &\quad - \left(\frac{\mathrm{e}_{r_2}(s,\sigma(t))}{r_1 - r_2}\mathrm{e}_{r_2}(\sigma(t),s) f(t) + \frac{r_2 \mathrm{e}_{r_2}(t,s)}{r_1 - r_2}\int_s^t \mathrm{e}_{r_2}(s,\sigma(\eta)) f(\eta)\Delta\eta\right) \\ &= \frac{r_1 \mathrm{e}_{r_1}(t,s)}{r_1 - r_2}\int_s^t \mathrm{e}_{r_1}(s,\sigma(\eta)) f(\eta)\Delta\eta - \frac{r_2 \mathrm{e}_{r_2}(t,s)}{r_1 - r_2}\int_s^t \mathrm{e}_{r_2}(s,\sigma(\eta)) f(\eta)\Delta\eta, \end{aligned} \tag{3.34}$$

$$\begin{aligned} y_{\mathsf{p}}^{\Delta^2}(t) &= \left(r_1\frac{\mathrm{e}_{r_1}(s,\sigma(t))}{r_1 - r_2}\mathrm{e}_{r_1}(\sigma(t),s) f(t) + r_1^2 \mathrm{e}_{r_1}(t,s)\int_s^t \frac{\mathrm{e}_{r_1}(s,\sigma(\eta)}{r_1 - r_2} f(\eta)\Delta\eta\right) \\ &\quad - \left(r_2\frac{\mathrm{e}_{r_2}(s,\sigma(t))}{r_1 - r_2}\mathrm{e}_{r_2}(\sigma(t),s) f(t) + r_2^2 \mathrm{e}_{r_2}(t,s)\int_s^t \frac{\mathrm{e}_{r_2}(s,\sigma(\eta)}{r_1 - r_2} f(\eta)\Delta\eta\right) \end{aligned}$$

$$= \int_s^t \frac{r_1^2 \mathrm{e}_{r_1}(t,\sigma(\eta)) - r_2^2 \mathrm{e}_{r_2}(t,\sigma(\eta))}{r_1 - r_2} f(\eta)\Delta\eta + f(t). \tag{3.35}$$

From (3.31), (3.32), (3.33) and (3.34), we get

$$\begin{aligned} y(s) &= y_{\mathsf{h}}(s) + y_{\mathsf{p}}(s) = -\frac{r_2 y_0 - y_1}{r_1 - r_2} + \frac{y_0 r_1 - y_1}{r_1 - r_2} = y_0 \\ y^{\Delta}(s) &= y_{\mathsf{h}}^{\Delta}(s) + y_{\mathsf{p}}^{\Delta}(s) = -r_1 \frac{r_2 y_0 - y_1}{r_1 - r_2} + r_2 \frac{y_0 r_1 - y_1}{r_1 - r_2} = y_1, \end{aligned}$$

i.e., y in (3.30) satisfies the initial conditions in (3.29). Finally, using the notation in (3.4), we can show that

$$L_2[y(t)] = L_2[y_{\mathsf{h}}(t) + y_{\mathsf{p}}(t)] = L_2[y_{\mathsf{h}}(t)] + L_2[y_{\mathsf{p}}(t)] = L_2[y_{\mathsf{p}}(t)]. \tag{3.36}$$

Substituting (3.32), (3.34) and (3.35) into (3.8), we compute

$$\begin{aligned} L_2[y_{\mathsf{p}}(t)] = &\left(\int_s^t \frac{r_1^2 \mathrm{e}_{r_1}(t,\sigma(\eta)) - r_2^2 \mathrm{e}_{r_2}(t,\sigma(\eta))}{r_1 - r_2} f(\eta)\Delta\eta + f(t) \right) \\ &- (r_1 + r_2) \int_s^t \frac{r_1 \mathrm{e}_{r_1}(t,\sigma(\eta)) - r_2 \mathrm{e}_{r_2}(t,\sigma(\eta))}{r_1 - r_2} f(\eta)\Delta\eta \\ &+ r_1 r_2 \int_s^t \frac{\mathrm{e}_{r_1}(t,\sigma(\eta)) - \mathrm{e}_{r_2}(t,\sigma(\eta))}{r_1 - r_2} f(\eta)\Delta\eta \\ = &\left(\int_s^t \frac{r_1^2 \mathrm{e}_{r_1}(t,\sigma(\eta)) - r_2^2 \mathrm{e}_{r_2}(t,\sigma(\eta))}{r_1 - r_2} f(\eta)\Delta\eta + f(t) \right) \\ &- \left(\int_s^t \frac{r_1^2 \mathrm{e}_{r_1}(t,\sigma(\eta)) - r_1 r_2 \mathrm{e}_{r_2}(t,\sigma(\eta))}{r_1 - r_2} f(\eta)\Delta\eta \right. \\ &\quad \left. + \int_s^t \frac{r_1 r_2 \mathrm{e}_{r_1}(t,\sigma(\eta)) - r_2^2 \mathrm{e}_{r_2}(t,\sigma(\eta))}{r_1 - r_2} f(\eta)\Delta\eta \right) \\ &+ \int_s^t \frac{r_1 r_2 \mathrm{e}_{r_1}(t,\sigma(\eta)) - r_1 r_2 \mathrm{e}_{r_2}(t,\sigma(\eta))}{r_1 - r_2} f(\eta)\Delta\eta \\ = f(t). & \end{aligned}$$

This shows that (3.30) is the unique solution of the IVP (3.29). □

Theorem 8 (Repeated Real Roots)**.** *Suppose that* (3.4) *is regressive and* $a_1^2 = a_2$. *Let*

$s \in \mathbb{T}$*, and* $y_0, y_1 \in \mathbb{R}$*. The unique solution of the initial value problem*

$$\begin{cases} y^{\Delta\Delta}(t) - 2a_1 y^{\Delta}(t) + a_2 y(t) = f(t) \\ y(s) = y_0 \quad \textit{and} \quad y^{\Delta}(s) = y_1 \end{cases} \tag{3.37}$$

is given by

$$\begin{aligned} y(t) = {} & y_0 \mathrm{e}_{r_1}(t,s) - (y_0 r_1 - y_1)\mathrm{m}_{r_1}(t,s)\mathrm{e}_{r_1}(t,s) \\ & + \int_s^t \mathrm{m}_{r_1}(t,\sigma(\eta))\mathrm{e}_{r_1}(t,\sigma(\eta)) f(\eta)\Delta\eta \end{aligned} \tag{3.38}$$

for $t \in \mathbb{T}$*, where* r_1 *and* r_2 *are defined as in* (3.9)*, i.e.,* $r_1 = r_2$*.*

Proof. We will show that (3.39) satisfies the IVP (3.37). To this end, we define

$$y_{\mathsf{h}}(t) := y_0 \mathrm{e}_{r_1}(t,s) - (y_0 r_1 - y_1)\mathrm{m}_{r_1}(t,s)\mathrm{e}_{r_1}(t,s), \tag{3.39}$$

$$y_{\mathsf{p}}(t) := \int_s^t \mathrm{m}_{r_1}(t,\sigma(\eta))\mathrm{e}_{r_1}(t,\sigma(\eta)) f(\eta)\Delta\eta. \tag{3.40}$$

First, from (3.12), (3.13) and (3.39), we compute

$$\begin{aligned} y_{\mathsf{h}}^{\Delta}(t) &= y_0 r_1 \mathrm{e}_{r_1}(t,s) - (y_0 r_1 - y_1)\big[\mathrm{e}_{r_1}(t,s) + r_1 \mathrm{m}_{r_1}(t,s)\mathrm{e}_{r_1}(t,s)\big] \\ &= y_1 \mathrm{e}_{r_1}(t,s) - r_1(y_0 r_1 - y_1)\mathrm{m}_{r_1}(t,s)\mathrm{e}_{r_1}(t,s), \\ y_{\mathsf{h}}^{\Delta^2}(t) &= y_1 r_1 \mathrm{e}_{r_1}(t,s) - r_1(y_0 r_1 - y_1)\big[\mathrm{e}_{r_1}(t,s) + r_1 \mathrm{m}_{r_1}(t,s)\mathrm{e}_{r_1}(t,s)\big] \\ &= -r_1(y_0 r_1 - 2y_1)\mathrm{e}_{r_1}(t,s) - r_1^2(y_0 r_1 - y_1)\mathrm{m}_{r_1}(t,s)\mathrm{e}_{r_1}(t,s). \end{aligned} \tag{3.41}$$

We rewrite (3.40) in the form

$$\begin{aligned} y_{\mathsf{p}}(t) = {} & \mathrm{e}_{r_1}(t,s) \int_s^t \mathrm{m}_{r_1}(s,\sigma(\eta))\mathrm{e}_{r_1}(s,\sigma(\eta)) f(\eta)\Delta\eta \\ & + \mathrm{m}_{r_1}(t,s)\mathrm{e}_{r_1}(t,s) \int_s^t \mathrm{e}_{r_1}(s,\sigma(\eta)) f(\eta)\Delta\eta. \end{aligned}$$

Next, we compute

$$
\begin{aligned}
y_{\mathsf{p}}^{\Delta}(t) &= r_1 \mathrm{e}_{r_1}(t,s) \int_s^t \mathrm{m}_{r_1}(s,\sigma(\eta))\mathrm{e}_{r_1}(s,\sigma(\eta))f(\eta)\Delta\eta \\
&\quad + \mathrm{e}_{r_1}(\sigma(t),s)\mathrm{m}_{r_1}(s,\sigma(t))\mathrm{e}_{r_1}(s,\sigma(t))f(t) \\
&\quad + \left[\mathrm{e}_{r_1}(t,s) + r_1 \mathrm{m}_{r_1}(t,s)\mathrm{e}_{r_1}(t,s)\right] \int_s^t \mathrm{e}_{r_1}(s,\sigma(\eta))f(\eta)\Delta\eta \\
&\quad + \mathrm{e}_{r_1}(\sigma(t),s)\mathrm{m}_{r_1}(\sigma(t),s)\mathrm{e}_{r_1}(s,\sigma(t))f(t) \\
&= r_1 \int_s^t \mathrm{m}_{r_1}(t,\sigma(\eta))\mathrm{e}_{r_1}(t,\sigma(\eta))f(\eta)\Delta\eta + \int_s^t \mathrm{e}_{r_1}(t,\sigma(\eta))f(\eta)\Delta\eta \\
&= r_1 y_{\mathsf{p}}(t) + \int_s^t \mathrm{e}_{r_1}(t,\sigma(\eta))f(\eta)\Delta\eta, \qquad (3.42) \\
y_{\mathsf{p}}^{\Delta^2}(t) &= r_1 y_{\mathsf{p}}^{\Delta}(t) + r_1 \mathrm{e}_{r_1}(t,s) \int_s^t \mathrm{e}_{r_1}(s,\sigma(\eta))f(\eta)\Delta\eta \\
&\quad + \mathrm{e}_{r_1}(\sigma(t),s)\mathrm{e}_{r_1}(s,\sigma(t))f(t) \\
&= r_1^2 y_{\mathsf{p}}(t) + 2r_1 \int_s^t \mathrm{e}_{r_1}(t,\sigma(\eta))f(\eta)\Delta\eta + f(t). \qquad (3.43)
\end{aligned}
$$

From (3.39), (3.40), (3.41) and (3.42), we get

$$
\begin{aligned}
y(s) &= y_{\mathsf{h}}(s) + y_{\mathsf{p}}(s) = y_0 \\
y^{\Delta}(s) &= y_{\mathsf{h}}^{\Delta}(s) + y_{\mathsf{p}}^{\Delta}(s) = y_1,
\end{aligned}
$$

i.e., y in (3.38) satisfies the initial conditions in (3.37). Finally, using the notation in (3.4), we can show that (3.36) holds. Substituting (3.40), (3.42) and (3.43) into (3.10), we compute

$$
\begin{aligned}
L_2[y_{\mathsf{p}}(t)] &= \left(r_1^2 y_{\mathsf{p}}(t) + 2r_1 \int_s^t \mathrm{e}_{r_1}(t,\sigma(\eta))f(\eta)\Delta\eta + f(t)\right) \\
&\quad - 2r_1\left(r_1 y_{\mathsf{p}}(t) + \int_s^t \mathrm{e}_{r_1}(t,\sigma(\eta))f(\eta)\Delta\eta\right) + r_1^2 y_{\mathsf{p}}(t) \\
&= f(t).
\end{aligned}
$$

This shows that (3.38) is the unique solution of the IVP (3.37). □

Theorem 9 (Roots in Complex Conjugates)**.** *Suppose that* (3.4) *is regressive and* $a_1^2 <$

a_2. *Let* $s \in \mathbb{T}$, *and* $y_0, y_1 \in \mathbb{R}$. *The unique solution of the initial value problem*

$$\begin{cases} y^{\Delta\Delta}(t) - 2a_1 y^{\Delta}(t) + a_2 y(t) = f(t) \\ y(s) = y_0 \quad \textit{and} \quad y^{\Delta}(s) = y_1 \end{cases} \tag{3.44}$$

is given by

$$\begin{aligned} y(t) = {} & y_0 \mathrm{e}_{\mathrm{Re}_{\mu}(r_1)}(t,s) \cos_{\mathrm{Im}_{\mu}(r_1)}(t,s) \\ & - \frac{y_0 \operatorname{Re}(r_1) - y_1}{\operatorname{Im}(r_1)} \mathrm{e}_{\mathrm{Re}_{\mu}(r_1)}(t,s) \sin_{\mathrm{Im}_{\mu}(r_1)}(t,s) \\ & + \frac{1}{\operatorname{Im}(r_1)} \int_s^t \mathrm{e}_{\mathrm{Re}_{\mu}(r_1)}(t,\sigma(\eta)) \sin_{\mathrm{Im}_{\mu}(r_1)}(t,\sigma(\eta)) f(\eta) \Delta\eta \end{aligned} \tag{3.45}$$

for $t \in \mathbb{T}$, *where* r_1 *and* r_2 *are defined as in* (3.14), *i.e.,* $r_1 = \overline{r_2}$.

Proof. We will show that (3.45) satisfies the IVP (3.44). To this end, we define

$$\begin{aligned} y_{\mathsf{h}}(t) := {} & y_0 \mathrm{e}_{\mathrm{Re}_{\mu}(r_1)}(t,s) \cos_{\mathrm{Im}_{\mu}(r_1)}(t,s) \\ & - \frac{y_0 \operatorname{Re}(r_1) - y_1}{\operatorname{Im}(r_1)} \mathrm{e}_{\mathrm{Re}_{\mu}(r_1)}(t,s) \sin_{\mathrm{Im}_{\mu}(r_1)}(t,s), \end{aligned} \tag{3.46}$$

$$y_{\mathsf{p}}(t) := \frac{1}{\operatorname{Im}(r_1)} \int_s^t \mathrm{e}_{\mathrm{Re}_{\mu}(r_1)}(t,\sigma(\eta)) \sin_{\mathrm{Im}_{\mu}(r_1)}(t,\sigma(\eta)) f(\eta) \Delta\eta. \tag{3.47}$$

First, from Lemma 8 and (3.46), we compute

$$\begin{aligned} y_{\mathsf{h}}^{\Delta}(t) = {} & y_0 \mathrm{e}_{\mathrm{Re}_{\mu}(r_1)}(t,s) \big[\operatorname{Re}(r_1) \cos_{\mathrm{Im}_{\mu}(r_1)}(t,s) - \operatorname{Im}(r_1) \sin_{\mathrm{Im}_{\mu}(r_1)}(t,s)\big] \\ & - \frac{y_0 \operatorname{Re}(r_1) - y_1}{\operatorname{Im}(r_1)} \mathrm{e}_{\mathrm{Re}_{\mu}(r_1)}(t,s) \big[\operatorname{Im}(r_1) \cos_{\mathrm{Im}_{\mu}(r_1)}(t,s) \\ & \qquad + \operatorname{Re}(r_1) \sin_{\mathrm{Im}_{\mu}(r_1)}(t,s)\big] \\ = {} & y_1 \mathrm{e}_{\mathrm{Re}_{\mu}(r_1)}(t,s) \cos_{\mathrm{Im}_{\mu}(r_1)}(t,s) \\ & - \frac{y_0 r_1 \overline{r}_1 - y_1 \operatorname{Re}(r_1)}{\operatorname{Im}(r_1)} \mathrm{e}_{\mathrm{Re}_{\mu}(r_1)}(t,s) \sin_{\mathrm{Im}_{\mu}(r_1)}(t,s) \end{aligned} \tag{3.48}$$

$$y_{\mathsf{h}}^{\Delta^2}(t) = y_1 \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\big[\mathrm{Re}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(t,s) - \mathrm{Im}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big]$$
$$- \frac{y_0 r_1 \overline{r}_1 - y_1 \mathrm{Re}(r_1)}{\mathrm{Im}(r_1)} \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\big[\mathrm{Im}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)$$
$$+ \mathrm{Re}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big]$$
$$= \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\Bigg[\big(2y_1 \mathrm{Re}(r_1) - y_0 r_1 \overline{r}_1\big)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)$$
$$- \left(\frac{\big(\mathrm{Re}(r_1)\big)^2 \big(y_0 \mathrm{Re}(r_1) - y_1\big)}{\mathrm{Im}(r_1)} + \mathrm{Im}(r_1)\big(y_0 \mathrm{Re}(r_1) + y_1\big)\right)$$
$$\times r \sin_{\mathrm{Im}_\mu(r_1)}(t,s)\Bigg].$$

By using Property 3, we rewrite (3.47) in the form

$$y_{\mathsf{p}}(t) = \frac{1}{\mathrm{Im}(r_1)} \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(\eta))$$
$$\times \big[\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))$$
$$+ \sin_{\mathrm{Im}_\mu(r_1)}(t,s)\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))\big] f(\eta)\Delta\eta$$
$$= \frac{1}{\mathrm{Im}(r_1)} \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)$$
$$\times \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(\eta))\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta)) f(\eta)\Delta\eta$$
$$+ \frac{1}{\mathrm{Im}(r_1)} \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)$$
$$\times \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta)) f(\eta)\Delta\eta.$$

Next, by Lemma 8, we compute

$$y_{\mathsf{p}}^{\Delta}(t)$$
$$= \frac{1}{\mathrm{Im}(r_1)}\Bigg[\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\sigma(t),s)\cos_{\mathrm{Im}_\mu(r_1)}(\sigma(t),s)$$
$$\times \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(t))\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(t)) f(t)$$
$$+ \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\big[\mathrm{Re}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(t,s) - \mathrm{Im}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big]$$
$$\times \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(\eta))\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta)) f(\eta)\Delta\eta\Bigg]$$
$$+ \frac{1}{\mathrm{Im}(r_1)}\Bigg[\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\sigma(t),s)\sin_{\mathrm{Im}_\mu(r_1)}(\sigma(t),s)$$

$$
\begin{aligned}
&\times \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(t)) \cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(t)) f(t) \\
&+ \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s) \big[\mathrm{Im}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(t,s) + \mathrm{Re}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big] \\
&\quad \times \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(\eta)) \cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta)) f(\eta)\Delta\eta \Big]
\end{aligned}
$$

$$
\begin{aligned}
= {} & \frac{1}{\mathrm{Im}(r_1)} \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta)) \big[\mathrm{Re}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta)) \cos_{\mathrm{Im}_\mu(r_1)}(t,s) \\
& \qquad - \mathrm{Im}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta)) \sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big] \\
& \qquad \times f(\eta)\Delta\eta \\
& + \frac{1}{\mathrm{Im}(r_1)} \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta)) \big[\mathrm{Im}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta)) \cos_{\mathrm{Im}_\mu(r_1)}(t,s) \\
& \qquad + \mathrm{Re}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta)) \sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big] \\
& \qquad \times f(\eta)\Delta\eta \\
= {} & \frac{1}{\mathrm{Im}(r_1)} \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta)) \Big[\mathrm{Re}(r_1)\big(\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta)) \cos_{\mathrm{Im}_\mu(r_1)}(t,s) \\
& \qquad + \cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta)) \sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big) \\
& \qquad + \mathrm{Im}(r_1)\big(\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta)) \cos_{\mathrm{Im}_\mu(r_1)}(t,s) \\
& \qquad - \sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta)) \sin_{\mathrm{Im}_\mu(r_1)}(t,s)\big)\Big] \\
& \qquad \times f(\eta)\Delta\eta \\
= {} & \frac{\mathrm{Re}(r_1)}{\mathrm{Im}(r_1)} \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta)) \sin_{\mathrm{Im}_\mu(r_1)}(t,\sigma(\eta)) f(\eta)\Delta\eta \\
& + \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta)) \cos_{\mathrm{Im}_\mu(r_1)}(t,\sigma(\eta)) f(\eta)\Delta\eta \\
= {} & \mathrm{Re}(r_1) y_{\mathsf{p}}(t) + \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta)) \cos_{\mathrm{Im}_\mu(r_1)}(t,\sigma(\eta)) f(\eta)\Delta\eta. \qquad (3.49)
\end{aligned}
$$

We rewrite (3.49) in the form

$$
\begin{aligned}
& y_{\mathsf{p}}^{\Delta}(t) \\
= {} & \mathrm{Re}(r_1) y_{\mathsf{p}}(t) \\
& + \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s) \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(\eta)) \big[\cos_{\mathrm{Im}_\mu(r_1)}(t,s) \cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))
\end{aligned}
$$

$$- \sin_{\mathrm{Im}_\mu(r_1)}(t,s)\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))\big] f(\eta)\Delta\eta$$

$$\begin{aligned}
= \ & \mathrm{Re}(r_1)y_{\mathsf{p}}(t) \\
& + \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))f(\eta)\Delta\eta \\
& - \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(\eta))\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))f(\eta)\Delta\eta.
\end{aligned}$$

It follows that

$$\begin{aligned}
& y_{\mathsf{p}}^{\Delta^2}(t) \\
= \ & \big(\mathrm{Re}(r_1)\big)^2 y_{\mathsf{p}}(t) \\
& + \mathrm{Re}(r_1)\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\cos_{\mathrm{Im}_\mu(r_1)}(t,s) \\
& \quad \times \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))f(\eta)\Delta\eta \\
& - \mathrm{Re}(r_1)\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)\sin_{\mathrm{Im}_\mu(r_1)}(t,s) \\
& \quad \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(\eta))\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))f(\eta)\Delta\eta \\
& + \Bigg(\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)[\mathrm{Re}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(t,s) - \mathrm{Im}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)] \\
& \qquad \times \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))f(\eta)\Delta\eta \\
& \qquad + \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\sigma(t),s)\cos_{\mathrm{Im}_\mu(r_1)}(\sigma(t),s)\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(t))\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(t))f(t)\Bigg) \\
& - \Bigg(\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,s)[\mathrm{Im}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(t,s) + \mathrm{Re}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)] \\
& \qquad \times \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(\eta))\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))f(\eta)\Delta\eta \\
& \qquad + \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(\sigma(t),s)\sin_{\mathrm{Im}_\mu(r_1)}(\sigma(t),s)\mathrm{e}_{\mathrm{Re}_\mu(r_1)}(s,\sigma(t))\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(t))f(t)\Bigg) \\
= \ & \big(\mathrm{Re}(r_1)\big)^2 y_{\mathsf{p}}(t) \\
& + \mathrm{Re}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))f(\eta)\Delta\eta \\
& - \mathrm{Re}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))f(\eta)\Delta\eta
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg([\mathrm{Re}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)-\mathrm{Im}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)]\\
&\quad\times\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))f(\eta)\Delta\eta\\
&\quad+\cos_{\mathrm{Im}_\mu(r_1)}(\sigma(t),s)\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(t))f(t)\Bigg)\\
&-\Bigg([\mathrm{Im}(r_1)\cos_{\mathrm{Im}_\mu(r_1)}(t,s)+\mathrm{Re}(r_1)\sin_{\mathrm{Im}_\mu(r_1)}(t,s)]\\
&\quad\times\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))f(\eta)\Delta\eta\\
&\quad+\sin_{\mathrm{Im}_\mu(r_1)}(\sigma(t),s)\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(t))f(t)\Bigg)\\
=&\big(\mathrm{Re}(r_1)\big)^2 y_{\mathsf{p}}(t)\\
&+2\,\mathrm{Re}(r_1)\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))\big[\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(t,s)\\
&\qquad\qquad-\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))\big]f(\eta)\Delta\eta\\
&-\mathrm{Im}(r_1)\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))[\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))\sin_{\mathrm{Im}_\mu(r_1)}(t,s)\\
&\qquad\qquad+\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(t,s)]f(\eta)\Delta\eta\\
&+\big[\cos_{\mathrm{Im}_\mu(r_1)}(\sigma(t),s)\cos_{\mathrm{Im}_\mu(r_1)}(s,\sigma(t))\\
&\quad-\sin_{\mathrm{Im}_\mu(r_1)}(\sigma(t),s)\sin_{\mathrm{Im}_\mu(r_1)}(s,\sigma(t))\big]f(t)\\
=&\big(\mathrm{Re}(r_1)\big)^2 y_{\mathsf{p}}(t)\\
&+2\,\mathrm{Re}(r_1)\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(t,\sigma(\eta))f(\eta)\Delta\eta\\
&-\mathrm{Im}(r_1)\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))\sin_{\mathrm{Im}_\mu(r_1)}(t,\sigma(\eta))f(\eta)\Delta\eta\\
&+f(t)\\
=&\Big[\big(\mathrm{Re}(r_1)\big)^2-\big(\mathrm{Im}(r_1)\big)^2\Big]y_{\mathsf{p}}(t)\\
&+2\,\mathrm{Re}(r_1)\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(t,\sigma(\eta))f(\eta)\Delta\eta \qquad (3.50)\\
&+f(t).
\end{aligned}
$$

Note that

$$y(s) = y_{\mathsf{h}}(s) + y_{\mathsf{p}}(s) = y_0$$
$$y^{\Delta}(s) = y_{\mathsf{h}}^{\Delta}(s) + y_{\mathsf{p}}^{\Delta}(s) = y_1,$$

i.e., y in (3.45) satisfies the initial conditions in (3.44). Finally, using the notation in (3.4), we can show that (3.36) holds. Substituting (3.47), (3.49) and (3.50) into (3.15), we compute

$$\begin{aligned}
L_2[y_{\mathsf{p}}(t)] &= \Bigg(\Big[\big(\mathrm{Re}(r_1)\big)^2 - \big(\mathrm{Im}(r_1)\big)^2\Big] y_{\mathsf{p}}(t) \\
&\qquad + 2\,\mathrm{Re}(r_1)\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(t,\sigma(\eta)) f(\eta)\Delta\eta + f(t)\Bigg) \\
&\quad - 2\,\mathrm{Re}(r_1)\Bigg(\mathrm{Re}(r_1) y_{\mathsf{p}}(t) \\
&\qquad\qquad + \int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(t,\sigma(\eta)) f(\eta)\Delta\eta\Bigg) \\
&\quad + r_1\bar{r}_1 y_{\mathsf{p}}(t) \\
&= \Bigg(\Big[\big(\mathrm{Re}(r_1)\big)^2 - \big(\mathrm{Im}(r_1)\big)^2\Big] - 2\big(\mathrm{Re}(r_1)\big)^2 + r_1\bar{r}_1\Bigg) y_{\mathsf{p}}(t) \\
&\quad + 2\big(\mathrm{Re}(r_1) - \mathrm{Re}(r_1)\big)\int_s^t \mathrm{e}_{\mathrm{Re}_\mu(r_1)}(t,\sigma(\eta))\cos_{\mathrm{Im}_\mu(r_1)}(t,\sigma(\eta)) f(\eta)\Delta\eta \\
&\quad + f(t) \\
&= f(t).
\end{aligned}$$

This shows that (3.45) is the unique solution of the IVP (3.44). □

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