DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ITERATED OSCILLATION TESTS FOR DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

by
Zehra ÖZSEVER

August, 2021
İZMİR

ITERATED OSCILLATION TESTS FOR DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

A Thesis Submitted to the
Graduate School of Natural And Applied Sciences of Dokuz Eylül University
In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

by
Zehra ÖZSEVER

August, 2021
İZMİR

M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**ITERATED OSCILLATION TESTS FOR DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS**" completed by **ZEHRA ÖZSEVER** under supervision of **PROF. DR. BAŞAK KARPUZ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

....

Prof. Dr. Başak KARPUZ

Supervisor

....

Assoc. Prof. Dr. Sevin GÜMGÜM

Jury Member

....

Assist. Prof. Dr. Melda DUMAN

Jury Member

Prof. Dr. Özgür ÖZÇELİK
Director
Graduate School of Natural and Applied Sciences

ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere gratitude to my supervisor Prof. Dr. Başak KARPUZ for his unending support, advises, his patience during the conduct of this study in answering questions and for being me an example with his discipline and vision.

I am very grateful to my family for their unconditional love, understanding, encouragement and supporting me throughout my life. Last but not the least, I would like to thank my little niece who gives me motivation and happiness even though she doesn’t know for now.

Zehra ÖZSEVER

ITERATED OSCILLATION TESTS FOR DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

ABSTRACT

In this thesis, we will reconsider the significant results on oscillation and nonoscillation of solutions of an important class of difference equations with variable coefficients in the literature and we will examine them with numerical examples. Later, we will give our new result for the oscillation of delay difference equations with variable coefficients and we will reinforce the importance of our result with an example where to the best of our knowledge all the oscillation results in the literature fail to give a positive answer. Lastly, we will state some of the other well-known iterative results on oscillation of solutions of delay difference equations to make our final comments.

Keywords: Oscillation, nonoscillation, delay difference equations

DEĞİŞKEN KATSAYILI FARK DENKLEMLERİN SALINIMI İÇİN YİNELEMELİ SALINIM TESTLERİ

ÖZ

Bu tezde, literatürdeki değişken katsayılı fark denklemlerin önemli bir sınıfının çözümlerinin salınımlı ve salınımsızlığına ilişkin önemli sonuçları yeniden ele alacağız ve sayısal örnekler üzerinden inceleyeceğiz. Daha sonra değişken katsayılı gecikmeli fark denklemlerin salınımı için yeni sonucumuzu vereceğiz ve sonucumuzun önemini bildiğimiz kadarıyla literatürde daha önceki hiçbir sonucun olumlu cevap veremediği bir sayısal örnekle pekiştireceğiz. Son olarak, gecikmeli fark denklemlerin çözümlerinin salınımı için bilinen diğer yinelemeli sonuçlardan bazılarını son açıklamalarımızı yapmak için ifade edeceğiz.

Anahtar kelimeler: Salınım, salınımsızlık, gecikmeli fark denklemleri

CONTENTS

CHAPTER ONE
INTRODUCTION

Ordinary difference equations have powerful outcomes and these outcomes help finding solution of many problems in the natural sciencies like physics, chemistry and biology. Nowadays, the ordinary difference equations appear in astronomy, mechanics and engineering. They take role in new inventions in technology, and also sending a vehicle into space. The most interesting applications of these equations are the theory of oscillations. For these results, today finding the new results in difference equations and its applications occupy an important place in mathematics.

Definition 1.0.1. *The difference equation of order* $(\tau + 1)$ *is in the form of*

$$x(n+1) = f\big(n, x(n), x(n-1), \cdots, x(n-\tau)\big) \quad \textit{for } n = 0, 1, \cdots$$

for a given function $f \in \mathrm{C}(\mathbb{N}_0 \times \mathbb{R}^{\tau+1}, \mathbb{R})$ *and* $\tau \in \mathbb{N}_0$*, where* $\mathbb{N}_0 := \{0, 1, \cdots\}$

Example 1. *Fibonacci first described his famous number sequence as the solution to a math problem: If a pair of rabbits are put together under certain conditions (no rabbits may leave the field), how many will there be in one year? This puzzle, posed by Fibonacci in the* 13*th-century, is the premise for Gravett's book.*

A pair of rabbits does not breed until they are 2 *months old. After they are* 2 *months old, each pair of rabbits produces another pair each month.Therefore, In the first month, there is* 1 *pair of rabbit. In the second month, that pair of rabbits mate, there is still* 1 *pair of rabbit. In the third month, there are* 2 *pairs of rabbits where one of them is newborn pair of rabbit and the old ones mate again. In the fourth month, there are* 3 *pairs of rabbits where the newborn pair of rabbits are produced because of mating of original pair at last month. In the* $(n+1)$*-st month, the number of pairs of rabbits are equal to the number of pairs in previous month i.e. the number of pairs in* n*-th month plus the number of pairs before the previous month i.e. the number of pairs in* $(n-1)$*-st month. Therefore, the mathematical formulation for the number of rabbits is*

$$x(n+1) = x(n) + x(n-1), \quad n = 0, 1, \cdots . \tag{1.1}$$

Definition 1.0.2. *The linear difference equation of order* $(\tau + 1)$ *is in the form of*

$$p_0(n)x(n+1)+p_1(n)x(n)+\cdots+p_{\tau+1}(n)x(n-\tau) = q(n), \quad p_0(n)p_{\tau+1}(n) \neq 0. \quad (1.2)$$

The equation (1.2) *is called an equation with constant coefficients, if the constants* $p_0(n), \cdots, p_{\tau+1}(n)$ *do* ***not*** *depend on* n. *Otherwise, it is called an equation with variables. If* $q(n) \equiv 0$*, then the equation* (1.2) *is homogenous. Otherwise, the equation* (1.2) *is nonhomogenous.*

Example 2. *Eq.* (1.1) *in Example 1, is the second-order homogenous difference equation with constant coefficients while the equation*

$$x(n+1) = (n-1)x(n) + x(n-1) + 2^n, \quad n = 0, 1, \cdots$$

is the second-order nonhomogenous difference equation with variable coefficients.

Definition 1.0.3. *A sequence* $\{x(n)\}$ *for which* (1.2) *is satisfied for* $n = 0, 1, \cdots$ *is called a solution of* (1.2). *It is known that for prescribed values* $\varphi_0, \varphi_1, \cdots, \varphi_\tau$, (1.2) *admits a unique solution* $\{x(n)\}$ *satisfying* $x(-j) = \varphi_j$ *for* $j = 0, 1, \cdots, \tau$.

Example 3. *Eq.* (1.1) *has the solution*

$$x(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \textit{for } n = -1, 0, \cdots,$$

where c_1 *and* c_2 *can be any real number. With the initial values* $x(-1) = 1$ *and* $x(0) = 1$*, we have a unique solution*

$$x(n) = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \textit{for } n = -1, 0, \cdots.$$

Precisely, we have

$$\{x(n)\} = \{\underbrace{1}_{x(-1)}, \underbrace{1}_{x(0)}, \underbrace{2}_{x(1)}, \underbrace{3}_{x(2)}, \underbrace{5}_{x(3)}, \underbrace{8}_{x(4)}, \underbrace{13}_{x(5)}, \underbrace{21}_{x(6)}, \underbrace{34}_{x(7)}, \cdots\}.$$

In this thesis, we advance a recent oscillation test for the oscillation of the delay difference equation

$$x(n+1) - x(n) + p(n)x(n-\tau) = 0 \quad \text{for } n = 0, 1, \cdots, \tag{1.3}$$

where $\{p(n)\} \subset [0, \infty)$ and $\tau \in \mathbb{N}_0$.

Definition 1.0.4. *A solution $\{x(n)\}$ of (1.3) is said to be eventually positive if*

$$\sup\{n : \; x(n) \leq 0\} < \infty.$$

Otherwise, if

$$\sup\{n : \; x(n) \geq 0\} < \infty,$$

then $\{x(n)\}$ is said to be eventually negative. A solution $\{x(n)\}$ of (1.3), which is neither eventually positive nor eventually negative is said to be oscillatory.

Example 4. *Consider the difference equation*

$$x(n+1) - x(n) + \frac{4}{27}x(n-2) = 0 \quad \textit{for } n = 0, 1, \cdots. \tag{1.4}$$

Note that

$$x_1(n) = \left(-\frac{1}{3}\right)^n, \quad x_2(n) = \left(\frac{2}{3}\right)^n \quad \textit{and} \quad x_3(n) = n\left(\frac{2}{3}\right)^n \quad \textit{for } n = -2, -1, \cdots$$

are three solutions of (1.4). Note that the solution $\{x_1(n)\}$ is oscillatory while $\{x_2(n)\}$ and $\{x_3(n)\}$ are nonoscillatory. Furthermore, the solution satisfying the initial condition $x(-2) = \frac{745}{81}$, $x(-1) = -\frac{697}{243}$ and $x(0) = \frac{793}{729}$ is

$$x_4(n) = \left(-\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^{n+6} \quad \textit{for } n = -2, -1, \cdots.$$

Explicitly, we have

$$\{x_4(n)\} = \Bigg\{ \underbrace{\frac{745}{81}}_{x_4(-2)}, \underbrace{-\frac{697}{243}}_{x_4(-1)}, \underbrace{\frac{793}{729}}_{x_4(0)}, \underbrace{-\frac{601}{2187}}_{x_4(1)}, \underbrace{\frac{985}{6561}}_{x_4(2)}, \underbrace{-\frac{217}{19683}}_{x_4(3)}, \underbrace{\frac{1753}{59049}}_{x_4(4)}, \underbrace{\frac{1319}{177147}}_{x_4(5)}, \underbrace{\frac{4825}{531441}}_{x_4(6)}, \cdots \Bigg\},$$

which is eventually positive since

$$\sup\{n : \ x_4(n) \leq 0\} = \sup\left\{n : \ \left(-\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^{n+6} \leq 0\right\} = 3 < \infty.$$

In the last few decades, the oscillatory character and the existence of positive solutions of difference equations with several deviating arguments have been extensively studied, see, for example, papers Erbe & Zhang (1989), Ladas et al. (1989a,b), Ladas (1991), Győri & Ladas (1991), Yu et al. (1994), Chen & Yu (1995), Tang & Yu (1999a,b), Tabor (2003), Berezansky & Braverman (2006), Chatzarakis & Stavroulakis (2006), Bohner et al. (2008), Chatzarakis et al. (2008), Malygina & Chudinov (2013), Karpuz (2017) and references cited therein. Our results will cover the general discussion in the mentioned references and complement them.

CHAPTER TWO
RESULTS IN THE LITERATURE

In this section, for the sake of convenience, we will quote some related results on the oscillation and nonoscillation of solutions to (1.3).

2.1 Preparatory Results

Before we give the proof of Theorem 2.2.3, we need the following lemma.

Lemma 2.1.1. *Assume that*

$$\limsup_{n\to\infty} \sum_{j=n-\tau}^{n-1} p(j) > 0. \tag{2.1}$$

Let $\{x(n)\}$ *be a nonoscillatory solution of* (1.3)*. Then,*

$$\liminf_{n\to\infty} \frac{x(n-\tau)}{x(n)} < \infty.$$

Proof. Consider, in view of (3.3), there exist an increasing divergent sequence $\{n_k\}$ and a constant $\varepsilon > 0$ such that

$$\sum_{j=n_k-\tau}^{n_k} p(j) \geq \sum_{j=n_k-\tau}^{n_k-1} p(j) \geq \varepsilon \quad \text{for all } k. \tag{2.2}$$

Define n_k^* to be the number between $(n_k - \tau)$ and n_k such that

$$\sum_{j=n_k-\tau}^{n_k^*-1} p(j) < \frac{\varepsilon}{2} \quad \text{and} \quad \sum_{j=n_k-\tau}^{n_k^*} p(j) \geq \frac{\varepsilon}{2} \quad \text{for all } k, \tag{2.3}$$

where we adopt the convention that sum over empty set is zero. Clearly, such a number exists. By (2.2) and (2.3), we get

$$\sum_{j=n_k^*}^{n_k} p(j) = \sum_{j=n_k-\tau}^{n_k} p(j) - \sum_{j=n_k-\tau}^{n_k^*-1} p(j) \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} \quad \text{for all } k. \tag{2.4}$$

From Eq. (1.3), (2.3) and eventually nonincreasing nature of $\{x(n)\}$, we have for all k that

$$\begin{aligned} x(n_k^*+1)-x(n_k-\tau) &= \sum_{j=n_k-\tau}^{n_k^*}[x(j+1)-x(j)] \\ &= -\sum_{j=n_k-\tau}^{n_k^*}p(j)x(j-\tau) \\ &\leq -\left(\sum_{j=n_k-\tau}^{n_k^*}p(j)\right)x(n_k^*-\tau) \\ &\leq -\frac{\varepsilon}{2}x(n_k^*-\tau). \end{aligned}$$

Hence,

$$\frac{\varepsilon}{2}x(n_k^*-\tau) \leq x(n_k-\tau) \quad \text{for all } k. \tag{2.5}$$

Similarly, from Eq. (1.3) and (2.4), we get for all k that

$$\begin{aligned} x(n_k+1)-x(n_k^*) &= \sum_{j=n_k^*}^{n_k}[x(j+1)-x(j)] \\ &= -\sum_{j=n_k^*}^{n_k}p(j)x(j-\tau) \\ &\leq -\left(\sum_{j=n_k^*}^{n_k}p(j)\right)x(n_k^*-\tau) \\ &\leq -\frac{\varepsilon}{2}x(n_k^*-\tau), \end{aligned}$$

and so

$$\frac{\varepsilon}{2}x(n_k-\tau) \leq x(n_k^*) \quad \text{for all } k. \tag{2.6}$$

From (2.5) and (2.6), we find that

$$\left(\frac{\varepsilon}{2}\right)^2 x(n_k^*-\tau) \leq x(n_k^*) \quad \text{for all } k,$$

i.e.,

$$\frac{x(n_k^*-\tau)}{x(n_k^*)} \leq \left(\frac{2}{\varepsilon}\right)^2 \quad \text{for all } k. \tag{2.7}$$

Then, (2.7) implies,

$$\liminf_{n\to\infty} \frac{x(n-\tau)}{x(n)} < \infty.$$

and the proof is complete. □

Before giving the proof of Theorem 2.2.5 (i), we need the following lemma.

Lemma 2.1.2. *If* (2.28) *holds, then*

$$\limsup_{n\to\infty} \max_{l-\tau\le n+1\le l} \left\{ \left(\sum_{j=l-\tau}^{n} p(j) \right) \left(\sum_{j=n+1}^{l} p(j) \right) \right\} > 0 \tag{2.8}$$

or

$$\limsup_{n\to\infty} \sum_{j=n-\tau}^{n} p(j) > 1. \tag{2.9}$$

Proof. We can find $N_1 \in \mathbb{N}$ and $\mu_0 > 1$ such that $\frac{1}{\lambda}\prod_{j=n-\tau}^{n}[1+\lambda p(j)] \ge \mu_0$ for all $\lambda \ge 1$ and $n \ge N_1$. Now, fix some $\lambda_0 \ge 1$ such that $\lambda_0 > \frac{1}{\mu_0 - 1}$. Then, we have

$$\begin{aligned} \left(1 + \frac{\lambda_0}{\tau+1}\sum_{j=n-\tau}^{n} p(j)\right)^{\tau+1} &= \left(\frac{1}{\tau+1}\sum_{j=n-\tau}^{n}[1+\lambda_0 p(j)]\right)^{\tau+1} \\ &\ge \prod_{j=n-\tau}^{n}[1+\lambda_0 p(j)] \ge \lambda_0\mu_0, \end{aligned}$$

for $n \ge N_1$, where we have applied the inequality of arithmetic and geometric means. Then, we have $\sum_{j=n-\tau}^{n} p(j) \ge \varepsilon$ for $n \ge N_1$, where $\varepsilon := \frac{\tau+1}{\lambda_0}\left((\lambda_0\mu_0)^{\frac{1}{\tau+1}} - 1\right) > 0$. Thus, $\max_{n-\tau\le j\le n}\{p(j)\} \ge \frac{\varepsilon}{\tau+1}$ for $n \ge N_1$. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of integers satisfying $p(n_k) \ge \frac{\varepsilon}{\tau+1}$ for $k = 1, 2, \cdots$. Now, consider the following two possible cases.

Case 1. Let $\limsup_{k\to\infty}\sum_{j=n_k-\tau}^{n_k-1} p(j) > 0$. Then, (2.8) immediately follows from

$$\limsup_{k\to\infty}\left[\left(\sum_{j=n_k-\tau}^{n_k-1} p(j)\right) p(n_k)\right] > 0.$$

Case 2. Let $\limsup_{k\to\infty}\sum_{j=n_k-\tau}^{n_k-1} p(j) = 0$. Then, each of the term of the sum tends to 0 as $k \to \infty$, i.e. $\lim_{k\to\infty} \prod_{j=n_k-\tau}^{n_k-1}[1+\lambda_0 p(j)] = 1$. Thus,

$$\liminf_{k\to\infty}\left(\frac{1}{\lambda_0}[1+\lambda_0 p(n_k)]\right) = \liminf_{k\to\infty}\left(\frac{1}{\lambda_0}\prod_{j=n_k-\tau}^{n_k}[1+\lambda_0 p(j)]\right) \geq \mu_0,$$

which yields $\liminf_{k\to\infty} p(n_k) \geq \mu_0 - \frac{1}{\lambda_0} > 1$, i.e., (2.9) holds. Therefore, the proof is complete. □

2.2 Main Results

To the best of our knowledge, one of the first results in this subject is given by L. H. Erbe and B. G. Zhang in 1989.

Theorem 2.2.1 ((Erbe & Zhang, 1989, Theorems 2.2 and 2.3)). *(i) Assume that*

$$\liminf_{n\to\infty} p(n) > \frac{\tau^\tau}{(\tau+1)^{\tau+1}}. \tag{2.10}$$

Then, every solution of (1.3) *oscillates.*

(ii) Assume that

$$p(n) \leq \frac{\tau^\tau}{(\tau+1)^{\tau+1}} \quad \textit{for all large } n. \tag{2.11}$$

Then, (1.3) *has an eventually positive solution.*

Proof. (i) Assume for the sake of contradiction that, there exists an eventually positive solution $\{x(n)\}$ of (1.3). Suppose that $x(n) > 0$ for $n \geq N_1$, where $N_1 \in \mathbb{N}$ is sufficiently large. Let $w(n) := \frac{x(n)}{x(n+1)} \geq 1$ for $n \geq N_1$. Dividing (1.3) by $x(n)$, we have

$$\frac{1}{w(n)} = 1 - p(n)w(n-\tau)\cdots w(n-1), \quad n \geq N_2, \tag{2.12}$$

where $N_2 \geq N_1 + \tau$. From (2.12), we have $p(n) > 0$ for $n \geq N_2$. Thus, $\{x(n)\}$ is nonincreasing on $\{N_2, N_2+1, \cdots\}$, and so, $w(n) \geq 1$ for $n \geq N_2$. Also $\{p(n)\}$ is bounded above. Otherwise, from (2.10) and (2.12), we get $w(n) < 0$ for all arbitrarily large n. If we set $w_* := \liminf_{n\to\infty} w(n)$, then from (2.12), we have

$$\limsup_{n\to\infty} \frac{1}{w(n)} = \frac{1}{w_*} = 1 - \liminf_{n\to\infty}\{p(n)w(n-\tau)\cdots w(n-1)\}$$
$$\leq 1 - w_*^\tau \liminf_{n\to\infty} p(n).$$

Thus, we have

$$\liminf_{n\to\infty} p(n) \leq \frac{w_* - 1}{w_*^{\tau+1}}.$$

Since $\max_{h\geq 1}\left\{\frac{h-1}{h^{\tau+1}}\right\} = \frac{\tau^\tau}{(\tau+1)^{\tau+1}}$, we have

$$\liminf_{n\to\infty} p(n) \leq \frac{\tau^\tau}{(\tau+1)^{\tau+1}},$$

which contradicts with (2.10).

(ii) We will show that

$$\frac{1}{w(n)} = 1 - p(n)w(n-\tau)\cdots w(n-1) \quad \text{for } n \geq N_1, \tag{2.13}$$

where $N_1 \in \mathbb{N}$ is sufficiently large, has a positive solution. For this purpose, we define

$$s(n) := \begin{cases} \dfrac{\tau+1}{\tau}, & N_1 - \tau \leq n < N_1 \\ \dfrac{1}{1 - p(n)s(n-\tau)\cdots s(n-1)}, & n \geq N_1. \end{cases} \tag{2.14}$$

From (2.13) and (2.14), it follows that $s(N_1) \leq \frac{\tau+1}{\tau}$. So, we define

$$s(N_1+1) = \frac{1}{1 - p(N_1+1)s(N_1+1-\tau)\cdots s(N_1)} \leq \frac{\tau+1}{\tau}.$$

By induction, $1 < s(n) \leq \frac{\tau+1}{\tau}$ for $n \geq N_1$ and $k \geq 1$. Thus, $\{s(n)\}$ satisfies (2.13) on $\{N_1, N_1+1, \cdots\}$. Next, defining

$$x(n) := \begin{cases} 1, & N_1 - \tau \leq n \leq N_1 \\ \dfrac{x(n-1)}{s(n-1)}, & n > N_1, \end{cases}$$

it follows that $\{x(n)\}$ satisfies (1.3). $\square$

Remark 1. *When there is a single constant coefficient, the equation*

$$x(n+1) - x(n) + \sum_{j=1}^{m} p_j x(n-\tau_j) = 0 \quad \textit{for } n = 0, 1, \cdots,$$

where $p_j \in \mathbb{R}^+ := (0, \infty)$ and $\tau_j \in \mathbb{N}_0$ for $j = 1, 2, \cdots, m$, in (Ladas et al., 1989a, Theorem 1) reads as

$$x(n+1) - x(n) + px(n-\tau) = 0 \quad \textit{for } n = 0, 1, \cdots, \tag{2.15}$$

where $p \in \mathbb{R}^+$ and $\tau \in \mathbb{N}_0$, whose characteristic equation is

$$\mu - 1 + p\mu^{-\tau} = 0. \tag{2.16}$$

Note that Eq. (2.16) ***cannot*** *hold if $\mu \in [1, \infty)$. Further, by simple calculus, we compute*

$$\min_{h \in (0,1)} \{h - 1 + ph^{-\tau}\} = \frac{\tau+1}{\tau}(\tau p)^{\frac{1}{\tau+1}} - 1,$$

which shows that Eq. (2.16) *fails to hold if $p > \frac{\tau^\tau}{(\tau+1)^{\tau+1}}$, and is fulfilled if $p \leq \frac{\tau^\tau}{(\tau+1)^{\tau+1}}$. Therefore, Theorem 2.2.1 extends (Ladas et al., 1989a, Theorem 1) to equations with a variable coefficient.*

Theorem 2.2.2 (Cf. (Erbe & Zhang, 1989, Theorem 2.5))**.** *Assume that there exists an increasing sequence $\{n_k\}$ of nonnegative integers such that*

$$\sum_{j=n_k-\tau}^{n_k} p(j) \geq 1 \quad \textit{for all } k.$$

Then, every solution of (1.3) *oscillates.*

Proof. Assume for the sake of contradiction that, there exists an eventually positive solution $\{x(n)\}$ of (1.3), i.e., $x(n) > 0$ for $n \geq N_1$, where $N_1 \in \mathbb{N}$ is sufficiently large. Then, $x(n-\tau) > 0$ for $n \geq N_2$, where $N_2 := N_1 + \tau$. This implies that $\{x(n)\}$ is nonincreasing on $\{N_2, N_2+1, \cdots\}$. There exists k_1 such that $n_{k_1} \geq N_2$. Now, we estimate that

$$\begin{aligned}x(n_k+1) &= x(n_k-\tau) + \sum_{j=n_k-\tau}^{n_k} [x(j+1)-x(j)] \\ &= x(n_k-\tau) - \sum_{j=n_k-\tau}^{n_k} p(j)x(j-\tau) \\ &\le x(n_k-\tau)\left(1 - \sum_{j=n_k-\tau}^{n_k} p(j)\right) \le 0\end{aligned}$$

for all $k \geq k_1$, which is a contradiction. □

Now, we will present two examples to show that Theorem 2.2.1 and Theorem 2.2.2 are **not** comparable.

Example 5. *Consider the equation*

$$x(n+1) - x(n) + \begin{Bmatrix} \frac{1}{6}, & 0 \equiv n \pmod 2 \\ \frac{1}{3}, & 1 \equiv n \pmod 2 \end{Bmatrix} x(n-2) = 0 \quad \textit{for } n = 0, 1, \cdots . \quad (2.17)$$

We compute that

$$\liminf_{n\to\infty} p(n) = \frac{1}{6} > \frac{2^2}{3^3} = \frac{4}{27}.$$

Thus, Theorem 2.2.1 (i) holds, i.e., every solution of (2.17) *oscillates. On the other hand, we compute that*

$$\sum_{j=n-2}^{n} p(j) = \begin{Bmatrix} \frac{2}{3}, & 0 \equiv n \pmod 2 \\ \frac{5}{6}, & 1 \equiv n \pmod 2 \end{Bmatrix} \not\geq 1 \quad \textit{for } n = 0, 1, \cdots .$$

This shows that the condition of Theorem 2.2.2 ***cannot*** *hold for any increasing sequence* $\{n_k\}$.

Example 6. *Consider the equation*

$$x(n+1) - x(n) + \begin{Bmatrix} \frac{1}{2}, & 0 \equiv n \pmod 2 \\ \frac{1}{10}, & 1 \equiv n \pmod 2 \end{Bmatrix} x(n-2) = 0 \quad \textit{for } n = 0, 1, \cdots . \quad (2.18)$$

On one hand, we compute that

$$\liminf_{n\to\infty} p(n) = \frac{1}{10} \ngtr \frac{2^2}{3^3} = \frac{4}{27}.$$

*Thus, Theorem 2.2.1 (i) does **not** hold. On the other hand, we consider*

$$\sum_{j=n-2}^{n} p(j) = \begin{cases} \frac{11}{10}, & 0 \equiv n \pmod 2 \\ \frac{7}{10}, & 1 \equiv n \pmod 2 \end{cases} \quad \text{for } n = 0, 1, \cdots. \tag{2.19}$$

By taking $n_k = 2k$ for $k \in \mathbb{N}$, we see that (2.19) *is equal to $\frac{11}{10}$, which is greater than* 1*. Thus, by Theorem 2.2.2, every solution of* (2.18) *oscillates.*

Theorem 2.2.1 (i) is improved by G. Ladas, Ch. G. Philos and Y. G. Sficas in 1989 by replacing the point-wise condition with the mean of consecutive τ-terms.

Theorem 2.2.3 ((Ladas et al., 1989b, Theorem 1))**.** *Assume that*

$$\liminf_{n\to\infty} \sum_{j=n-\tau}^{n-1} p(j) > \left(\frac{\tau}{\tau+1}\right)^{\tau+1}. \tag{2.20}$$

Then, every solution of (1.3) *oscillates.*

Proof of Theorem 2.2.3. Assume to the contrary that $\{x(n)\}$ is a nonoscillatory solution of (1.3). Assume that $\{x(n)\}$ is eventually positive, i.e., $x(n), x(n-\tau) > 0$ for $n \geq N_1$, where $N_1 \in \mathbb{N}$ is sufficiently large. It follows from (1.3) that

$$x(n+1) - x(n) + w(n)p(n)x(n) = 0, \quad \text{where} \quad w(n) := \frac{x(n-\tau)}{x(n)},$$

for $n \geq N_1$. It follows that

$$\begin{aligned} w(n) &= \frac{1}{\prod_{j=n-\tau}^{n-1}[1-w(j)p(j)]} \\ &\geq \frac{1}{\left(1-\frac{1}{\tau}\sum_{j=n-\tau}^{n-1}w(j)p(j)\right)^{\tau}} \\ &\geq \frac{1}{\left(1-\frac{z(n)}{\tau}\sum_{j=n-\tau}^{n-1}p(j)\right)^{\tau}} \\ &= \frac{1}{\frac{z(n)}{\tau}\sum_{j=n-\tau}^{n-1}p(j)\left(1-\frac{z(n)}{\tau}\sum_{j=n-\tau}^{n-1}p(j)\right)^{\tau}}\frac{z(n)}{\tau}\sum_{j=n-\tau}^{n-1}p(j) \\ &\geq \left(\frac{\tau+1}{\tau}\right)^{\tau+1}\sum_{j=n-\tau}^{n-1}p(j)z(n), \end{aligned}$$

where $z(n) := \min_{n-\tau\leq j\leq n-1}\{w(j)\}$ (here, we have used the fact that $\max_{h\in[0,1]}\{h(1-h)^r\} \leq \frac{r^r}{(r+1)^{r+1}}$ for $r > 0$) and $n \geq N_1$. By Lemma 2.1.1, we see that w_* is a positive number, where $w_* := \liminf_{n\to\infty} w(n)$. Note that $\liminf_{n\to\infty} z(n) = w_*$. Then, we obtain

$$w_* \geq \liminf_{n\to\infty}\left\{\left(\frac{\tau+1}{\tau}\right)^{\tau+1}\right\}\sum_{j=n-\tau}^{n-1}p(j)w_*$$

or equivalently

$$\liminf_{n\to\infty}\sum_{j=n-\tau}^{n-1}p(j) \leq \left(\frac{\tau}{\tau+1}\right)^{\tau+1},$$

which is a contradiction. □

Remark 2. *Let us justify that Theorem 2.2.3 improves Theorem 2.2.1 (i). We estimate that*

$$\begin{aligned} \liminf_{n\to\infty}\frac{1}{\tau}\sum_{j=n-\tau}^{n-1}p(j) &= \liminf_{n\to\infty}\frac{1}{\tau}\sum_{k=1}^{\tau}p(n-k) \\ &\geq \frac{1}{\tau}\sum_{k=1}^{\tau}\liminf_{n\to\infty}p(n-k) \\ &= \frac{1}{\tau}\sum_{k=1}^{\tau}\liminf_{n\to\infty}p(n) \\ &= \liminf_{n\to\infty}p(n). \end{aligned}$$

This proves that (2.20) *improves* (2.10).

Next, we give an example, where Theorem 2.2.1 and Theorem 2.2.2 fail to apply but Theorem 2.2.3 does.

Example 7. *Consider the equation*

$$x(n+1) - x(n) + \left\{ \begin{matrix} \frac{1}{8}, & 0 \equiv n \,(\mathrm{mod}\, 2) \\ \frac{1}{4}, & 1 \equiv n \,(\mathrm{mod}\, 2) \end{matrix} \right\} x(n-2) = 0 \quad \textit{for } n = 0, 1, \cdots . \tag{2.21}$$

We compute that

$$\liminf_{n\to\infty} p(n) = \frac{1}{8} \not> \frac{2^2}{3^3} = \frac{4}{27}.$$

Thus, Theorem 2.2.1 (i) does ***not*** *hold. Simply, we have*

$$\sum_{j=n-2}^{n} p(j) = \left\{ \begin{matrix} \frac{1}{2}, & 0 \equiv n \,(\mathrm{mod}\, 2) \\ \frac{5}{8}, & 1 \equiv n \,(\mathrm{mod}\, 2) \end{matrix} \right\} \not\geq 1 \quad \textit{for } n = 0, 1, \cdots .$$

That is, the condition of Theorem 2.2.2 ***cannot*** *hold for any increasing sequence* $\{n_k\}$*. Finally, we compute*

$$\sum_{j=n-2}^{n-1} p(j) = \left\{ \begin{matrix} \frac{3}{8}, & 0 \equiv n \,(\mathrm{mod}\, 2) \\ \frac{3}{8}, & 1 \equiv n \,(\mathrm{mod}\, 2) \end{matrix} \right\} \equiv \frac{3}{8} \quad \textit{for } n = 0, 1, \cdots ,$$

which yields

$$\liminf_{n\to\infty} \sum_{j=n-2}^{n-1} p(j) = \frac{3}{8} > \left(\frac{2}{3}\right)^3 = \frac{8}{27}.$$

Therefore, by Theorem 2.2.3, every solution of (2.21) *oscillates.*

Next, J. S. Yu, B. G. Zhang and Z. C. Wang in 1994 explored a very important approach, which improves the above result by replacing the sum with a product. Their approach also allowed to prove a new nonoscillation test, which improves Theorem 2.2.1 (ii).

Theorem 2.2.4 ((Yu et al., 1994, Theorem 1))**.** *(i) Assume that*

$$\liminf_{n\to\infty} \inf_{\lambda\in\Lambda} \left\{ \frac{1}{\lambda \prod_{j=n-\tau}^{n-1} [1 - \lambda p(j)]} \right\} > 1, \tag{2.22}$$

where

$$\Lambda := \{\lambda > 0 : \ 1 - \lambda p(n) > 0 \quad \textit{for all large } n\}. \tag{2.23}$$

Then, every solution of (1.3) *oscillates.*

(ii) Assume that there exists $\lambda_0 \in \Lambda$ *such that*

$$\frac{1}{\lambda_0 \prod_{j=n-\tau}^{n-1}[1 - \lambda_0 p(j)]} \leq 1 \quad \textit{for all large } n. \tag{2.24}$$

Then, (1.3) *has an eventually positive solution.*

Proof. (i) Assume to the contrary that $\{x(n)\}$ is a nonoscillatory solution of (1.3). Assume that $x(n), x(n-\tau) > 0$ for $n \geq N_1$, where $N_1 \in \mathbb{N}$ is sufficiently large. It follows from (1.3) that

$$x(n+1) - x(n) + w(n)p(n)x(n) = 0, \quad \text{where} \quad w(n) := \frac{x(n-\tau)}{x(n)},$$

for $n \geq N_1$. It follows that

$$\begin{aligned} w(n) &= \frac{1}{\prod_{j=n-\tau}^{n-1}[1 - w(j)p(j)]} \\ &\geq \frac{1}{\prod_{j=n-\tau}^{n-1}[1 - z(n)p(j)]} \\ &= \frac{1}{z(n)\prod_{j=n-\tau}^{n-1}[1 - z(n)p(j)]} z(n) \\ &\geq \inf_{\lambda \in \Lambda} \left\{ \frac{1}{\lambda \prod_{j=n-\tau}^{n-1}[1 - \lambda p(j)]} \right\} z(n), \end{aligned} \tag{2.25}$$

where $z(n) := \min_{n-\tau \leq j \leq n-1}\{w(j)\}$ and $n \geq N_1$. One can show that (2.22) implies (2.1). Indeed, if (2.1) fails, then $\lim_{n\to\infty} p^*(n) = 0$, where $p^*(n) := \max_{n-\tau \leq j \leq n-1}\{p(j)\}$. Thus,

$$\begin{aligned} \liminf_{n\to\infty} \inf_{\lambda \in \Lambda} \left\{ \frac{1}{\lambda \prod_{j=n-\tau}^{n-1}[1 - \lambda p(j)]} \right\} &\leq \liminf_{n\to\infty} \inf_{\lambda \in \Lambda} \left\{ \frac{1}{\lambda \prod_{j=n-\tau}^{n-1}[1 - \lambda p^*(n)]} \right\} \\ &= \lim_{n\to\infty} \frac{(\tau+1)^{\tau+1}}{\tau^\tau} p^*(n) = 0, \end{aligned}$$

which contradicts (2.22). It follows from Lemma 2.1.1 that w_* is a positive number, where $w_* := \liminf_{n\to\infty} w(n)$. Note that $\liminf_{n\to\infty} z(n) = w_*$. Then, from (2.25), we obtain

$$w_* \geq \liminf_{n\to\infty} \inf_{\lambda\in\Lambda} \left\{ \frac{1}{\lambda \prod_{j=n-\tau}^{n-1}[1-\lambda p(j)]} \right\} w_*,$$

or equivalently

$$\liminf_{n\to\infty} \inf_{\lambda\in\Lambda} \left\{ \frac{1}{\lambda \prod_{j=n-\tau}^{n-1}[1-\lambda p(j)]} \right\} \leq 1,$$

which is a contradiction.

(ii) By (2.24), we choose a positive integer N_1 such that $N_1 \geq \tau$ and

$$\lambda_0 \prod_{j=n-\tau}^{n-1} [1-\lambda_0 p(j)] \geq 1, \quad n \geq N_1.$$

Define

$$y(n) := \begin{cases} 1, & N_1 - \tau \leq n < N_1 \\ \dfrac{1}{\lambda_0 \prod_{j=n-\tau}^{n-1}[1-\lambda_0 y(j) p(j)]}, & n \geq N_1. \end{cases}$$

Then,

$$y(N_1) = \frac{1}{\lambda_0 \prod_{j=N_1-\tau}^{N_1-1}[1-\lambda_0 y(j) p(j)]} = \frac{1}{\lambda_0 \prod_{j=N_1-\tau}^{N_1-1}[1-\lambda_0 p(j)]} \leq 1.$$

In general, by induction, we obtain

$$y(n) := \frac{1}{\lambda_0 \prod_{j=n-\tau}^{n-1}[1-\lambda_0 y(j) p(j)]} \leq 1, \quad n \geq N_1.$$

Thus, $\{y(n)\}$ is defined. Also, we define

$$z(n) := 1 - \lambda_0 y(n) p(n), \quad n \geq N_1.$$

Then, $z(n) > 0$ for $n \geq N_1 - \tau$ and

$$z(n) = 1 - \frac{p(n)}{\prod_{j=n-\tau}^{n-1} z(j)}, \quad n \geq N_1. \tag{2.26}$$

Define

$$x(n) := \begin{cases} 1, & N_1 - \tau \le n < N_1 \\ \displaystyle\prod_{j=N_1-1}^{n-1} z(j), & n \ge N_1. \end{cases}$$

Then we have by (2.26)

$$\frac{x(n+1)}{x(n)} - 1 + p(n)\frac{x(n-\tau)}{x(n)} = 0.$$

That is,

$$x(n+1) - x(n) + p(n)x(n-\tau) = 0.$$

Thus, we obtain a positive solution $\{x(n)\}$ of equation (1.3). □

Remark 3. *Let us show that Theorem 2.2.4 (i) improves Theorem 2.2.3. We estimate by using the inequality of arithmetic-geometric means that*

$$\begin{aligned}\frac{1}{\lambda\prod_{j=n-\tau}^{n-1}[1-\lambda p(j)]} &\ge \frac{1}{\lambda\Big(1-\frac{\lambda}{\tau}\sum_{j=n-\tau}^{n-1}p(j)\Big)^{\tau}} \\ &\ge \frac{1}{\lambda\Big(1-\frac{\lambda}{\tau}\sum_{j=n-\tau}^{n-1}p(j)\Big)^{\tau}}\Bigg|_{\lambda\to\frac{1}{\frac{\tau+1}{\tau}\sum_{j=n-\tau}^{n-1}p(j)}} \\ &= \left(\frac{\tau+1}{\tau}\right)^{\tau+1}\sum_{j=n-\tau}^{n-1}p(j)\end{aligned}$$

for $\lambda \in \Lambda$ and all large n. This proves that (2.22) *improves* (2.20).

On the other hand, Theorem 2.2.4 (ii) also improves Theorem 2.2.1 (ii). For justification, suppose that $p(n) \le \frac{\tau^\tau}{(\tau+1)^{\tau+1}}$ for all large n. We can find $M \in \mathbb{R}^+$ such that $M \le \frac{\tau^\tau}{(\tau+1)^{\tau+1}}$, $p(n) \le M$ for all large n, and $1-\lambda_0 M > 0$, where $\lambda_0 := \frac{1}{(\tau+1)M}$, i.e., $\lambda_0 \in \Lambda$. It follows that

$$\lambda_0\prod_{i=n-\tau}^{n-1}[1-\lambda_0 p(i)] \ge \lambda_0\prod_{i=n-\tau}^{n-1}[1-\lambda_0 M] = \frac{\tau^\tau}{(\tau+1)^{\tau+1}}\frac{1}{M} \ge 1 \quad \textit{for all large } n.$$

This proves that (2.24) *improves* (2.11).

Example 8. *Consider the equation*

$$x(n+1)-x(n)+\begin{Bmatrix}\frac{1}{4}, & 0\equiv n\,(\text{mod}\,2)\\ \frac{1}{28}, & 1\equiv n\,(\text{mod}\,2)\end{Bmatrix}x(n-2)=0 \quad \textit{for } n=0,1,\cdots. \tag{2.27}$$

We compute that

$$\sum_{j=n-2}^{n-1} p(j)=\begin{Bmatrix}\frac{2}{7}, & 0\equiv n\,(\text{mod}\,2)\\ \frac{2}{7}, & 1\equiv n\,(\text{mod}\,2)\end{Bmatrix}\equiv\frac{2}{7} \quad \textit{for } n=0,1,\cdots,$$

which yields

$$\liminf_{n\to\infty}\sum_{j=n-2}^{n-1} p(j)=\frac{2}{7}\not>\left(\frac{2}{3}\right)^3.$$

Thus, Theorem 2.2.3 fails. On the other hand, we compute that $\Lambda=(0,4)$ *and*

$$\frac{1}{\lambda\prod_{j=n-2}^{n-1}[1-\lambda p(j)]}=\begin{Bmatrix}\frac{1}{\lambda(1-\frac{\lambda}{28})(1-\frac{\lambda}{4})}, & 0\equiv n\,(\text{mod}\,2)\\ \frac{1}{\lambda(1-\frac{\lambda}{4})(1-\frac{\lambda}{28})}, & 1\equiv n\,(\text{mod}\,2)\end{Bmatrix}$$
$$=\frac{1}{\lambda\big(1-\frac{\lambda}{28}\big)\big(1-\frac{\lambda}{4}\big)}$$

for $n=0,1,\cdots$. *Thus, we see that*

$$\inf_{\lambda\in(0,4)}\left\{\frac{1}{\lambda\big(1-\frac{\lambda}{28}\big)\big(1-\frac{\lambda}{4}\big)}\right\}=\frac{1}{\lambda\big(1-\frac{\lambda}{28}\big)\big(1-\frac{\lambda}{4}\big)}\bigg|_{\lambda\to\frac{4}{3}(8-\sqrt{43})}$$
$$=\frac{1}{504}\big(260+43\sqrt{43}\big)>1$$

for $n=0,1,\cdots$. *That is, by Theorem 2.2.4 (i), every solution of* (2.27) *oscillates.*

Finally, we would like to quote the following results from Karpuz (2017). We will be confine our attention on the oscillation part of this recent result.

Theorem 2.2.5 ((Karpuz, 2017, Theorems 1 and 2)). *(i) Assume that*

$$\liminf_{n\to\infty}\inf_{\lambda\geq 1}\left\{\frac{1}{\lambda}\prod_{j=n-\tau}^{n}\big[1+\lambda p(j)\big]\right\}>1. \tag{2.28}$$

Then, every solution of (1.3) *oscillates.*

(ii) Assume that there exists $\lambda_0 \geq 1$ *such that*

$$\frac{1}{\lambda_0} \prod_{j=n-\tau}^{n} \big[1 + \lambda_0 p(j)\big] \leq 1 \quad \textit{for all large } n. \tag{2.29}$$

Then, (1.3) *has an eventually positive solution.*

Now, we present the proof of Theorem 2.2.5.

Proof of Theorem 2.2.5. (i) Assume to the contrary that $\{x(n)\}$ is a nonoscillatory solution of (1.3). Assume that $x(n), x(n-\tau) > 0$ for $n \geq N_1$, where $N_1 \in \mathbb{N}$ is sufficiently large. By Lemma 3.1.1, (2.9) **cannot** hold. So, we have to assume (2.8). Let $w(n) := \frac{x(n-\tau)}{x(n+1)}$ for $n \geq N_1$. Now, we claim that

$$1 \leq \ell := \liminf_{n\to\infty} w(n) < \infty. \tag{2.30}$$

Let $N_2 \in \mathbb{N}$ satisfy $N_2 \geq N_1 + 2\tau$. From (1.3), for $n \geq N_2$, we have

$$\begin{aligned} x(n+1) > x(n+1) - x(l-1) &= -\sum_{j=n+1}^{l} [x(j+1) - x(j)] \\ &= \sum_{j=n+1}^{l} p(j)x(j-\tau) \geq \Bigg(\sum_{j=n+1}^{l} p(j)\Bigg) x(l-\tau) \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} x(l-\tau) > x(l-\tau) - x(n+1) &= -\sum_{j=l-\tau}^{n} [x(j+1) - x(j)] \\ &= \sum_{j=l-\tau}^{n} p(j)x(j-\tau) \geq \Bigg(\sum_{j=l-\tau}^{n} p(j)\Bigg) x(n-\tau), \end{aligned} \tag{2.32}$$

where l satisfies $l - \tau \leq n + 1 \leq l$. Combining (2.31) and (2.32), we get

$$1 \leq w(n) < \Bigg[\Bigg(\sum_{j=l-\tau}^{n} p(j)\Bigg)\Bigg(\sum_{j=n+1}^{l} p(j)\Bigg)\Bigg]^{-1} \text{for } n \geq N_2, \tag{2.33}$$

Considering (2.8), we take inferior limit as $n \to \infty$ in (2.33) after taking minimum over l to obtain (2.30). On the other hand, from (1.3), we get

$$[1 + w(n)p(n)]x(n+1) - x(n) = 0 \quad \text{for } n \geq N_2$$

or equivalently

$$w(n) = \prod_{j=n-\tau}^{n} \frac{x(j)}{x(j+1)} = \prod_{j=n-\tau}^{n} [1 + w(j)p(j)] \quad \text{for } n \geq N_2.$$

Thus, we have

$$w(n) \geq \prod_{j=n-\tau}^{n} [1 + w_*(n)p(j)] \quad \text{for } n \geq N_2, \tag{2.34}$$

where

$$w_*(n) := \min_{n-\tau \leq j \leq n} w(j) \geq 1 \quad \text{for } n \geq N_2.$$

Clearly, $\liminf_{n\to\infty} w_*(n) = \ell$. Taking inferior limit of both sides of (2.34), we get

$$\ell \geq \liminf_{n\to\infty} \prod_{j=n-\tau}^{n} [1 + \ell p(j)]$$

or equivalently

$$\liminf_{n\to\infty} \left(\frac{1}{\ell} \prod_{j=n-\tau}^{n} [1 + \ell p(j)] \right) \leq 1,$$

which contradicts (2.28) since $\ell \geq 1$. Therefore, the proof is complete.

(ii) Assume that (2.29) holds for $n \geq N_1$, where $N_1 \in \mathbb{N}$. Note that, (2.29) implies

$$\begin{aligned} 1 - p(n) \prod_{j=n-\tau}^{n-1} [1 + \lambda_0 p(j)] &= 1 - \frac{p(n)}{1 + \lambda_0 p(n)} \prod_{j=n-\tau}^{n} [1 + \lambda_0 p(j)] \\ &\geq 1 - \frac{\lambda_0 p(n)}{1 + \lambda_0 p(n)} = \frac{1}{1 + \lambda_0 p(n)} > 0 \end{aligned}$$

for $n \geq N_1$. Further, by (2.29), we have $\frac{1}{\lambda_0}[1 + \lambda_0 p(j)] \leq 1$ for $n \geq N_1$, which yields

$1 - p(n) \geq \frac{1}{\lambda_0} > 0$ for $n \geq N_1$, i.e., $0 \leq p(n) < 1$ for $n \geq N_1$. Now, we define

$$y(n) := \begin{cases} 1, & N_1 - \tau \leq n < N_1 \\ \dfrac{\prod_{j=n-\tau}^{n-1}[1 + \lambda_0 y(j)p(j)]}{\lambda_0\big(1 - p(n)\prod_{j=n-\tau}^{n-1}[1 + \lambda_0 y(j)p(j)]\big)}, & n \geq N_1. \end{cases} \tag{2.35}$$

First, we claim that $y(n) > 0$ for $n \geq N_1$. Assume the contrary that, $y(l) \leq 0$ for some integer $l \geq N_1$. Without loss of generality, we may assume that $y(n) > 0$ for $N_1 - \tau \leq n < l$. Then, we have

$$y(l) \geq \frac{\prod_{j=l-\tau}^{l-1}[1 + \lambda_0 y(j)p(j)]}{\lambda_0(1 - p(l))} > 0,$$

which is a contradiction. Thus, $y(n) > 0$ for $n \geq N_1 - \tau$. Next, we claim that, $y(n) \leq 1$ for $n \geq N_1$. Assume to the contrary that, $y(l) > 1$ for some integer $l \geq N_1$. Without loss of generality, we may assume that $y(n) \leq 1$ for $N_1 - \tau \leq n < l$. Then, we have

$$\begin{aligned} y(l) &\leq \frac{\prod_{j=l-\tau}^{l-1}[1 + \lambda_0 p(j)]}{\lambda_0\big(1 - p(l)\prod_{j=l-\tau}^{l-1}[1 + \lambda_0 p(j)]\big)} \\ &\leq \frac{\frac{\lambda_0}{1+\lambda_0 p(l)}}{\lambda_0\Big(1 - \frac{\lambda_0 p(l)}{1+\lambda_0 p(l)}\Big)} = 1, \end{aligned}$$

which is a contradiction. Thus, $y(n) \leq 1$ for $n \geq N_1 - \tau$. From (2.35), we have

$$y(n) = \frac{1}{\lambda_0} \prod_{j=n-\tau}^{n} [1 + \lambda_0 y(j)p(j)] \quad \text{for } n \geq N_1. \tag{2.36}$$

Finally, we define

$$x(n) := \prod_{j=N_1-\tau}^{n-1} \frac{1}{1 + \lambda_0 y(j)p(j)} \quad \text{for } n \geq N_1 - \tau. \tag{2.37}$$

We iterate (1.3) in the backwards direction to define $x(n)$ for $-\tau \leq n < N_1 - \tau$, i.e.,

$$x(n) := \begin{cases} -\dfrac{x(n+\tau+1) - x(n+\tau)}{p(n+\tau)}, & p(n+\tau) \neq 0 \text{ and } N_1 - \tau < n \leq -\tau \\ 1, & p(n+\tau) = 0 \text{ and } n = N_1 - \tau < n \leq -\tau. \end{cases}$$

Clearly, $0 < x(n) \leq 1$ for $n \geq N_1 - \tau$. By (2.36) and (2.37), we see that

$$\begin{aligned} 0 &= 1 - [1 + \lambda_0 y(n)p(n)] + \lambda_0 y(n)p(n) \\ &= 1 - \frac{x(n)}{x(n+1)} + p(n)\frac{x(n-\tau)}{x(n+1)} \end{aligned}$$

for $n \geq N_1$. Thus, $x(n)$ is eventually positive and satisfies (1.3). □

Remark 4. *Suppose that* (2.28) *holds for* (2.15)*, i.e.,*

$$\frac{1}{\lambda}\prod_{j=n-\tau}^{n}[1+\lambda p] \equiv \frac{1}{\lambda}(1+\lambda p)^{\tau+1} > 1 \quad \textit{for all } \lambda \geq 1. \tag{2.38}$$

Note that (2.38) *trivially holds for* $1 > \lambda > 0$*, then*

$$\begin{aligned} & \frac{1}{\lambda}(1+\lambda p)^{\tau+1} > 1 \quad \textit{for all } \lambda > 0 \\ \iff & (1+\lambda p)^{\tau+1} > \lambda \quad \textit{for all } \lambda > 0 \\ \iff & -\lambda + (1+\lambda p)^{\tau+1} > 0 \quad \textit{for all } \lambda > 0 \\ \iff & -\frac{p\lambda}{1+\lambda p} + p(1+\lambda p)^{\tau} > 0 \quad \textit{for all } \lambda > 0 \\ \iff & \frac{1}{1+\lambda p} - 1 + p(1+\lambda p)^{\tau} > 0 \quad \textit{for all } \lambda > 0 \\ \iff & \mu - 1 + p\mu^{-\tau} > 0 \quad \textit{for all } 1 > \mu := \frac{1}{1+\lambda p} > 0, \end{aligned}$$

i.e., the characteristic equation (2.16) *has no roots when* $1 > \mu > 0$*. Further, the characteristic equation* (2.16) *has no roots in the case* $\mu \geq 1$ *either since* $p \in \mathbb{R}^{+}$*. As a result, the characteristic equation* (2.16) *has no positive roots. Therefore, Theorem 2.2.5 (i) extends (Ladas et al., 1989a, Theorem 1) to equations with a variable coefficient.*

Remark 5. *Theorem 2.2.5 (ii) improves Theorem 2.2.1 (ii). To show this, suppose that*

$p(n) \leq \frac{\tau^\tau}{(\tau+1)^{\tau+1}}$ *for all large* n*. Then, we estimate that*

$$\begin{aligned}\frac{1}{\lambda_0}\prod_{j=n-\tau}^{n}[1+\lambda_0 p(j)] &\leq \frac{1}{\lambda_0}\left(1+\frac{\lambda_0}{\tau+1}\sum_{j=n-\tau}^{n}p(j)\right)^{\tau+1}\\ &\leq \frac{1}{\lambda_0}\left(1+\frac{\lambda_0}{\tau+1}\sum_{j=n-\tau}^{n}p(j)\right)^{\tau+1}\Bigg|_{\lambda_0\to\frac{\tau+1}{\tau}\frac{1}{\sum_{j=n-\tau}^{n}p(j)}}\\ &= \left(\frac{\tau+1}{\tau}\right)^{\tau}\sum_{j=n-\tau}^{n}p(j)\\ &\leq \left(\frac{\tau+1}{\tau}\right)^{\tau}\sum_{j=n-\tau}^{n}\frac{\tau^\tau}{(\tau+1)^{\tau+1}}=1\end{aligned}$$

for all large n*. Thus,* (2.11) *implies* (2.29).

From the proof in Remark 5, we can give the following corollary of Theorem 2.2.5 (ii).

Corollary 2.2.5.1. *Assume that*

$$\sum_{j=n-\tau}^{n}p(j) \leq \left(\frac{\tau}{\tau+1}\right)^{\tau} \quad \textit{for all large } n.$$

Then, Eq. (1.3) *has a nonoscillatory solution.*

Below, we give four examples to illustrate that Theorem 2.2.5 (i) and Theorem 2.2.5 (ii) **cannot** be compared with Theorem 2.2.4 (i) and Theorem 2.2.4 (ii), respectively. The following example includes a numerical equation, where Theorem 2.2.4 (i) applies but Theorem 2.2.5 (i) fails.

Example 9. *Consider the equation*

$$x(n+1)-x(n)+\left\{\begin{array}{ll}\frac{17}{64}, & 0\equiv n \pmod 2\\ \frac{5}{64}, & 1\equiv n \pmod 2\end{array}\right\}x(n-2)=0 \quad \textit{for } n=0,1,\cdots. \tag{2.39}$$

We compute that

$$\begin{aligned}
&\liminf_{n\to\infty}\inf_{\lambda\geq 1}\left\{\frac{1}{\lambda}\prod_{j=n-2}^{n}[1+\lambda p(j)]\right\}\\
&=\liminf_{n\to\infty}\inf_{\lambda\geq 1}\begin{cases}\frac{1}{\lambda}\left(1+\lambda\frac{17}{64}\right)^2\left(1+\lambda\frac{5}{64}\right), & 0\equiv n \pmod 2\\ \frac{1}{\lambda}\left(1+\lambda\frac{17}{64}\right)\left(1+\lambda\frac{5}{64}\right)^2, & 1\equiv n \pmod 2\end{cases}\\
&=\liminf_{n\to\infty}\begin{cases}\frac{1}{\lambda}\left(1+\lambda\frac{17}{64}\right)^2\left(1+\lambda\frac{5}{64}\right)\Big|_{\lambda\to\frac{16}{85}(\sqrt{969}-17)}, & 0\equiv n \pmod 2\\ \frac{1}{\lambda}\left(1+\lambda\frac{17}{64}\right)\left(1+\lambda\frac{5}{64}\right)^2\Big|_{\lambda\to\frac{16}{85}(705-5)}, & 1\equiv n \pmod 2\end{cases}\\
&=\liminf_{n\to\infty}\begin{cases}\frac{3}{2560}\left(537+19\sqrt{969}\right), & 0\equiv n \pmod 2\\ \frac{3}{8704}\left(1329+47\sqrt{705}\right), & 1\equiv n \pmod 2\end{cases}\\
&\approx\liminf_{n\to\infty}\begin{cases}1.3224, & 0\equiv n \pmod 2\\ 0.88819, & 1\equiv n \pmod 2\end{cases}=0.88819\not>1.
\end{aligned}$$

Thus, Theorem 2.2.5 (i) does ***not*** *apply for Eq.* (2.39). *On the other hand, we have* $\Lambda:=(0,\frac{64}{17})$ *and compute that*

$$\begin{aligned}
\liminf_{n\to\infty}\inf_{\lambda\in\Lambda}\left\{\frac{1}{\lambda\prod_{j=n-2}^{n-1}[1-\lambda p(j)]}\right\}&=\liminf_{n\to\infty}\inf_{0<\lambda<\frac{64}{17}}\left\{\frac{1}{\lambda\left(1-\lambda\frac{17}{64}\right)\left(1-\lambda\frac{5}{64}\right)}\right\}\\
&=\frac{1}{\lambda\left(1-\lambda\frac{17}{64}\right)\left(1-\lambda\frac{5}{64}\right)}\Bigg|_{\lambda\to\frac{64}{255}(22-\sqrt{229})}\\
&=\frac{195075}{128\left(229\sqrt{229}-2233\right)}\approx 1.23>1.
\end{aligned}$$

This shows that Theorem 2.2.4 (i) applies for Eq. (2.39)*, i.e., every solution of Eq.* (2.39) *is oscillatory.*

The following example includes a numerical equation, where Theorem 2.2.5 (i) applies but Theorem 2.2.4 (i) fails.

Example 10. *Consider the equation*

$$x(n+1)-x(n)+\begin{Bmatrix} \dfrac{35}{128}, & 0 \equiv n \pmod 2 \\ \dfrac{61}{265}, & 1 \equiv n \pmod 2 \end{Bmatrix} x(n-1)=0 \quad \textit{for}\, n=0,1,\cdots. \tag{2.40}$$

Note that $\Lambda := (0, \frac{128}{35})$ *and we compute*

$$\begin{aligned}
&\liminf_{n\to\infty} \inf_{\lambda\in\Lambda} \left\{ \frac{1}{\lambda \prod_{j=n-1}^{n-1}[1-\lambda p(j)]} \right\} \\
&= \liminf_{n\to\infty} \inf_{0<\lambda<\frac{128}{35}} \begin{Bmatrix} \dfrac{1}{\lambda(1-\frac{61}{265}\lambda)}, & 0 \equiv n \pmod 2 \\ \dfrac{1}{\lambda(1-\frac{35}{128}\lambda)}, & 1 \equiv n \pmod 2 \end{Bmatrix} \\
&= \liminf_{n\to\infty} \begin{Bmatrix} \left.\dfrac{1}{\lambda(1-\frac{61}{265}\lambda)}\right|_{\lambda\to\frac{128}{61}}, 0 \equiv n \pmod 2 \\ \left.\dfrac{1}{\lambda(1-\frac{35}{128}\lambda)}\right|_{\lambda\to\frac{64}{35}}, 1 \equiv n \pmod 2 \end{Bmatrix} \\
&= \liminf_{n\to\infty} \begin{Bmatrix} \dfrac{61}{64}, & 0 \equiv n \pmod 2 \\ \dfrac{35}{32}, & 1 \equiv n \pmod 2 \end{Bmatrix} = \frac{61}{64} \not> 1
\end{aligned}$$

Thus, Theorem 2.2.4 (i) does ***not*** *apply for Eq.* (2.40)*. On the other hand, the estimation*

$$\begin{aligned}
\liminf_{n\to\infty} \inf_{\lambda\geq 1} \left\{ \frac{1}{\lambda} \prod_{j=n-1}^{n} [1+\lambda p(j)] \right\} &= \liminf_{n\to\infty} \inf_{\lambda\geq 1} \left\{ \frac{1}{\lambda}\left(1+\frac{61}{265}\lambda\right)\left(1+\frac{35}{128}\lambda\right) \right\} \\
&= \frac{1}{\lambda}\left(1+\frac{61}{265}\lambda\right)\left(1+\frac{35}{128}\lambda\right)\bigg|_{\lambda\to 12\sqrt{\frac{2}{2135}}} \\
&= \frac{1}{256}\left(131+2\sqrt{4270}\right) \approx 1.02223 > 1
\end{aligned}$$

showing that Theorem 2.2.5 (i) applies for Eq. (2.40)*, i.e., every solution of Eq.* (2.40) *is oscillatory.*

The following example includes a numerical equation, where Theorem 2.2.4 (ii) applies but Theorem 2.2.5 (ii) fails.

Example 11. *Consider the equation*

$$x(n+1)-x(n)+\begin{Bmatrix} \dfrac{57}{256}, & 0\equiv n \pmod 2 \\ \dfrac{5}{128}, & 1\equiv n \pmod 2 \end{Bmatrix} x(n-2)=0 \quad \textit{for}\, n=0,1,\cdots. \tag{2.41}$$

We compute that

$$\begin{aligned}
\frac{1}{\lambda}\prod_{j=n-2}^{n}[1+\lambda p(j)] &= \begin{Bmatrix} \dfrac{1}{\lambda}\left(1+\dfrac{5}{128}\lambda\right)\left(1+\dfrac{57}{256}\lambda\right)^2, & 0\equiv n \pmod 2 \\ \dfrac{1}{\lambda}\left(1+\dfrac{5}{128}\lambda\right)^2\left(1+\dfrac{57}{256}\lambda\right), & 1\equiv n \pmod 2 \end{Bmatrix} \\
&\geq \begin{Bmatrix} \dfrac{1}{\lambda}\left(1+\dfrac{5}{128}\lambda\right)\left(1+\dfrac{57}{256}\lambda\right)^2\Bigg|_{\lambda\to\frac{32}{285}(\sqrt{7809}-57)}, & 0\equiv n \pmod 2 \\ \dfrac{1}{\lambda}\left(1+\dfrac{5}{128}\lambda\right)^2\left(1+\dfrac{57}{256}\lambda\right)\Bigg|_{\lambda\to\frac{64}{285}(\sqrt{1165}-5)}, & 1\equiv n \pmod 2 \end{Bmatrix} \\
&= \begin{Bmatrix} \dfrac{8951+137\sqrt{7809}}{20480}, & 0\equiv n \pmod 2 \\ \dfrac{9323+233\sqrt{1165}}{29184}, & 1\equiv n \pmod 2 \end{Bmatrix} \\
&\approx \begin{Bmatrix} 1.0282, & 0\equiv n \pmod 2 \\ 0.591961, & 1\equiv n \pmod 2 \end{Bmatrix} \not\leq 1
\end{aligned}$$

for all $\lambda \geq 1$. *That is, Theorem 2.2.5 (ii) does **not** apply for Eq.* (2.41). *On the other hand, we see that* $\Lambda := (0, \frac{256}{57})$, *and with* $\lambda_0 := \frac{17}{8}$, *we have*

$$\frac{1}{\lambda_0\prod_{j=n-\tau}^{n-1}[1-\lambda_0 p(j)]} = \frac{1}{\frac{17}{8}\left(1-\frac{17}{8}\frac{57}{256}\right)\left(1-\frac{17}{8}\frac{5}{128}\right)} = \frac{16777216}{17224077} \leq 1$$

for $n=0,1,\cdots$. *By Theorem 2.2.4 (ii), Eq.* (2.41) *has a nonoscillatory solution.*

The following example includes a numerical equation, where Theorem 2.2.5 (ii) applies but Theorem 2.2.4 (ii) fails.

Example 12. *Consider the equation*

$$x(n+1)-x(n)+\left\{\begin{matrix} \dfrac{17}{64}, & 0\equiv n \pmod 3 \\ \dfrac{5}{64}, & 0\not\equiv n \pmod 3 \end{matrix}\right\} x(n-2)=0 \quad \textit{for } n=0,1,\cdots. \tag{2.42}$$

Clearly, $\Lambda := (0, \frac{64}{17})$ *and*

$$\begin{aligned}
\frac{1}{\lambda\prod_{j=n-2}^{n-1}[1-\lambda p(j)]} &= \left\{\begin{matrix} \dfrac{1}{\lambda\big(1-\lambda\frac{5}{64}\big)^2}, & 0\equiv n \pmod 3 \\ \dfrac{1}{\lambda\big(1-\lambda\frac{5}{64}\big)\big(1-\lambda\frac{17}{64}\big)}, & 0\not\equiv n \pmod 3 \end{matrix}\right\} \\
&\geq \left\{\begin{matrix} \left.\dfrac{1}{\lambda\big(1-\lambda\frac{5}{64}\big)^2}\right|_{\lambda\to\frac{64}{17}}, & 0\equiv n \pmod 3 \\ \left.\dfrac{1}{\lambda\big(1-\lambda\frac{5}{64}\big)\big(1-\lambda\frac{17}{64}\big)}\right|_{\lambda\to\frac{64}{255}(22-\sqrt{229})}, & 0\not\equiv n \pmod 3 \end{matrix}\right\} \\
&= \left\{\begin{matrix} \dfrac{4913}{9216}, & 0\equiv n \pmod 3 \\ \dfrac{195075}{128\big(229\sqrt{229}-2233\big)}, & 0\not\equiv n \pmod 3 \end{matrix}\right\} \\
&\approx \left\{\begin{matrix} 0.533, & 0\equiv n \pmod 3 \\ 1.237, & 0\not\equiv n \pmod 3 \end{matrix}\right\} \not\leq 1
\end{aligned}$$

for all $\lambda \geq 1$. *This shows that Theorem 2.2.4 (ii) does* ***not*** *apply for Eq.* (2.42). *However, with* $\lambda_0 := 5$*, we compute that*

$$\frac{1}{\lambda_0}\prod_{j=n-2}^{n}[1+\lambda_0 p(j)] = \frac{1}{5}\left(1+5\frac{17}{64}\right)\left(1+5\frac{5}{64}\right)^2 \approx 0.9 \leq 1 \quad \textit{for } n=0,1,\cdots.$$

Thus, Theorem 2.2.5 (ii), Eq. (2.42) *has a nonoscillatory solution.*

CHAPTER THREE
THE ORIGINAL RESULT

In this section, we state our new result on the oscillation of (1.3) and lemmas, which are required in the proof of the our main result Theorem 3.0.1. The connection between these three lemmas are interesting on their own.

Theorem 3.0.1. *Assume that there exists $\ell \in \mathbb{N}$ such that*

$$\liminf_{n\to\infty} \beta_\ell(n) > 1, \tag{3.1}$$

where

$$\beta_k(n) := \begin{cases} 1, & k = 0 \\ \inf\limits_{\lambda\geq 1}\left\{\dfrac{1}{\lambda}\prod\limits_{j=n-\tau}^{n}\left[1+\lambda\beta_{k-1}(j)p(j)\right]\right\}, & k \in \mathbb{N}. \end{cases} \tag{3.2}$$

Then, every solution of (1.3) *oscillates.*

Remark 6. *Theorem 3.0.1 with $\ell = 1$ covers Theorem 2.2.5 (i).*

3.1 Preparatory Results

Lemma 3.1.1. *If* (1.3) *has a nonoscillatory solution, then*

$$\sum_{j=n-\tau}^{n} p(j) < 1 \quad \textit{for all large } n. \tag{3.3}$$

Proof. The claim follows from Theorem 2.2.2. □

Lemma 3.1.2. *Assume*

$$\limsup_{n\to\infty} \sum_{j=n-\tau}^{n} p(j) < \infty \tag{3.4}$$

and

$$\lim_{n\to\infty}\left(p(n)\sum_{j=n-\tau}^{n-1} p(j)\right) = 0. \tag{3.5}$$

Then,

$$\limsup_{n\to\infty} \beta_k(n) \le \left(\limsup_{n\to\infty} \sum_{j=n-\tau}^{n} p(j)\right)^k \quad \textit{for } k \in \mathbb{N}. \tag{3.6}$$

Proof. It follows from (3.5) that

$$\lim_{n\to\infty} \big(p(n)p(n-j)\big) = 0 \quad \text{for } j = 1, 2, \cdots, \tau. \tag{3.7}$$

By expanding the product in (3.2), we write

$$\begin{aligned}\beta_k(n) = \inf_{\lambda\ge 1}\Bigg\{ \frac{1}{\lambda}\Bigg(1 + \lambda \sum_{j=n-\tau}^{n} \beta_{k-1}(j)p(j) + \lambda^2 \sum_{j=n-\tau}^{n-1} \beta_{k-1}(j)p(j) \\ \times \sum_{i=j+1}^{n} \beta_{k-1}(i)p(i) + \cdots + \lambda^{\tau+1} \prod_{j=n-\tau}^{n} \beta_{k-1}(j)p(j)\Bigg)\Bigg\}\end{aligned} \tag{3.8$_k$}$$

for all large n. From (3.7) and (3.8) with $k = 1$, we see that

$$\beta_1(n) = \inf_{\lambda\ge 1}\left\{ \frac{1}{\lambda} + \sum_{j=n-\tau}^{n} p(j) + o(1)(\lambda + \cdots + \lambda^\tau)\right\} \quad \text{for all large } n, \tag{3.9}$$

where o is the so-called "little-o notation" meaning that the coefficients of $\lambda, \lambda^2, \cdots, \lambda^\tau$ tend to 0 as $n \to \infty$. It follows from (3.4) and (3.9) that

$$\limsup_{n\to\infty} \beta_1(n) \le \limsup_{n\to\infty} \sum_{j=n-\tau}^{n} p(j) =: M, \tag{3.10}$$

i.e., $\{\beta_1(n)\}$ is bounded. From (3.7), (3.8) with $k = 2$ and (3.10), we see that

$$\beta_2(n) = \inf_{\lambda\ge 1}\left\{ \frac{1}{\lambda} + \sum_{j=n-\tau}^{n} \beta_1(j)p(j) + o(1)(\lambda + \cdots + \lambda^\tau)\right\} \quad \text{for all large } n. \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\limsup_{n\to\infty} \beta_2(n) \le \limsup_{n\to\infty} \sum_{j=n-\tau}^{n} \beta_1(j)p(j) \le M \limsup_{n\to\infty} \sum_{j=n-\tau}^{n} p(j) \le M^2, \tag{3.12}$$

i.e., $\{\beta_2(n)\}$ is bounded. By induction, we obtain $\limsup_{n\to\infty} \beta_k(n) \le M^k$ for $k \in \mathbb{N}$, which proves (3.6). □

Lemma 3.1.3. *Let* $\{x(n)\}$ *be a nonoscillatory solution of* (1.3)*. If*

$$\limsup_{n\to\infty}\left(p(n)\sum_{j=n-\tau}^{n-1}p(j)\right)>0, \tag{3.13}$$

then

$$\liminf_{n\to\infty}\frac{x(n-\tau)}{x(n+1)}<\infty.$$

Proof. Without loss of generality, assume that $x(n), x(n-\tau)>0$ for $n\geq N_1$, where $N_1\in\mathbb{N}$ is sufficiently large. Then, $\{x(n)\}$ is nonincreasing on $\{N_1, N_1+1,\cdots\}$. In view of (3.13), there exist an increasing divergent sequence $\{n_k\}\subset\{N_1, N_1+1,\cdots\}$ and a constant $\varepsilon>0$ such that

$$p(n_k)\sum_{j=n_k-\tau}^{n_k-1}p(j)\geq\varepsilon\quad\text{for all } k\in\mathbb{N}. \tag{3.14}$$

It follows from (1.3) that

$$x(n_k)>x(n_k)-x(n_k+1)=p(n_k)x(n_k-\tau)\quad\text{for all } k\in\mathbb{N}. \tag{3.15}$$

Also, from (1.3), we have

$$\begin{aligned}x(n_k-\tau)>x(n_k-\tau)-x(n_k)&=-\sum_{j=n_k-\tau}^{n_k-1}[x(j+1)-x(j)]\\&=\sum_{j=n_k-\tau}^{n_k-1}p(j)x(j-\tau)\geq\sum_{j=n_k-\tau}^{n_k-1}p(j)x(n_k-1-\tau)\end{aligned} \tag{3.16}$$

for all $k\in\mathbb{N}$. Combining (3.14), (3.15) and (3.16) yields

$$\frac{x(n_k-1-\tau)}{x(n_k)}<\frac{1}{p(n_k)\sum_{j=n_k-\tau}^{n_k-1}p(j)}\leq\frac{1}{\varepsilon}\quad\text{for all } k\in\mathbb{N},$$

which completes the proof. □

CHAPTER FOUR
THE PROOF

Proof of Theorem 3.0.1. Assume the contrary that $\{x(n)\}$ is an nonoscillatory solution of (1.3). Without loss of generality, we suppose that $\{x(n)\}$ is eventually positive. Then, there exists $N_1 \in \mathbb{N}$ such that $x(n)$ and $x(n-\tau)$ are positive for $n \geq N_1$. By (1.3), $\{x(n)\}$ is a nonincreasing sequence on $\{N_1, N_1+1, \cdots\}$. Define $w(n) := \frac{x(n-\tau)}{x(n+1)}$ for $n \geq N_1$. Note that $w(n) \geq 1$ for $n \geq N_1$. From (1.3), we write

$$x(n+1) - x(n) + w(n)p(n)x(n+1) = 0 \quad \text{for } n \geq N_1,$$

which yields

$$w(n) = \prod_{j=n-\tau}^{n} \big[1 + w(j)p(j)\big] \quad \text{for } n \geq N_2, \tag{4.1}$$

where $N_2 := N_1 + \tau$. Now, we define

$$z_k(n) := \begin{cases} w(n), & k = 0 \\ \min\{z_{k-1}(j) : \ j = n-\tau, n-\tau+1, \cdots, n\}, & k = 1, 2, \cdots, \ell \end{cases} \tag{4.2}$$

for $n \geq N_2$. By (4.1), (4.2) with $k = 1$ and $w(n) \geq 1$ for $n \geq N_2$, it follows that $z_1(n) \geq 1$ for $n \geq N_3$, where $N_3 := N_2 + \tau$. By (3.2) with $k = 1$,

$$\begin{aligned} w(n) &\geq \prod_{j=n-\tau}^{n} [1 + z_1(n)p(j)] \\ &= \left(\frac{1}{z_1(n)} \prod_{j=n-\tau}^{n} [1 + z_1(n)p(j)]\right) z_1(n) \\ &\geq \beta_1(n) z_1(n) \end{aligned}$$

for $n \geq N_3$. From (4.1) and (4.2) with $k = 2$, we know that $z_2(n) \leq z_1(n)$ for $n \geq N_4$, and by definition $z_2(n) \geq 1$ for $n \geq N_4$, where $N_4 := N_3 + \tau$. By (3.2) with $k = 2$,

$$\begin{aligned} w(n) &\geq \prod_{j=n-\tau}^{n} [1 + z_1(j)\beta_1(j)p(j)] \\ &\geq \prod_{j=n-\tau}^{n} [1 + z_2(n)\beta_1(j)p(j)] \\ &= \left(\frac{1}{z_2(n)} \prod_{j=n-\tau}^{n} [1 + z_2(n)\beta_1(j)p(j)] \right) z_2(n) \\ &\geq \beta_2(n) z_2(n) \end{aligned}$$

for $n \geq N_4$. By induction, it follows from (3.2) with $k = \ell$, (4.1) and (4.2) with $k = \ell$ that

$$w(n) \geq \beta_\ell(n) z_\ell(n) \quad \text{for } n \geq N_5. \tag{4.3}$$

where $N_5 := N_4 + \ell\tau$. By Lemma 3.1.1, Lemma 3.1.2 and Lemma 3.1.3, we obtain $\omega_* := \liminf_{n\to\infty} w(n) < \infty$. Note that $\liminf_{n\to\infty} z_\ell(n) = \omega_*$. Thus, taking inferior limits on both sides of (4.3), we get

$$\begin{aligned} w_* &\geq \liminf_{n\to\infty} \beta_\ell(n) \liminf_{n\to\infty} z_\ell(n) \\ &= \liminf_{n\to\infty} \beta_\ell(n) w_*, \end{aligned}$$

which yields $\liminf_{n\to\infty} \beta_\ell(n) \leq 1$ contradicting (3.1). This completes the proof. $\square$

In the example below, we will show the importance of Theorem 3.0.1, where Theorem 2.2.1, Theorem 2.2.3, Theorem 2.2.4 and Theorem 2.2.5 **cannot** deliver an answer on the oscillatory behavior of solutions but Theorem 3.0.1 applies and gives a positive answer.

Example 13. *Consider the equation*

$$x(n+1)-x(n)+\begin{Bmatrix} \frac{15}{100}, & 0\equiv n \,(\mathrm{mod}\, 4) \\ \frac{17}{100}, & 1\equiv n \,(\mathrm{mod}\, 4) \\ \frac{14}{100}, & 2\equiv n \,(\mathrm{mod}\, 4) \\ \frac{15}{100}, & 3\equiv n \,(\mathrm{mod}\, 4) \end{Bmatrix} x(n-2)=0 \quad \text{for } n=0,1,\cdots. \tag{4.4}$$

- *We have* $\Lambda := (0, \frac{100}{17})$*, which is defined in* (2.23)*, and*

$$\frac{1}{\lambda\prod_{j=n-2}^{n-1}[1-\lambda p(j)]}=\begin{cases} \frac{1}{\lambda(1-\lambda\frac{15}{100})(1-\lambda\frac{14}{100})}, & 0\equiv n \,(\mathrm{mod}\, 4) \\ \frac{1}{\lambda(1-\lambda\frac{15}{100})^2}, & 1\equiv n \,(\mathrm{mod}\, 4) \\ \frac{1}{\lambda(1-\lambda\frac{17}{100})(1-\lambda\frac{15}{100})}, & 2\equiv n \,(\mathrm{mod}\, 4) \\ \frac{1}{\lambda(1-\lambda\frac{14}{100})(1-\lambda\frac{17}{100})}, & 3\equiv n \,(\mathrm{mod}\, 4) \end{cases}$$

for $n=0,1,\cdots$. *Simply, we compute*

$$\begin{aligned}\inf_{\lambda\in(0,\frac{100}{17})}\left\{\frac{1}{\lambda(1-\lambda\frac{15}{100})(1-\lambda\frac{14}{100})}\right\} &= \frac{1}{\lambda(1-\lambda\frac{15}{100})(1-\lambda\frac{14}{100})}\bigg|_{\lambda\to\frac{10}{63}(29-\sqrt{211})} \\ &= \frac{1}{50}\big(211\sqrt{211}-3016\big)\approx\frac{98}{100}\not>1,\end{aligned}$$

*which shows that Theorem 2.2.4 (i) fails. This implies that Theorem 2.2.1 (i) and Theorem 2.2.3 also **cannot** apply.*

- *We have*

$$\frac{1}{\lambda}\prod_{j=n-2}^{n}[1+\lambda p(j)]=\begin{cases} \frac{1}{\lambda}(1+\lambda\frac{15}{100})^2(1+\lambda\frac{14}{100}), & 0\equiv n \,(\mathrm{mod}\, 4) \\ \frac{1}{\lambda}(1+\lambda\frac{17}{100})(1+\lambda\frac{15}{100})^2, & 1\equiv n \,(\mathrm{mod}\, 4) \\ \frac{1}{\lambda}(1+\lambda\frac{14}{100})(1+\lambda\frac{17}{100})(1+\lambda\frac{15}{100}), & 2\equiv n \,(\mathrm{mod}\, 4) \\ \frac{1}{\lambda}(1+\lambda\frac{15}{100})(1+\lambda\frac{14}{100})(1+\lambda\frac{17}{100}), & 3\equiv n \,(\mathrm{mod}\, 4) \end{cases}$$

for $n = 0, 1, \cdots$. *Simply, we compute*

$$\inf_{\lambda \geq 1}\left\{\frac{1}{\lambda}\left(1+\lambda\frac{15}{100}\right)^2\left(1+\lambda\frac{14}{100}\right)\right\} \approx \frac{1}{\lambda}\left(1+\lambda\frac{15}{100}\right)^2\left(1+\lambda\frac{14}{100}\right)\Big|_{\lambda \to \frac{341}{100}}$$
$$\approx \frac{99}{100} \not> 1,$$

which shows that Theorem 2.2.5 fails too.

- *First, we compute*

$$\beta_1(n) = \inf_{\lambda \geq 1}\left\{\frac{1}{\lambda}\prod_{j=n-2}^{n}[1+\lambda p(j)]\right\}$$
$$\approx \begin{cases} \frac{1}{\lambda}(1+\lambda\frac{15}{100})^2(1+\lambda\frac{14}{100})\big|_{\lambda\to\frac{341}{100}}, & 0 \equiv n \pmod 4 \\ \frac{1}{\lambda}(1+\lambda\frac{17}{100})(1+\lambda\frac{15}{100})^2\big|_{\lambda\to\frac{319}{100}}, & 1 \equiv n \pmod 4 \\ \frac{1}{\lambda}(1+\lambda\frac{14}{100})(1+\lambda\frac{17}{100})(1+\lambda\frac{15}{100})\big|_{\lambda\to\frac{327}{100}}, & 2 \equiv n \pmod 4 \\ \frac{1}{\lambda}(1+\lambda\frac{15}{100})(1+\lambda\frac{14}{100})(1+\lambda\frac{17}{100})\big|_{\lambda\to\frac{327}{100}}, & 3 \equiv n \pmod 4 \end{cases}$$
$$\approx \begin{cases} \frac{99}{100}, & 0 \equiv n \pmod 4 \\ \frac{106}{100}, & 1 \equiv n \pmod 4 \\ \frac{103}{100}, & 2 \equiv n \pmod 4 \\ \frac{103}{100}, & 3 \equiv n \pmod 4 \end{cases}$$

for $n = 0, 1, \cdots$. *Next, we compute*

$$\beta_2(n) = \inf_{\lambda \geq 1} \left\{ \frac{1}{\lambda} \prod_{j=n-2}^{n} \left[1 + \lambda \beta_1(j) p(j) \right] \right\}$$

$$\approx \begin{cases} \frac{1}{\lambda}(1+\lambda\frac{99}{100}\frac{15}{100})(1+\lambda\frac{103}{100}\frac{15}{100})(1+\lambda\frac{103}{100}\frac{14}{100})\big|_{\lambda\to\frac{336}{100}}, & 0 \equiv n \pmod 4) \\ \frac{1}{\lambda}(1+\lambda\frac{106}{100}\frac{17}{100})(1+\lambda\frac{99}{100}\frac{15}{100})(1+\lambda\frac{103}{100}\frac{15}{100})\big|_{\lambda\to\frac{311}{100}}, & 1 \equiv n \pmod 4) \\ \frac{1}{\lambda}(1+\lambda\frac{103}{100}\frac{14}{100})(1+\lambda\frac{106}{100}\frac{17}{100})(1+\lambda\frac{99}{100}\frac{15}{100})\big|_{\lambda\to\frac{318}{100}}, & 2 \equiv n \pmod 4) \\ \frac{1}{\lambda}(1+\lambda\frac{103}{100}\frac{15}{100})(1+\lambda\frac{103}{100}\frac{14}{100})(1+\lambda\frac{106}{100}\frac{17}{100})\big|_{\lambda\to\frac{314}{100}}, & 3 \equiv n \pmod 4) \end{cases}$$

$$\approx \begin{cases} \frac{101}{100}, & 0 \equiv n \pmod 4) \\ \frac{109}{100}, & 1 \equiv n \pmod 4) \\ \frac{106}{100}, & 2 \equiv n \pmod 4) \\ \frac{108}{100}, & 3 \equiv n \pmod 4) \end{cases}$$

for $n = 0, 1, \cdots$. *This yields* $\liminf_{n\to\infty} \beta_2(n) = \frac{101}{100} > 1$, *i.e., Theorem 3.0.1 applies with* $\ell = 2$.

Therefore, every solution of (4.4) *is oscillatory.*

CHAPTER FIVE
CONCLUSIONS

In the literature, there exist other iterative tests for the oscillation of solutions of delay difference equations. In this direction, we quote below one of the first important results by X. H. Tang and J. S. Yu.

Theorem 5.0.1 ((Tang & Yu, 1999b, Corollary 1)). *Assume that there exists* $\ell \in \mathbb{N}$ *such that*

$$\liminf_{n\to\infty} p_\ell(n) > \left(\frac{\tau}{\tau+1}\right)^{\ell(\tau+1)} \tag{5.1}$$

where

$$p_k(n) := \begin{cases} 1, & k=0 \\ \displaystyle\sum_{j=n+1}^{n+\tau} p_{k-1}(j)p(j), & k \in \mathbb{N}. \end{cases}$$

Then, every solution of (1.3) *oscillates.*

Proof. By Theorem 2.2.2, we know that $p_1(n) < 1$ for all large n. Then,

$$\liminf_{n\to\infty} p_i(n) \le \liminf_{n\to\infty} p_1(n), \quad \text{for } i = 1, 2, \cdots .$$

This shows by (5.1) that

$$\sum_{j=0}^{\infty} p(j) = p(0) + \sum_{k=0}^{\infty} p_1(k\tau) = \infty, \tag{5.2}$$

and there exists $\varepsilon > 0$ such that

$$\left(\frac{\tau+1}{\tau}\right)^{\ell} \big(p_\ell(n)\big)^{\frac{1}{\tau+1}} - 1 \ge \varepsilon \quad \text{for all large } n. \tag{5.3}$$

Thus, (5.2) and (5.3) imply that

$$\sum_{j=0}^{\infty} p(j)\left[\left(\frac{\tau+1}{\tau}\right)^{\ell} \big(p_\ell(j)\big)^{\frac{1}{\tau+1}} - 1\right] = \infty. \tag{5.4}$$

By (Tang & Yu, 1999b, Theorem 1), every solution of (1.3) is oscillatory. □

Remark 7. *Recall that Theorem 5.0.1 includes Theorem 2.2.3 with* $\ell = 1$.

Next, we quote a special case of another iterative result by M. Bohner, B. Karpuz and Ö. Öcalan, which is extracted from Bohner et al. (2008) for the discrete time scale nonnegative integers.

Theorem 5.0.2 (Cf. (Bohner et al., 2008, Theorem 2.3))**.** *Assume that there exists* $\ell \in \mathbb{N}$ *such that*

$$\liminf_{n\to\infty} \alpha_\ell(n) > 1, \tag{5.5}$$

where

$$\alpha_k(n) := \begin{cases} 1, & k = 0 \\ \inf\limits_{\lambda\in\Lambda_k} \left\{ \dfrac{1}{\lambda \prod_{j=n-\tau}^{n-1}[1 - \lambda\alpha_{k-1}(j)p(j)]} \right\}, & k \in \mathbb{N} \end{cases}$$

and

$$\Lambda_k := \{\lambda > 0 : \ 1 - \lambda\alpha_{k-1}(n)p(n) > 0 \textit{ for all large } n\} \quad \textit{for } k \in \mathbb{N}.$$

Then, every solution of (1.3) *oscillates.*

Proof. Assume the contrary that $\{x(n)\}$ is an nonoscillatory solution of (1.3). Without loss of generality, we suppose that $\{x(n)\}$ is eventually positive. Then, there exists $N_1 \in \mathbb{N}$ such that $x(n+1), x(n)$ and $x(n-\tau)$ are positive for $n \geq N_1$. By (1.3), $\{x(n)\}$ is a nonincreasing sequence on $\{N_1, N_1+1, \cdots\}$. Define $w(n) := \frac{x(n-\tau)}{x(n)}$ for $n \geq N_1$,. We rewrite (1.3) in the following form

$$x(n+1) - x(n) + w(n)p(n)x(n) = 0 \quad \text{for } n \geq N_1, \tag{5.6}$$

Then, we have

$$\begin{aligned} 0 &= x(n+1) - x(n) + w(n)p(n)x(n) \\ &> -x(n)[1 - w(n)p(n)], \end{aligned}$$

which implies $w(n)p(n) < 1$. From (5.6), we see that

$$w(n) = \frac{1}{\prod_{j=n-\tau}^{n-1}[1 - w(j)p(j)]} \quad \text{for } n \geq N_2, \tag{5.7}$$

where $N_2 := N_1 + \tau$. Now, we define

$$z_k(n) := \begin{cases} w(n), & k = 0 \\ \min\{z_{k-1}(j) : \ j = n-\tau, n-\tau+1, \cdots, n\}, & k = 1, 2, \cdots, \ell \end{cases} \tag{5.8}$$

for $n \geq N_2$. By (5.7), (5.8) with $k = 1$ for $n \geq N_2$, it follows that $w(n) \geq z_1(n)$ for $n \geq N_2$. Thus $z_1(n) \in \Lambda_1$ for $n \geq N_3$, where $N_3 := N_2 + \tau$. Thus,

$$\begin{aligned} w(n) &\geq \frac{1}{\prod_{j=n-\tau}^{n-1}[1 - z_1(n)p(j)]} \\ &= \frac{z_1(n)}{z_1(n)\prod_{j=n-\tau}^{n-1}[1 - z_1(n)p(j)]} \\ &\geq \alpha_1(n)z_1(n) \end{aligned}$$

for $n \geq N_3$. From (5.6), we get

$$\begin{aligned} 0 &\geq x(n+1) - x(n) + z_1(n)\alpha_1(n)p(n)x(n) \\ &> -x(n)[1 - z_1(n)\alpha_1(n)p(n)], \end{aligned}$$

which implies that $z_1(n)\alpha_1(n)p(n) < 1$ for $n \geq N_3$, i.e., $z_1(n) \in \Lambda_2$ for $n \geq N_3$. From (5.7) and (5.8) with $k = 2$, since $z_2(n) \leq z_1(n)$ for $n \geq N_4$, we have, $z_2(n) \in \Lambda_2$ for $n \geq N_4$, where $N_4 := N_3 + \tau$. Thus,

$$\begin{aligned} w(n) &\geq \frac{1}{\prod_{j=n-\tau}^{n-1}[1 - z_1(n)\alpha_1(j)p(j)]} \\ &\geq \frac{1}{\prod_{j=n-\tau}^{n-1}[1 - z_2(n)\alpha_1(j)p(j)]} \\ &= \frac{z_2(n)}{z_2(n)\prod_{j=n-\tau}^{n-1}[1 - z_2(n)\alpha_1(j)p(j)]} \\ &\geq \alpha_2(n)z_2(n) \end{aligned}$$

for $n \geq N_4$. By induction, it follows from (5.7) and (5.8) with $k = \ell$ that

$$w(n) \geq \alpha_\ell(n) z_\ell(n) \quad \text{for } n \geq N_5. \tag{5.9}$$

where $N_5 := N_4 + \ell\tau$. From Lemma 3.1.3, taking inferior limits on both sides of (5.9), we get

$$\begin{aligned}
\liminf_{n\to\infty} w(n) &\geq \liminf_{n\to\infty}[\alpha_\ell(n) z_\ell(n)] \\
&\geq \liminf_{n\to\infty} \alpha_\ell(n) \liminf_{n\to\infty} z_\ell(n) \\
&= \liminf_{n\to\infty} \alpha_\ell(n) \liminf_{n\to\infty} w(n),
\end{aligned} \tag{5.10}$$

which yields $\liminf_{n\to\infty} \alpha_\ell(n) \leq 1$ contradicting (5.5). This completes the proof. □

Remark 8. *Theorem 5.0.1 and Theorem 5.0.2 are* ***not*** *comparable.*

Example 14. *Consider the difference equation*

$$x(n+1) - x(n) + \begin{Bmatrix} \frac{1345}{16384}, & 0 \equiv n \pmod 3 \\ \frac{209}{2560}, & 1 \equiv n \pmod 3 \\ \frac{205}{2496}, & 2 \equiv n \pmod 3 \end{Bmatrix} x(n-4) = 0 \quad \textit{for } n = 0, 1, \cdots. \tag{5.11}$$

We compute that

$$p_1(n) = \begin{Bmatrix} \frac{1046339}{3194880} \approx 0.327505, & 0 \equiv n \pmod 3 \\ \frac{1047907}{3194880} \approx 0.327996, & 1 \equiv n \pmod 3 \\ \frac{523891}{1597440} \approx 0.327957, & 2 \equiv n \pmod 3 \end{Bmatrix} \quad \textit{for } n = 0, 1, \cdots$$

$$p_2(n) = \begin{Bmatrix} \frac{1096021915273}{10207258214400} \approx 0.107377, & 0 \equiv n \pmod 3 \\ \frac{1097632233449}{10207258214400} \approx 0.107534, & 1 \equiv n \pmod 3 \\ \frac{548561398937}{5103629107200} \approx 0.107485, & 2 \equiv n \pmod 3 \end{Bmatrix} \quad \textit{for } n = 0, 1, \cdots,$$

which shows that

$$\liminf_{n\to\infty} p_2(n) = 0.107377 > \left(\frac{4}{4+1}\right)^{2(4+1)} \approx 0.107374.$$

That is, every solution of (5.11) *oscillates by Theorem 5.0.1.*

On the other hand, we compute

$$\begin{aligned}
\alpha_1(n) &= \inf_{\lambda\in(0,\frac{2496}{205})}\left\{\frac{1}{\lambda\prod_{j=n-\tau}^{n-1}[1-\lambda p(j)]}\right\} \\
&= \inf_{\lambda\in(0,\frac{2496}{205})}\left\{\begin{array}{ll}\frac{1}{\lambda(1-\lambda\frac{205}{2496})^2(1-\lambda\frac{1345}{16384})(1-\lambda\frac{209}{2560})}, & 0\equiv n\,(\mathrm{mod}\,3)\\ \frac{1}{\lambda(1-\lambda\frac{205}{2496})(1-\lambda\frac{1345}{16384})^2(1-\lambda\frac{209}{2560})}, & 1\equiv n\,(\mathrm{mod}\,3)\\ \frac{1}{\lambda(1-\lambda\frac{205}{2496})(1-\lambda\frac{1345}{16384})(1-\lambda\frac{209}{2560})^2}, & 2\equiv n\,(\mathrm{mod}\,3)\end{array}\right\} \\
&= \left\{\begin{array}{ll}1.00096, & 0\equiv n\,(\mathrm{mod}\,3)\\ 1.00084, & 1\equiv n\,(\mathrm{mod}\,3)\\ 0.999467, & 2\equiv n\,(\mathrm{mod}\,3)\end{array}\right\} \not> 1 \quad \textit{for}\ n=0,1,\cdots \\
\alpha_2(n) &\approx \inf_{\lambda\in(0,12.1697)}\left\{\frac{1}{\lambda\prod_{j=n-\tau}^{n-1}[1-\lambda\alpha_1(j)p(j)]}\right\} \\
&\approx \inf_{\lambda\in(0,12.1697)}\left\{\begin{array}{ll}\frac{1}{\lambda(1-\lambda 0.0821714)(1-\lambda 0.0820876)^2(1-\lambda 0.0817096)}, & 0\equiv n\,(\mathrm{mod}\,3)\\ \frac{1}{\lambda(1-\lambda 0.0821714)^2(1-\lambda 0.0820876)(1-\lambda 0.0817096)}, & 1\equiv n\,(\mathrm{mod}\,3)\\ \frac{1}{\lambda(1-\lambda 0.0821714)(1-\lambda 0.0820876)(1-\lambda 0.0817096)^2}, & 2\equiv n\,(\mathrm{mod}\,3)\end{array}\right\} \\
&\approx \left\{\begin{array}{ll}1.00115, & 0\equiv n\,(\mathrm{mod}\,3)\\ 1.0014, & 1\equiv n\,(\mathrm{mod}\,3)\\ 0.999996, & 2\equiv n\,(\mathrm{mod}\,3)\end{array}\right\} \not> 1 \quad \textit{for}\ n=0,1,\cdots,
\end{aligned}$$

which shows that Theorem 5.0.2 fails.

Example 15. *When* $\ell = 1$, *Theorem 5.0.2 is better than Theorem 5.0.1 since Theorem 2.2.4 is better than Theorem 2.2.3.*

As the final sentence, we would like to mention that our main result Theorem 3.0.1

complements Theorem 5.0.2 in a similar manner that Theorem 2.2.5 complements Theorem 2.2.4.

REFERENCES

Berezansky, L., & Braverman, E. (2006). On existence of positive solutions for linear difference equations with several delays. *Advances in Dynamical Systems and Applications*, *1*(1), 29–47.

Bohner, M., Karpuz, B., & Öcalan, O. (2008). Iterated oscillation criteria for delay dynamic equations of first order. *Advances in Difference Equations*, Art. ID 458687, 12.

Chatzarakis, G. E., Koplatadze, R., & Stavroulakis, I. P. (2008). Oscillation criteria of first order linear difference equations with delay argument. *Nonlinear Analysis: Theory, Methods & Applications*, *68*(4), 994–1005.

Chatzarakis, G. E., & Stavroulakis, I. P. (2006). Oscillations of first order linear delay difference equations. *The Australian Journal of Mathematical Analysis and Applications*, *3*(1), Art. 14, 11.

Chen, M.-P., & Yu, J. S. (1995). Oscillations of delay difference equations with variable coefficients. In *Proceedings of the first international conference on difference equations* (105–114). Luxembourg: Gordon and Breach.

Erbe, L. H., & Zhang, B. G. (1989). Oscillation of discrete analogues of delay equations. *Differential and Integral Equations*, *2*(3), 300–309.

Győri, I., & Ladas, G. (1991). *Oscillation theory of delay differential equations with applications*. Oxford Mathematical Monographs. New York: The Clarendon Press & Oxford University Press.

Karpuz, B. (2017). Sharp oscillation and nonoscillation tests for linear difference equations. *Journal of Difference Equations and Applications*, *23*(12), 1929–1942.

Ladas, G. (1991). Recent developments in the oscillation of delay difference equations. In *Lecture notes in pure and applied mathematics* (vol. 127) (321–332). New York: Dekker.

Ladas, G., Philos, C. G., & Sficas, Y. G. (1989a). Necessary and sufficient conditions for the oscillation of difference equations. *Libertas Mathematica*, *9*, 121–125.

Ladas, G., Philos, C. G., & Sficas, Y. G. (1989b). Sharp conditions for the oscillation of delay difference equations. *Journal of Applied Mathematics and Simulation*, *2*(2), 101–111.

Malygina, V., & Chudinov, K. (2013). Explicit conditions for the nonoscillation of difference equations with several delays. *Electronic Journal of Qualitative Theory of Differential Equations*, No. 46, 12.

Tabor, J. (2003). Oscillation of linear difference equations in Banach spaces. *Journal of Differential Equations*, *192*(1), 170–187.

Tang, X. H., & Yu, J. S. (1999a). A further result on the oscillation of delay difference equations. *Computers & Mathematics with Applications*, *38*(11-12), 229–237.

Tang, X. H., & Yu, J. S. (1999b). Oscillation of delay difference equation. *Computers & Mathematics with Applications*, *37*(7), 11–20.

Yu, J. S., Zhang, B. G., & Wang, Z. C. (1994). Oscillation of delay difference equations. *pplicable Analysis*, *53*(1-2), 117–124.