DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

FINITENESS CONDITIONS IN RINGS AND MODULES WITH RESPECT TO ISOMORPHISM CLASSES

by
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İZMİR

FINITENESS CONDITIONS IN RINGS AND MODULES WITH RESPECT TO ISOMORPHISM CLASSES

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**FINITENESS CONDITIONS IN RINGS AND MODULES WITH RESPECT TO ISOMORPHISM CLASSES** " completed by **HAKAN ŞANAL** under supervision of **ASSOC. PROF. DR. SALAHATTIN ÖZDEMİR** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

This thesis is a survey on Artinian, Noetherian and simple modules with respect to isomorphism classes, that includes some of the results from the papers Facchini & Nazemian (2016), Facchini & Nazemian (2017), Alvarado-García et al. (2020) and Chaturvedi & Prakash (2021). Also, open problems are given at the end of each chapter. So, this thesis will be a useful reference for the people dealing with this subject.

Facchini & Nazemian (2016) generalize the idea of Artinian and Noetherian modules considering these chain conditions up to isomorphism. For any ring R and an R-module M, M is said to be isoartinian if, for every descending chain $M \geq M_1 \geq M_2 \geq \cdots$ of submodules of M, there exists an index $n \geq 1$ such that M_n is isomorphic to M_i for every $i \geq n$. The term isonoetherian is defined dually. Similarly in the same paper, an isosimple module is defined as a non-zero module whose all non-zero submodules are isomorphic to it. Dually, in Alvarado-García et al. (2020), co-isosimple modules are defined as non-zero modules whose all non-zero quotients are isomorphic to them. In the recent paper Chaturvedi & Prakash (2021), the notions of Ei/Nei Artinian and Noetherian modules are defined as generalizations of isoartinian and isonoetherian modules. In this thesis, we investigate these classes of modules.

Keywords: Isoartinian, isonoetherian, isosimple

İZOMORFİZM SINIFLARINA GÖRE HALKALAR VE MODÜLLERDE SONLULUK KOŞULLARI

ÖZ

Bu tez, Facchini & Nazemian (2016), Facchini & Nazemian (2017), Alvarado-García et al. (2020) ve Chaturvedi & Prakash (2021) makalelerinden bazı sonuçları içeren, izomorfizm sınıflarına göre Artin, Noether ve basit modüller üzerine bir araştırmadır. Ayrıca her bölümün sonunda açık problemlere yer verilmiştir. Dolayısıyla bu tez, bu konuyla ilgilenen kişiler için faydalı bir referans olacaktır.

Facchini & Nazemian (2016), Artin ve Noether modülleri, bu zincir koşullarını izomorfizme göre düşünerek genelleştirirler. Herhangi bir R halkası ve bir R-modül M için, M modülüne izo-artin denir eğer, M nin alt modüllerinin her azalan M $\geq M_1 \geq M_2 \geq \cdots$ zinciri için, bir $n \geq 1$ indeksi varsa öyle ki her $i \geq$ n için M_n M_i ye izomorfiktir. İzo-noether terimi de dual bir şekilde tanımlanmıştır. Aynı makalede, benzer şekilde, izobasit modül, sıfırdan farklı olmak üzere, tüm sıfırdan farklı alt modülleri kendisine izomorfik olan modül şeklinde tanımlanmıştır. Dual bir şekilde co-izobasit modüller, sıfırdan farklı olmak üzere, tüm sıfırdan farklı bölümleri kendilerine izomorfik olan modüller şeklinde tanımlanmıştır Alvarado-García et al. (2020). Chaturvedi & Prakash (2021) adlı son çalışmada Nei/Ei Artin ve Noether modüller gibi kavramlar izo-artin ve izo-noether modüllerin genellemesi şeklinde tanımlanmıştır. Bu tezde, bu modül sınıflarını inceleyeceğiz.

Anahtar kelimeler: İzo-artin, izo-noether, izobasit

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CHAPTER ONE
INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Throughout this thesis, R denotes an associative ring with unity and all modules are unitary right R-modules unless otherwise stated.

A. Facchini and Z. Nazemian (2016, 2017) generalize the idea of Artinian and Noetherian modules by considering these chain conditions up to isomorphism. Simple modules with respect to isomorphisms, called isosimple modules, are also studied in these papers. The term isosimple module is also called a virtually simple module in M. Behboodi & Vedadi (2018) and called an iso-retractable module in Chaturvedi (2019). Alvarado-García et al. (2020) studied co-isosimple modules, which is a dual concept of isosimple modules. Co-isosimple modules are also called anti-Hopfian modules in Hirano & Mogami (1987). In Chaturvedi & Prakash (2021), Nei/Ei Artinian and Noetherian modules are defined as a generalization of iso version of Artinian and Noetherian modules. In this survey thesis, we deal with these classes of modules. We study these papers from which we include some of the results with their proofs, but sometimes with more explanations. We also give proofs for some unproven results.

Some details about the classical module and ring theory, which are needed throughout, will be examined in Chapter 1. We will give some basic definitions and some results, mostly included in the books Kasch (1982) and Lam (2001), which will be useful in this thesis.

Chapter 2 includes some results on isosimple, isoartinian, and isonoetherian modules and rings from the paper by Facchini & Nazemian (2016). We will give some results and examples comparing simple and isosimple, Artinian and isoartinian, Noetherian and isonoetherian modules. We also give some results both from the paper Alvarado-García et al. (2020) on co-isosimple modules (a dual concept of isosimple module),

and from the paper Chaturvedi & Prakash (2021) on Ei/Nei Artinian and Noetherian modules.

In Facchini & Nazemian (2016), a module M is called *isoartinian* if, for every descending chain $M \geq M_1 \geq M_2 \geq \cdots$ of submodules of M, there exists an index $n \geq 1$ such that M_n is isomorphic to M_i for every $i \geq$ n. Dually, M is called *isonoetherian* if, for every ascending chain $M_1 \leq M_2 \leq \cdots$ of submodules of M, there exists an index $n \geq 1$ such that M_n is isomorphic to M_i for every $i \geq n$. For example $\mathbb{Z}_{\mathbb{Z}}$ is an isoartinian and isonoetherian module but not Artinian. Similarly, an *isosimple* module is defined as a non-zero module whose all non-zero submodules are isomorphic to it. As an example, $\mathbb{Z}[i]_{\mathbb{Z}[i]}$ is an isosimple module, but not simple, where $\mathbb{Z}[i]$ is the ring of Gaussian integers. Dually, in Alvarado-García et al. (2020), a *co-isosimple* module is defined as a non-zero module whose all non-zero quotients are isomorphic to it. For instance, $\mathbb{Z}_p$ is a co-isosimple module over itself, where p is a prime integer. In Chaturvedi & Prakash (2021), a module M is called *nei-noetherian* if for all ascending chains $M_1 \leq M_2 \leq \cdots$ of non-essential submodules of M, there can be found an index n such that $M_i \cong M_n$, for each $i \geq n$ and a module M is called *ei-noetherian* if, for all ascending chains $M_1 \leq M_2 \leq \cdots$ of essential submodules of M, there can be found an index n such that $M_i \cong M_n$, for each $i \geq n$. *Nei-artinian* and *ei-artinian* modules are also defined as dual versions of nei-noetherian and ei-noetherian modules, respectively. As an example, $\mathbb{Z}_{\mathbb{Z}}$ is an nei-noetherian, nei-artinian, ei-noetherian and ei-artinian module.

In chapter 3, the isoradical and isosocle of modules are investigated and some results from the paper Facchini & Nazemian (2017) are stated. The (right) $isoradical$ I-$rad(M)$ of a module M is defined as the intersection of the kernels of all homomorphisms $h : M \to I$, where I is an isosimple module. For instance, I -$rad(\mathbb{Z}_{\mathbb{Z}}) = 0$. The $isosocle$ I-$soc(M)$ of a module M is defined as the sum of the images of all homomorphisms $h : U \to M$, where U is an isosimple module. For example, I-$soc(\mathbb{Z}_{\mathbb{Z}}) = \mathbb{Z}$ and $soc(\mathbb{Z}_{\mathbb{Z}}) = 0$. We also investigate co-isoradical of modules, which were defined in Alvarado-García et al. (2020). For a module M, the intersection of all isomaximal submodules is called co-$isoradical$ of M and denoted

by $CI\text{-}rad(M)$. As an example, $CI\text{-}rad(\mathbb{Z}_{\mathbb{Z}}) = 0$. We give some results from that paper.

1.2 Preliminaries

In this section, some basic facts about classical module and ring theory will be examined which will be useful for the rest. For this section, we utilize some books such as Kasch (1982), Lam (1999), and Lam (2001), and for further reading one can consult these books.

1.2.1 Basic Notions in Modules

Definition 1.2.1.1. *A non-zero module M is called **indecomposable** if it cannot be written as a direct sum of any two proper submodules of it.*

Definition 1.2.1.2. *A non-zero right R-module M is called **simple** if and only if M has no non-zero proper submodules.*

Definition 1.2.1.3. *For an R-module M, the ring of all homomorphisms from M to M, under the pointwise addition and function composition, is called the **endomorphism ring** of M, denoted by $End(M)$.*

Lemma 1.2.1.1. *(Schur's Lemma) The endomorphism ring of a simple module is a division ring.*

Example 1.2.1.1. *$\mathbb{Z}_p$ is a simple $\mathbb{Z}$-module, where p is a prime. The endomorphism ring End($\mathbb{Z}_p$) $\cong \mathbb{Z}_p$, is a field, and so it is a division ring.*

Definition 1.2.1.4. *A submodule N of an R-module M is called **maximal** if N is not contained in a proper submodule of M except N itself.*

Proposition 1.2.1.1. *Let M be an R-module and N be a submodule of M. Then, N is a maximal submodule of M if and only if M/N is a simple module.*

Theorem 1.2.1.1. *(Kasch (1982), Theorem 8.1.3) For a R-module M, the following are equivalent:*

(1) All submodules of M are the sum of simple submodules.

(2) M is a sum of simple submodules.

(3) M is a direct sum of simple submodules.

(4) All submodules of M are direct summands of M.

If a module M satisfies one of the above equivalent conditions, then it is called ***semisimple****.*

Theorem 1.2.1.2. *(Lam (2001), Corollary 3.7) For any ring R, ${}_RR$ is semisimple if and only if R_R is semisimple.*

Example 1.2.1.2. *(Kasch, 1982) (1) Every $V = V_k$ over a division ring k is semisimple:*

$$V_k = \sum_{x \in V} xk, \; xk \text{ is simple for a non-zero } x.$$

(2) n is square-free or equals $\pm$ 1 if and only if $\mathbb{Z}/n\mathbb{Z}$ with a non-zero n is semisimple as a $\mathbb{Z}$-module.

(3) Since $\mathbb{Z}_{\mathbb{Z}}$ and $\mathbb{Q}_{\mathbb{Z}}$ have no simple submodules, they are not semisimple.

1.2.2 Essential and Small Submodules

Definition 1.2.2.1. *For a right R-module M, a submodule N of M is called a* ***small submodule of M****, if, for any submodule U of M, $N + U = M$ implies $U = M$.*

Definition 1.2.2.2. *Let M be an R-module and N a non-zero submodule of M. N is called an* ***essential submodule of M****, and we will write $N \leq_e M$, if $N \cap X \neq 0$ for any non-zero submodule X of M.*

Example 1.2.2.1. *(Kasch, 1982) (1) For every module M, 0 is a small submodule of M and M is an essential submodule of M.*
(2) If M is a semisimple module, then the only small submodule of M is 0 and the only essential submodule of M is M itself.
(3) 0 is the only small ideal of $\mathbb{Z}$, but all non-zero ideals are essential in $\mathbb{Z}$.
(4) Every finitely generated submodule of $\mathbb{Q}_{\mathbb{Z}}$ is small in $\mathbb{Q}_{\mathbb{Z}}$.

Definition 1.2.2.3. *Let R be a ring, and M be a right R-module. Define the* ***annihilator of M*** *to be the following subset of R:*

$$ann_R(M) := \{r \in R \mid mr = 0 \ for \ all \ m \ in \ M\}.$$

Note that $ann_R(M)$ is an ideal of R.

Definition 1.2.2.4. *Let M be a right R-module and x in M. Then the set*

$$Z(M) := \{x \in M \mid ann(x) \leq_e R\}$$

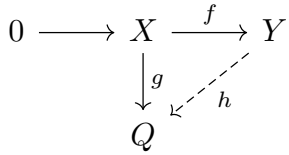
is called the ***singular submodule*** *of M.*
If Z(M)=M, then M is called ***singular****.*
If Z(M)=0, then M is called ***non-singular****.*

Example 1.2.2.2. *(1) $\mathbb{Z}_{\mathbb{Z}}$ is a non-singular module.*
(2) Let R be $\mathbb{Z}/4\mathbb{Z}$. Then, $2R$ is a singular R-module.

Proposition 1.2.2.1. *((Lam, 1999), Chapter 3.7, Exercise 3) For any submodule N in a non-singular module M, N is essential in M if and only if M/N is singular.*

1.2.3 Injective and Projective Modules

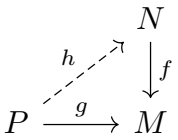
Definition 1.2.3.1. *A right module Q over the ring R is **injective** if, whenever X and Y are right R-modules, $f : X \to Y$ is an injective module homomorphism and $g : X \to Q$ is an arbitrary module homomorphism, there can be found a module homomorphism $h : Y \to Q$ such that $hf = g$:*



Example 1.2.3.1. *(1) Trivially, the zero module $\{0\}$ is injective.*
(2) $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules.
(3) $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ are not injective $\mathbb{Z}$-modules.

Definition 1.2.3.2. *Let R be a ring, M be an R-module and N be a submodule of M. Then, M is said to be an **essential extension** of N if $N \leq_e M$.*

Definition 1.2.3.3. *An R-module P is called **projective** if for every surjective module homomorphism $f : N \to M$ and every module homomorphism $g : P \to M$, there can be found a homomorphism $h : P \to N$ such that $fh = g$.*



Example 1.2.3.2. *(1) $\mathbb{Z}$ is a projective $\mathbb{Z}$-module.*
(2) $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are projective $\mathbb{Z}/6\mathbb{Z}$-modules.
(3) $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

Definition 1.2.3.4. *If a module M has a basis, then it is called a free module.*

Theorem 1.2.3.1. *(Kasch (1982), Theorem 5.4.1) A module M is isomorphic to a direct summand of a free module if and only if it is projective.*

Theorem 1.2.3.2. *(Kasch (1982), Corollary 8.2.2) For a ring R, the following are equivalent:*

(1) R is semisimple.

(2) All right (resp. left) R-modules are injective.

(3) All right (resp. left) R-modules are projective.

(4) All right (resp. left) R-modules are semisimple.

1.2.4 Artinian and Noetherian Modules

Definition 1.2.4.1. *Let R be a ring and M be a right R-module. M is called **Artinian** if, for every descending chain $M_1 \geq M_2 \geq M_3 \geq \cdots$ of submodules of M, there can be found an index $n \geq 1$ such that $M_n = M_i$ for every $i \geq n$.*

Definition 1.2.4.2. *Let R be a ring and M be a right R-module. M is called **Noetherian** if, for every ascending chain $M_1 \leq M_2 \leq M_3 \leq \cdots$ of submodules of M, there exists an index $n \geq 1$ such that $M_n = M_i$ for every $i \geq n$.*

Definition 1.2.4.3. *A ring R is called **right Noetherian** (resp. Artinian) if and only if R_R is Noetherian (resp. Artinian).*

Example 1.2.4.1. *(1) Every PID is Noetherian. In particular, $\mathbb{Z}_{\mathbb{Z}}$ is Noetherian but not Artinian.*

(2) As a $\mathbb{Z}$-module $\mathbb{Z}(p^{\infty})$ is Artinian but not Noetherian.

(3) $\mathbb{Q}_p/\mathbb{Z}$ is Artinian but not Noetherian as a $\mathbb{Z}$-module, where p is a prime number and $\mathbb{Q}_p = \{m/p^r | r, m \in \mathbb{Z}, r \geq 0\}$.

(4) $\mathbb{Q}$ is both Artinian and Noetherian as a ring.

(5) Over the ring $\mathbb{Z}$, $\mathbb{Z}[1/2]/\mathbb{Z}$ is Artinian but not Noetherian where $\mathbb{Z}[1/2] = \{a/b | a \in \mathbb{Z}, b = 2^m, m \geq 0\}$ and $\mathbb{Q}_{\mathbb{Z}}$ is neither Noetherian nor Artinian.

Definition 1.2.4.4. *Let M be a module over a ring R. Given a chain of submodules of M of the form*

$$M_0 \subsetneqq M_1 \subsetneqq \cdots \subsetneqq M_n = M,$$

*n is called the **length** of the chain. The largest length of any of its chains defines the length of M. However, if there is no such largest length, then M has infinite length.*

Theorem 1.2.4.1. *(Kasch (1982), Theorem 6.1.2) Let M be a right R-module and N be a submodule of M.*
(I) The following are equivalent:
(1) M is Artinian.
(2) N and M/N are Artinian.
(II) The following are equivalent:
(1) M is Noetherian.
(2) N and M/N are Noetherian.
(3) Each submodule of M is finitely generated.
(III) The following are equivalent:
(1) M is Artinian and Noetherian.
(2) M is of finite length.

Lemma 1.2.4.1. *(Kasch (1982), Corollary 6.1.3) If M is a finite sum of Noetherian submodules, then M is Noetherian. It also applies to the Artinian case.*

Theorem 1.2.4.2. *(Cohn (2005), Theorem 5.3.9) Let R be a right Artinian ring and M be a right R-module. Then the following are equivalent:*
(1) M is Noetherian.
(2) M is Artinian.

Corollary 1.2.4.1. *Every (right) Artinian ring is (right) Noetherian.*

Theorem 1.2.4.3. *(Kasch (1982), Theorem 6.5.1) The following are equivalent:*
(1) R_R is Noetherian.
(2) Each direct sum of injective R-modules is injective.

Theorem 1.2.4.4. *(Kasch (1982), Theorem 8.1.6) For a semisimple right R-module M, the following conditions are equivalent:*

(1) M is a sum of finitely many simple modules.

(2) M is a direct sum of finitely many simple modules.

(3) M has finite length.

(4) M is Artinian.

(5) M is Noetherian.

(6) M is finitely generated.

1.2.5 The Socle and The Radical

Definition 1.2.5.1. *The **Jacobson radical** of a module M is the intersection of all maximal submodules of M,*

$$rad(M) = \bigcap\{N \mid N \; is \; a \; maximal \; submodule \; of \; M\}.$$

Definition 1.2.5.2. *For any ring R, the **radical** of R is the intersection of all maximal right (and left) ideals of the ring, denoted by $Rad(R) = Rad(R_R) = Rad({}_RR)$.*

Example 1.2.5.1. *(1) $Rad(\mathbb{Z}) = 0$.*

(2) $Rad(\mathbb{Z}/12\mathbb{Z}) = 6\mathbb{Z}/12\mathbb{Z}$, where $6\mathbb{Z}/12\mathbb{Z} = 2\mathbb{Z}/12\mathbb{Z} \cap 3\mathbb{Z}/12\mathbb{Z}$.

(3) Rad($\mathbb{Q}_{\mathbb{Z}}$) = $\mathbb{Q}$ since for every $q \in \mathbb{Q}$, $q\mathbb{Z}$ is small in $\mathbb{Q}$, equivalently, $\mathbb{Q}$ has no maximal submodules.

Definition 1.2.5.3. *A ring R is called **semiprimary** if $rad(R)$ is nilpotent and $R/rad(R)$ is semisimple.*

Theorem 1.2.5.1. *(Hopkins-Levitzki) Let R be a semiprimary ring. Then for any right R-module M, the following are equivalent:*

(1) M is Noetherian.

(2) M is Artinian.

(3) M has a composition series.

Corollary 1.2.5.1. *A ring R is right Artinian if and only if it is right Noetherian and semiprimary.*

Definition 1.2.5.4. *The **socle** of a module M is defined to be the sum of the simple submodules of M. In the set notation,*

$$soc(M) = \sum\{N \mid N \; is \; a \; simple \; submodule \; of \; M\}.$$

Example 1.2.5.2. *(1) $soc(\mathbb{Z}) = 0$, $\mathbb{Z}$ as a module over itself.*
(2) $soc(\mathbb{Z}/p^n\mathbb{Z}) = p^{n-1}\mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$, where p is a prime number.

Lemma 1.2.5.1. *(Kasch, 1982) Let M be a right R-module. Then the following hold:*
(1) $soc(soc(M)) = soc(M)$.
(2) $Z(M)soc(M) = \{0\}$.

Theorem 1.2.5.2. *For any R-module M, the following hold:*
(1) $Msoc(R_R) \leq soc(M)$.
(2) For every submodule N of M, $soc(N) = soc(M) \cap N$
(3) If $N \leq_e M$, then $soc(N) = soc(M)$.
(4) For a maximal submodule N of M, $N \geq soc(M)$ if and only if N is essential in M.

Definition 1.2.5.5. *A module M is called **semiartinian** if every non-zero homomorphic image of M contains a simple submodule, i.e., $soc(M/N)$ is non-zero for every proper submodule N of M. A ring R is called right/left **semiartinian** if every non-zero R-module has a simple submodule.*

Proposition 1.2.5.1. *(Kasch (1982), Chapter 9, Exercise 7) For a right R-module M, the following hold:*
(1) If M is semiartinian, then for each submodule N of M, M/N is also semiartinian.
(2) Over a right Noetherian ring, if a module is semiartinian, then it is the sum of its Artinian submodules.

1.2.6 Some Special Rings

In this section, we give some definitions and basic properties of some rings which are needed.

Definition 1.2.6.1. *A ring R is called* ***right non-singular*** *if $Z(R_R) = \{0\}$.*

Proposition 1.2.6.1. *(Lam (1999), Theorem 7.21) Let R be a right non-singular ring, and M be any right R-module. Then, $Z(M/Z(M)) = 0$.*

Example 1.2.6.1. *(1) If R is a semisimple ring, then $Z(R) = \{0\}$*
(2) If R is the ring of upper triangular $n \times n$ matrices over a semisimple ring k, then R is non-singular.
(3) If k is a semisimple ring, then

$$R = \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

is non-singular.

Theorem 1.2.6.1. *(Anderson & Fuller (1992), Chapter 6.24, Exercise 15) The following conditions are equivalent for a ring R:*
(1) R is self-injective on one side and Noetherian on one side.
(2) R is self-injective on one side and Artinian on one side.
(3) Every left or right projective R-module is injective.
(4) Every left or right injective R-module is projective.
If a ring R satisfies one of the above equivalent conditions, then it is called ***Quasi-Frobenius ring (or QF-ring)****.*

Example 1.2.6.2. *(1) Because over a semisimple ring every module is both projective and injective, semisimple rings are Quasi-Frobenius.*
(2) $\mathbb{Z}/n\mathbb{Z}$ is Quasi-Frobenius for any positive integer $n > 1$.

Theorem 1.2.6.2. *(Lam (2001), Theorem 4.23) For any ring R, the following are equivalent:*

(1) For each element $a \in R$, there can be found some $x \in R$ such that $a = axa$.

(2) Each principal left ideal is a direct summand of ${}_RR$.

(3) All finitely generated left ideals are direct summands of ${}_RR$.

If a ring R satisfies any of the above equivalent conditions, R is called a ***von-Neumann Regular ring (or VNR ring for short)****.*

Example 1.2.6.3. *(1) Every field is a VNR ring.*

(2) Every direct product of VNR rings is also VNR ring.

Proposition 1.2.6.2. *(Lam (2001), Proposition 4.27) The endomorphism ring of a semisimple module is a VNR ring.*

Theorem 1.2.6.3. *(Lam (2001), Theorem 4.25) A semisimple ring is exactly a left (resp. right) Noetherian VNR ring.*

Definition 1.2.6.2. *A ring R is called a right* ***V-ring*** *if each simple right R-module is injective.*

Theorem 1.2.6.4. *(Kaplansky) A commutative ring is $von\text{-}Neumann\ Regular$ if and only if it is a V-ring.*

Definition 1.2.6.3. *A proper ideal I in a ring R is said to be a* ***prime ideal*** *if for ideals $A, B \leq R$, $AB \leq I$ implies that $A \leq I$ or $B \leq I$.*

Definition 1.2.6.4. *A ring R is called a* ***prime*** *(resp.,* ***semiprime****) ring if (0) is a* ***prime*** *(resp.,* ***semiprime****) ideal.*

Note that, all prime rings are semiprime.

Example 1.2.6.4. *(Lam (2001), Examples 10.17) (1) Any domain is a prime ring.*

(2) Over an integral domain, each matrix ring is prime.

(3) The direct product of two or more non-zero rings is never a prime ring.

(4) All direct products of semiprime rings are semiprime.

(5) The ideal $< 6 >$ of the ring of integers $\mathbb{Z}$ is not prime, but it is semiprime.

Definition 1.2.6.5. *For a non-zero R-module M, if all pairs of non-zero submodules of M have non-zero intersection, then M is called* ***uniform****.*

Definition 1.2.6.6. *The* ***uniform dimension*** *of a module M denoted $u.dim(M)$, is defined to be n if there can be found a finite set of uniform submodules U_i such that $\oplus_{i=1}^{n} U_i \leq_e M$.*

Definition 1.2.6.7. *For a ring R, if $u.dim(R_R) < \infty$, and R satisfies the ACC on right annihilators of subsets of R, then it is called a right* ***Goldie ring****.*

Proposition 1.2.6.3. *(Lam (1999), 11.10) All right Noetherian rings are right Goldie. However, the converse is not true, since all commutative domains are also right Goldie.*

Definition 1.2.6.8. *Let R be a domain. R is called a* ***right Ore domain*** *if any two non-zero elements of R have a non-zero common right multiple, that is, for all pairs of non-zero x and y, there can be found a pair of elements r and s of R such that $xr = ys \neq 0$.*

Theorem 1.2.6.5. *(Goldie Theorem) (Lam (1999), Theorem 10.22) The following are equivalent for a domain R:*
(1) R is right Ore.
(2) $u.dim(R_R) = 1$.
(3) $u.dim(R_R) < \infty$

Corollary 1.2.6.1. *(Lam (1999), Corollary 10.23) A right Noetherian domain is a right Ore domain.*

Proposition 1.2.6.4. *(Lam (1999), Corollary 11.18) If R is a prime (resp. semiprime) right Goldie ring, then so is $R_n := \mathbb{M}_n(R)$.*

Theorem 1.2.6.6. *(Lam (1999), Corollary 11.19) Let $S = R[x]$. Then R is prime (resp. semiprime) right Goldie if and only if S is.*

Corollary 1.2.6.2. *A domain R is right Goldie if and only if it is right Ore.*

Proposition 1.2.6.5. *(V.Dung (1992), Proposition 1) A ring R is right semiartinian if and only if every homomorphic image of R_R has an essential socle.*

Definition 1.2.6.9. *A ring R is called right **self-injective** if it is an injective as a right module over itself.*

Theorem 1.2.6.7. *(Lam (2001), Theorem 19.1) For any non-zero ring R, the following are equivalent:*

(1) R has a unique maximal left ideal.

(2) R has a unique maximal right ideal.

(3) $R/rad(R)$ is a division ring.

*A ring R is called **local** if any of the above conditions holds.*

Definition 1.2.6.10. *A ring R is called a **discrete valuation ring (DVR)** if it is a principal ideal domain with exactly one non-zero maximal ideal.*

Proposition 1.2.6.6. *(Dummit & Foote (2004), Theorem 7) The following properties of a ring R are equivalent:*

(1) R is a DVR,

(2) R is PID with a non-zero unique maximal ideal,

(3) R is a Noetherian integral domain that is also a local ring whose unique maximal ideal is non-zero and principal.

CHAPTER TWO
ARTINIAN, NOETHERIAN AND SIMPLE MODULES WITH RESPECT TO ISOMORPHISM

In this chapter, the "iso" definitions will be included for Artinian and Noetherian modules and rings, which were defined first in Facchini & Nazemian (2016). Also, co-isosimple modules and Nei/Ei Artinian and Noetherian modules will be mentioned which were studied in Alvarado-García et al. (2020) and Chaturvedi & Prakash (2021), respectively. Some properties of these concepts will be given.

In Section 2.1, isosimple modules and their endomorphism rings will be considered. They will be compared with simple modules and be analyzed where to coincide. In Section 2.2, isoartinian modules will be investigated. Besides, the definition of Artinian dimension will be given and used for characterizing isoartinian modules. In Section 2.3, isonoetherian modules will be contained and compared with isoartinian definition. In Section 2.4, isoartinian and isonoetherian rings will be involved and associated with the classical Artinian and Noetherian rings. Moreover, they will be characterized by using some special rings. In Section 2.5, co-isosimple modules and their endomorphism rings will be included. Lastly, in Section 2.6, Nei/Ei Artinian and Noetherian modules will be mentioned.

2.1 Isosimple Modules

Recall that a non-zero module M is said to be **simple** if it has no submodules other than 0 and M.

Definition 2.1.1. *An **isosimple** module is defined as a non-zero module whose all non-zero submodules are isomorphic to it.*

It can be said by the above definition that a simple module is always isosimple. However, the converse does not satisfy in general.

Example 2.1.0.1. *The $\mathbb{Z}$-module $\mathbb{Z}$ is an isosimple module because $n\mathbb{Z} \cong \mathbb{Z}$ for every natural number n. But, $\mathbb{Z}$ is not simple since, for example $2\mathbb{Z}$ is a proper submodule of $\mathbb{Z}$.*

Proposition 2.1.0.1. *(Facchini & Nazemian (2016), Remark 2.2) (1) Isosimple modules are cyclic, Noetherian, and uniform.*
(2) The module R_R is isosimple when R is a principal right ideal domain (PRID).
(3) For an R-module M, simple and isosimple concepts coincide if M is Artinian. In particular, if R is semiartinian, then simple and isosimple R-modules coincide. Besides, over a commutative V- ring, isosimple modules are simple.

Proof. (1) Let M_R be an isosimple module. First, take a non-zero element $a \in M$ and consider the submodule generated by a, say aR. Since M_R is isosimple and $aR \neq 0$, then $aR \cong M_R$, and so M_R is cyclic. Next, all submodules of M_R are cyclic since M is isosimple so, M_R is Noetherian. Finally, M contains a uniform module U, and by isosimplicity, $U \cong M$. Thus, M_R is uniform.

(2) By (1), every submodule of R_R is cyclic and uniform. Now, for any non-zero $a \in R$, aR is projective since $aR \cong R_R$ as R_R is isosimple. Then, the epimorphism $\phi : R_R \to aR$ are given by $\phi(r) = ar$ splits. Therefore, the kernel of ϕ, $ann_r(a)$ is a direct summand of R_R. Since $a \neq 0$ and R_R is uniform, $aR \cap ann_R(a) = 0$ implies that $ann_R(a) = 0$. Now the conclusion follows.

(3) Let M_R be an isosimple Artinian module. Since any Artinian module has a simple submodule, M_R has a simple submodule, say S. So, by isosimplicity, $M_R \cong S$. Therefore, M_R is simple.
Now, by the definition of a semiartinian ring, all non-zero modules M_R has a simple submodule. Let M be an isosimple module over a semiartinian ring R. So, M has a simple submodule, then M is simple.
And lastly, let R be a commutative V-ring and M be an isosimple R-module. Since isosimple modules are uniform and uniform modules are indecomposable, M is an indecomposable module over R. By Theorem 1.2.6.4, R is a commutative VNR ring

and by Couchot (2003), Theorem 2.13, it can be concluded that over a commutative V-ring, isosimple modules are simple. □

Proposition 2.1.0.2. *(Facchini & Nazemian (2016), Lemma 3.1) All non-zero endomorphisms of an isosimple module are injective mapping.*

Proof. For a non-zero endomorphism f of an isosimple module M, we have $M/ker(f) \cong f(M) \cong M$. Let $\phi : f(M) \to M$ be an isomorphism and $g = \phi$ o f. Then, $ker(g) = ker(f)$ and g is onto. Now if $ker(f) \neq 0$, then we have a strictly ascending chain $ker(g) \lneqq ker^2(g) \lneqq ker^3(g) \lneqq \cdots$, contradicting the fact that M is Noetherian. □

Corollary 2.1.0.1. *(Facchini & Nazemian, 2016) The endomorphism ring of an isosimple module is a domain.*

Proof. Let f and g be two non-zero elements of $End(M)$, where M is an isosimple module, and let m be a non-zero element of M. Then, by Proposition 2.1.0.2, $g(m) \neq 0$, and similarly, $f(g(m)) \neq 0$. Thus, $End(M)$ is a domain. □

Theorem 2.1.0.1. *(Facchini & Nazemian (2016), Lemma 3.2) The endomorphism ring of an isosimple module is a right Ore domain.*

Definition 2.1.2. *(1) Let S be a ring and M be a non-zero left S-module. M is called **completely prime** if the left annihilator $l.ann(m)$ of m in S equal to the left annihilator $l.ann(M)$ of M in S for every non-zero element m in M.*
*(2) A non-zero proper submodule N of M is called **completely prime** if M/N is a completely prime S-module.*

Proposition 2.1.0.3. *Let R be a ring and I be a right ideal of R. The endomorphism ring of the non-zero cyclic R-module R/I is canonically isomorphic to E_I/I, where*

$$E_I := \{r \in R \mid rI \leq I\} \text{ is the idealizer of } I$$

.

Proof. Let us define a map $\phi : E_I/I \to End_R(R/I, R/I)$ by sending $\bar{r} \in E_I/I$ to the left multiplication map $\phi(\bar{r}) : R/I \to R/I$ given by $\phi(\bar{r})(\bar{x}) = \bar{r}\bar{x}$ for all $\bar{x} \in R/I$. Then, by the similar argument in Lam (2001), Example 1.12, we have $\phi(\bar{r}) \in End_R(R/I, R/I)$ and ϕ is a ring isomorphism. □

Proposition 2.1.0.4. *(Facchini & Nazemian (2016), Proposition 3.3) Let R be a ring. An R-module M is isosimple if and only if $M_R \cong R/I$, where*
(a) I is a proper right ideal of R,
(b) for each right ideal J of R containing I, there exists $r \in E_I$ with $J = rR + I$, and
(c) I is a completely prime submodule of R, where the module R is considered as a left module over the ring E_I.

Proof. (⇒)Let M_R be an isosimple module. Then, $M_R \cong R/I$ for some proper right ideal I of R. To show (b), consider the submodules of R/I, which are of the form J/I, where J is a right ideal of R containing I. So, either $J = I$ or $R/I \cong J/I$. If $J = I$, then there can be found an element r in E_I such that $J = rR + I$, where r is the zero element of E_I. If $R/I \cong J/I$, then there can be found an endomorphism of R/I whose image is J/I. Therefore, there is an element r in E_I such that $J = rR+I$. Now, to show that I is a completely prime submodule of R, we must show that R/I is a completely prime left E_I - module, that is if an element e in E_I annihilates a non-zero element m in R/I, then m also annihilates R/I. If $em = 0$, then left multiplication by e induces an endomorphism of R_R $/I$ whose kernel contains the non-zero element m, hence is not injective, so that the endomorphism is the zero endomorphism. Hence, e is in I which makes e is an element of $l.ann(R/I)$.
(⇐)Now, let I be a right ideal that satisfies (a), (b), and (c). To show that $R/I \cong M_R$ is isosimple, we will consider the submodules of R/I which are of the form J/I, where J is a right ideal of R containing I. Let us choose a non-zero submodule J/I. We know by (b), $J = rR + I$ for some r in E_I. Then, consider the endomorphism of R/I induced by left multiplication by r, whose image is J/I. To get the isomorphism between R/I and J/I, we just need to show that the kernel of this map is zero. If m $\in R/I$ is in the kernel, rm must be zero, so that r $\in l.ann(m)$, then either
$r \in l.ann(R/I) = I$ or $m = 0$. If $r \in I$, then J must be equal to I which is a

contradiction because we fixed J/I as a non-zero submodule of R/I. So, $m = 0$. Hence, the kernel is zero, as desired. □

As an example, as it is indicated from the previous proposition, if R/I is an isosimple module, then I is a completely prime ideal in its idealizer E_I since $E_I/I \cong End(R/I)$ is a domain.

2.2 Isoartinian Modules

Definition 2.2.1. *Let R be a ring and M be a right R-module. M is called **isoartinian** if, for every descending chain $M \geq M_1 \geq M_2 \cdots$ of submodules of M, there can be found an index $n \geq 1$ such that $M_n \cong M_i$ for every $i \geq n$.*

By definition, it is clear, if a right R-module M is Artinian, then it is isoartinian. However, the converse does not satisfy in general. For example, $\mathbb{Z}_{\mathbb{Z}}$ is isoartinian, but not an Artinian module.

Proposition 2.2.0.1. *(Facchini & Nazemian (2016), Lemma 2.1) M is an isoartinian module if and only if for each non-empty set F of submodules of M, there is a submodule N of M in F such that, for each $N' \leq N$, if N' is in F, then $N \cong N'$. Identically, M is an isoartinian module if and only if for each non-empty chain C of submodules of M, there is a submodule N of M in C such that, for each $N' \leq N$, if N' is in C, then $N \cong N'$.*

Definition 2.2.2. *(Facchini & Nazemian (2017), Definition 2.1) For any ring R, let ζ be the class of all Artinian R-modules. Let ζ_0 be the set that contains only the zero module, and $\zeta_1 = \zeta$. Inductively, for each ordinal number $\alpha > 1$, let ζ_α be the class of all R-modules M such that for each submodule N of M, either $N \cong M$ or $N \in \bigcup_{\beta<\alpha} \zeta_\beta$. If a module M belongs to some ζ_α, for a minimum α, M has the Artinian dimension of α, and denoted by $art.dim(M)$. If there is no such α, then M does not have an Artinian dimension.*

Theorem 2.2.0.1. *(Facchini & Nazemian (2017), Theorem 2.2) A module is isoartinian if and only if it has an Artinian dimension.*

Proof. ($\Leftarrow$) Let M be a module with Artinian dimension, then for each $N \leq N' \leq M$, $art.dim(N) \leq art.dim(N')$. Now, let us consider the descending chain $M_1 \geq M_2 \geq \cdots$ of submodules of M with $art.dim(M_1) \geq art.dim(M_2) \geq \cdots$ so that there exists an ordinal $n \geq 1$ with $art.dim(M_n) = art.dim(M_i)$ for all $i \geq n$. Now, it is enough to consider M_n with an Artinian dimension bigger than 1 since, otherwise, M_n is Artinian so is isoartinian. Let $art.dim(M_n) > 1$. Then, $M_n \cong M_i$ by definition for all $i \geq n$, so M is isoartinian.

($\Rightarrow$) Now, let M be an isoartinian module and assume by the contrary that it does not have an Artinian dimension and so is not Artinian. Hence, there can be found a submodule M_1 of M which is not isomorphic to M and does not have an Artinian dimension. Similarly, we can find a submodule M_2 of M_1 which is not isomorphic to M_1 and does not have an Artinian dimension. Continuing in this way leads to a descending chain of submodules $M_1 > M_2 > \cdots$ which is strict, where M_i does not have an Artinian dimension and M_i is not isomorphic to M_{i+1} for every positive integer i. This contradicts with M being isoartinian. □

Let M be an isosimple that is not simple right R-module for an arbitrary ring R. Then, $M \in \zeta_2$ since it is isomorphic to all its non-zero submodules. Therefore, an isosimple module that is not simple is of Artinian dimension 2.

The following result stated in Facchini & Nazemian (2016) without proof.

Proposition 2.2.0.2. *For a domain D, the followings are equivalent:*
(1) D is right isoartinian,
(2) D_D is isosimple,
(3) D is $PRID$.

Proof. We have proved for any ring R, the module R_R is isosimple if R is $PRID$, which proves $(3 \Rightarrow 2)$, and let D be a domain, and D_D is isosimple, so is cyclic. Therefore D is $PRID$ which proves $(2 \Rightarrow 3)$. By definitions of isosimple and isoartinian modules it is clear that D is right isoartinian if D_D is isosimple, which proves $(2 \Rightarrow 1)$ So, it is enough to show that D_D is isosimple if D is right isoartinian. $(1 \Rightarrow 2)$ Now, let D be a right isoartinian domain, so D_D is isoartinian. Therefore, it has an isosimple submodule, say A. Let us pick a non-zero element a of A. Then, $aD \cong A$ since A is isosimple. Then, we have $D_D \cong aD \cong A$. So, D_D is isosimple. □

Example 2.2.0.1. *As one more example of an isosimple module, let R be the ring of Gaussian integers which is denoted by $\mathbb{Z}[i]$. It is known that R is a principal ideal domain, so a principal right ideal domain. Therefore, R_R is an isosimple right module, so it is Noetherian. Since R is not of finite length, R_R cannot be Artinian by the Hopkins-Levitzki lemma. Therefore, $\mathbb{Z}[i]_{\mathbb{Z}[i]}$ is an isosimple module, but not simple.*

Theorem 2.2.0.2. *(Facchini & Nazemian (2016), Theorem 2.9) Every isoartinian module M contains an essential submodule that is a direct sum of isosimple modules.*

Proof. Let C denote the set of all families of independent isosimple submodules of M. By Zorn's Lemma, C has a maximal member $T = \{ W_\alpha \mid \alpha \in I \}$. Since every non-zero submodule of M contains an isosimple module, from the maximality of T it follows that $W = \oplus_{\alpha \in I} W_\alpha$ is essential in M. □

2.3 Isonoetherian Modules

Definition 2.3.1. *Let R be a ring and M be a right R-module. M is called* ***isonoetherian*** *if, for every ascending chain $M_1 \leq M_2 \leq \cdots$ of submodules of M, there can be found an index $n \geq 1$ such that $M_n \cong M_i$ for every $i \geq n$.*

Proposition 2.3.0.1. *All submodules of an isonoetherian (resp., isoartinian) module are also isonoetherian (resp., isoartinian).*

Proof. Let M be an isonoetherian module and N be a non-zero submodule of M. Assume by the contrary that, N is not isonoetherian, so there exists an ascending chain $N_1 \leq N_2 \leq N_3 \leq \cdots$ of submodules of N which does not stop up to isomorphism. However, since $N \leq M$, this chain is also an ascending chain of submodules of M, which is isonoetherian which leads to a contradiction. Therefore, all submodules of an isonoetherian module are also isonoetherian. □

Proposition 2.3.0.2. *(Facchini & Nazemian (2016), Proposition 5.1) All isonoetherian modules have finite Goldie dimension.*

Proof. Let M be an isonoetherian module. Assume by the contrary that M has not finite Goldie dimension, thus it contains a direct sum of infinitely many non-zero submodules M_i. Now, take a non-zero element a_i in M_i for each i. Then, the infinite direct sum of a_iR is an isonoetherian module. Now, let us consider the ascending chain

$$a_1R \leq a_1R \oplus a_2R \leq a_1R \oplus a_2R \oplus a_3R \leq \cdots$$

Then, there exists $n_1 \geq 1$ such that

$$a_1R \oplus .... \oplus a_{n_1}R \cong a_1R \oplus ... \oplus a_{n_1}R \oplus a_{n_1+1}R$$

so that there exists $1 \leq j \leq n_1$ such that a_jR is not Noetherian. Set $b_1 := a_j$. Similarly, $\oplus_{i=n_1+1}^{\infty} a_iR$ is isonoetherian so that there exists $j \geq n_1 + 1$ such that a_jR is not Noetherian. By setting $b_2 := a_j$ and continuing in this manner, we get $b_1, b_2, \cdots$ where

$\oplus_{i=1}^{\infty} b_i R$ is isonoetherian and every $b_i R$ is not Noetherian. Let K_i be an infinitely generated submodule of $b_i R$ for every index $i \geq 1$, and let us consider the ascending chain

$$K_1 \leq b_1 R \leq b_1 R \oplus K_2 \leq b_1 R \oplus b_2 R \leq b_1 R \oplus b_2 R \oplus K_3 \leq \cdots$$

of submodules of $\oplus_{i=1}^{\infty} b_i R$. Then, we can find $n \geq 1$ satisfying

$$b_1 R \oplus \cdots \oplus b_n R \oplus K_{n+1} \cong b_1 R \oplus \cdots \oplus b_{n+1} R.$$

which implies that K_{n+1} is finitely generated implying a contradiction. So, M has a finite Goldie dimension. □

The following corollary is given in Facchini & Nazemian (2016) without proof. So, we will prove this corollary.

Corollary 2.3.0.1. *(Facchini & Nazemian (2016), Corollary 5.3) Every right isonoetherian domain is a right Ore domain.*

Proof. Let D be a right isonoetherian domain, then D_D has finite Goldie dimension by the previous theorem. So, by Theorem 1.2.6.5, it can be concluded that D is a right Ore domain. □

Proposition 2.3.0.3. *(Facchini & Nazemian (2017), Proposition 2.8) For a module M of finite Artinian dimension, the following are equivalent :*

1. *M satisfies the ACC on its Artinian submodules.*
2. *M is isonoetherian.*

Proof. (2 $\Rightarrow$ 1) Let $M_1 \lneqq M_2 \lneqq \cdots$ be an ascending chain of Artinian submodules of M. Then, there can be found an index n satisfying $M_n \cong M_{n+1} \cong \cdots$. But, M_{n+1} is Artinian and it cannot be isomorphic to a proper submodule which contradicts the fact that M is of finite Artinian dimension, i.e., M is isoartinian.

(1 $\Rightarrow$ 2) Now, let us assume that M satisfies ACC on its Artinian submodules, and contrarily assume that M is not isonoetherian. So, we can find an ascending chain

$$M_1 \leq M_2 \leq \cdots$$

such that $M_i \ncong M_{i+1}$ for all $i \geq 1$. If M_1 is not Artinian, the Artinian dimension of it must be bigger than 1. Therefore, we get, for all $n \geq 1$, $art.dim(M_n) > n$, which contradicts the fact that M is of finite Artinian dimension. □

2.4 Isoartinian and Isonoetherian Rings

Definition 2.4.1. *For any ring R, if R_R is isoartinian (isonoetherian, respectively), then R is called* ***right isoartinian*** *(isonoetherian, respectively).*

Every Noetherian ring is isonoetherian, but in general the converse does not hold.

Example 2.4.0.1. *Let D be a discrete valuation ring (DVR) that is not a field and Q be the field of fractions of D. Then, the following matrix ring T is not right Noetherian but is right isonoetherian ring by Facchini & Nazemian (2016), Example 2.6(3)*

$$T = \begin{pmatrix} D & Q \\ 0 & D \end{pmatrix}$$

It is well known that a right Artinian ring is right Noetherian but in general the converse does not hold. But, there are right isoartinian rings which are not right isonoetherian.

Example 2.4.0.2. *(Facchini & Nazemian (2019), Remark 3.15) Let k be a field and V a vector space over k of countable dimension. If*

$$R = \begin{pmatrix} k & V \\ 0 & k \end{pmatrix}$$

then, R is right isoartinian by Lam (2001), Proposition 1.17. Since R_R is not of finite uniform dimension (or Goldie dimension) because of V, then R is not right isonoetherian by the following theorem.

Theorem 2.4.0.1. *(Facchini & Nazemian (2016), Proposition 5.1) Each isonoetherian module has finite Goldie dimension.*

Proof. Let M be an isonoetherian module. Assume by the contrary that it has not finite Goldie dimension so that it has a submodule which is of the form $\oplus_{i=1}^{\infty} M_i$, where each M_i is non-zero. Let us take a non-zero element $a_i \in M_i$ for every i, so $\oplus_{i=1}^{\infty} a_i R$ is an isonoetherian module. Then, let us consider the ascending chain $a_1R \leq a_1R \oplus a_2R \leq a_1R \oplus a_2R \oplus a_3R \leq \cdots$ So, it can be found $n_1 \geq 1$ where $a_1R \oplus \cdots \oplus a_{n_1}R \cong a_1R \oplus \cdots \oplus a_{n_1}R \oplus a_{n_1+1}R$. Therefore, for some $1 \leq j \leq n_1$, a_jR is not Noetherian. By setting $b_1 := a_j$ and then setting $b_2 := a_j$, we get submodules which are not Noetherian for some index. So, by continuing in this manner, we get a sequence $b_1, b_2, b_3, \cdots$ where, $\oplus_{i=1}^{\infty} b_i R$ is isonoetherian but each b_iR is not Noetherian. Now, for each i, let K_i be a infinitely generated submodule of b_iR and let us consider the ascending chain $K_1 \leq b_1R \leq b_1R \oplus K_2 \leq b_1R \oplus b_2R \leq b_1R \oplus b_2R \oplus K_3 \leq \cdots$ of submodules of $\oplus_{i=1}^{\infty} b_i R$. So, it can be found an index n where, $b_1R \oplus \cdots \oplus b_nR \oplus K_{n+1} \cong b_1R \oplus \cdots \oplus b_{n+1}R$ implying that K_{n+1} is finitely generated which leads to a contradiction. Therefore, each isonoetherian module has a finite Goldie dimension. □

Proposition 2.4.0.1. *(Facchini & Nazemian (2016), Proposition 4.1) $R := \prod_{i \in I} R_i$ is right isoartinian (isonoetherian, respectively) if and only if I is finite and each R_i is right isoartinian (isonoetherian, respectively).*

Proof. Let $R := \prod_{i \in I} R_i$ be a finite direct product of rings. Let $i_0, i_1, \cdots$ be an infinite sequence of distinct elements of I. For all $n \geq 0$, let $T_n := \prod R_{i_t}$ for all $t \geq n$. The ideals then form a strictly descending chain, which are pairwise non-isomorphic as R-modules due to the fact that they have different annihilators: $T_0 \supsetneq T_1 \supsetneq \cdots \supsetneq T_{n-1} \supsetneq T_n \supsetneq \cdots$ Therefore, R cannot be isoartinian, so I must be finite.

Conversely, let R_{i_j}'s be the ideals of R_i's, respectively, where $j \in J$, J is a finite index set. Since all R_i's are right isoartinian, all R_{i_j}'s must also be right isoartinian. Let us consider a descending chain of ideals of $\prod_{i \in I} R_i$

$$R_1 \times R_2 \times \cdots \times R_n \geq R_{1_{j_1}} \times R_{2_{j_2}} \times \cdots \times R_{n_{j_n}} \geq \cdots$$

Since all R_i's are isoartinian, there is a positive integer m such that

$$R_{1_{j_1}} \times \cdots \times R_{m_{j_m}} \cong R_{1_{t_1}} \times \cdots \times R_{t_{j_t}} for\ all\ t \geq m$$

Therefore, $\prod_{i \in I} R_i$ is right isoartinian. □

Proposition 2.4.0.2. *(Facchini & Nazemian (2016), Proposition 4.2) For a right isoartinian ring, the intersection of the annihilators of the isosimple R-modules is zero if and only if R is semiprime.*

Proof. Let I be the intersection of the annihilators of all isosimple R-modules. Now, assume contrarily, I is non-zero and R is semiprime. Since I_R is isoartinian, then it contains an isosimple right ideal of R, say C. Therefore, $C^2 = 0$ since $C \leq I$, so $C = 0$ since it is an isosimple right ideal of R, a contradiction. Now, suppose $I = 0$ and let K be an ideal with $K^2 = 0$. Take any isosimple module M. If $MK = 0$, by assumption $K \leq I = 0$, and we are done. Else, $MK \cong M$ so that $0 = MK^2 \cong MK \cong M$, a contradiction. So, R is semiprime. □

Lemma 2.4.0.1. *(Facchini & Nazemian (2016), Lemma 4.3) If a ring R is semiprime and right isoartinian, then it is right non-singular.*

Proof. Let R be a semiprime and right isoartinian ring and contrarily assume R is not non-singular. Since a right isoartinian ring has an isosimple right ideal, $Z(R_R)$ has an isosimple right ideal, say I. If $I^2 = 0$, then $I = 0$ since R is semiprime, but by definition, I must be non-zero. So $I^2 \neq 0$, and there can be found an element a in I such that $aI \neq 0$. So, $I \cong aI \cong I/(r.ann(a) \cap I)$. Since I is Noetherian, $r.ann(a) = 0$, a contradiction because $r.ann(a)$ is essential. □

Lemma 2.4.0.2. *(Facchini & Nazemian (2016), Lemma 4.4) For a right non-singular ring R and an isosimple right ideal S of R, if I_S is the sum of all right ideals of R isomorphic to S, then I_S is a two-sided ideal of R.*

Proof. We claim that $xJ \leq I_S$ for all x in R, where $J \cong S$. If $xJ = 0$, then $xJ \leq I_S$. Let xJ be non-zero. Then $xJ \cong J/(r.ann(x) \cap J)$ since J is an isosimple right ideal of R. Since R is non-singular and J is uniform, $r.ann(x) \cap J$ must be 0, so $xJ \cong J \cong S$. Therefore, $xJ \leq I_S$. □

If I is an ideal in a semiprime ring R, then $r.ann(I) = l.ann(I) = ann(I)$. By *annihilator ideal*, we mean an ideal of form $ann(I)$ for some ideal I of R.

Lemma 2.4.0.3. *(Facchini & Nazemian (2016), Lemma 4.5) If a ring R is semiprime and right isoartinian, then it satisfies the ascending chain condition on annihilator ideals.*

Proof. The proof follows from Facchini & Nazemian (2016), Theorem 4.6 and Lam (1999), 11.13. □

Theorem 2.4.0.2. *(Facchini & Nazemian (2016), Theorem 4.6) For a semiprime and right isoartinian ring R, the following are equivalent:*

1. R is right Noetherian.

2. R is right Goldie.

3. R_R has a finite Goldie dimension.

Proof. (1 $\Rightarrow$ 2) This follows just by the definitions of right Goldie ring and finite uniform dimension.

(2 $\Rightarrow$ 3) Now, assume that R is right Goldie. By definition, it has a finite Goldie dimension, i.e., R_R has a finite Goldie dimension.

(3 $\Rightarrow$ 1) This follows by Facchini & Nazemian (2017), Theorem 4.10. □

Lemma 2.4.0.4. *(Facchini & Nazemian (2016), Lemma 4.10) For any commutative ring R, the following hold:*
(1) If R is isoartinian, then it satisfies the ACC (DCC) on annihilators.
(2) If R is self-injective, then it is Artinian if and only if it is isoartinian.

Proof. (1) Let $I_1 \leq I_2 \leq \cdots$ be an ascending chain of R that are annihilators of subsets of R. So, there is a descending chain $ann(I_1) \geq ann(I_2) \geq \cdots$. Since R is isoartinian, then there is an index n such that $ann(I_i) \cong ann(I_n)$ for every $i \geq n$. Then, by taking the annihilators, we have that $I_i = I_n$ for $i \geq n$.
(2) Let R be a self-injective ring. If R is Artinian, then it is already isoartinian. Now, it is known that a right self-injective ring with the ACC on the right annihilators is a Quasi-Frobenius ring. Therefore, R is Artinian by the definition of Quasi-Frobenius ring. □

We now give some facts about Hilbert's Basis Theorem.

Theorem 2.4.0.3. *(Hilbert's Basis Theorem) If R is a right Noetherian ring, then the polynomial ring $R[x]$ is also a right Noetherian ring.*

We know that if R is right Artinian, $R[x]$ need not be right Artinian.

Example 2.4.0.3. *Let F be a field. Then it's Artinian. However, the descending chain $(x) \supsetneqq (x^2) \supsetneqq (x^3) \supsetneqq \cdots$ of ideals of $F[x]$ does not stop. Therefore, $F[x]$ is not Artinian.*

Similarly, the Hilbert's Basis Theorem holds for the isonoetherian case by Prakash & Chaturvedi (2019), Theorem 2.1, but does not need to hold for the isoartinian case.

Example 2.4.0.4. *Let us consider the ring of integers $\mathbb{Z}$, which is isosimple so isoartinian. We claim that $\mathbb{Z}[x]$ is not isoartinian. We know by Proposition 2.2.0.2, a domain D is right isoartinian if and only if D is $PRID$. Since the domain $\mathbb{Z}[x]$ is not $PRID$, it cannot be isoartinian.*

However, if R is a field, then $R[x]$ is a PID. Therefore, $R[x]$ is a right isoartinian ring.

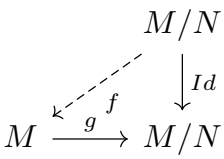
2.5 Co-isosimple Modules

In this section, co-isosimple modules and their endomorphism rings will be studied. Co-isosimple modules will be compared with simple and isosimple modules.

Definition 2.5.1. *If a non-zero module M is isomorphic to all its non-zero quotients, then it is called* ***co-isosimple****.*

Proposition 2.5.0.1. *(Alvarado-García et al. (2020), Proposition 2.6) For a left projective module M, M is simple if and only if it is co-isosimple.*

Proof. Let M be a projective module and take a proper submodule N of M. Since M is co-isosimple, it is isomorphic to quotient M/N so that M/N is also a projective module. Therefore, it can be found a homomorphism f from M/N to M which leads to $M = N \oplus Im(f)$, where the natural epimorphism g is from M to M/N.



Then, $M = Im(f)$ since M is hollow, that is, every proper submodule is small, so that $N = \{0\}$. Therefore, M is simple. □

Corollary 2.5.0.1. *(Alvarado-García et al. (2020), Corollary 2.7) A ring R is a division ring if and only if it is co-isosimple as a left module over itself.*

Proof. ($\Rightarrow$) Let R be a division ring, so is simple. Therefore, ${}_RR$ is simple and so is a co-isosimple left module over itself.

($\Leftarrow$) Let ${}_RR$ be a co-isosimple module. Then, it is Artinian and so that R is an Artinian ring. Then, each non-zero divisor in R is a unit which implies that R is a division ring.

□

Proposition 2.5.0.2. *(Alvarado-García et al. (2020), Proposition 2.8) Let M be an R-module. If $End(M)$ is a VNR ring, then M is simple if and only if it is a co-isosimple module.*

Proof. ($\Leftarrow$) Let M be a co-isosimple module and E denotes the endomorphism ring of M which is $Von\text{-}Neumann\ regular$. Let us take a proper submodule N of M, then we know that M is isomorphic to M/N, i.e., it can be found an isomorphism f from M to M/N. So, there exists $g \in E$ such that $ker(g) = N$ which implies N is a direct summand of M. Therefore, $N = \{0\}$ since M is hollow and, thus M is simple. □

Proposition 2.5.0.3. *(Alvarado-García et al. (2020), Proposition 2.9) All non-zero endomorphisms of co-isosimple modules are epimorphisms.*

Proof. Let M be a co-isosimple module and E denotes the endomorphism ring of M. Let f be a homomorphism from M to M itself, so it can be found an isomorphism $g : M \to f(M)$ which leads to an injective homomorphism ig for an inclusion mapping i. Hence, $M = f(M)$, that is, f is an automorphism so that it is an epimorphism. □

Corollary 2.5.0.2. *(Alvarado-García et al. (2020), Corollary 2.10) The endomorphism ring of a co-isosimple module is a local domain.*

Proof. Let M be a co-isosimple module and E denotes the endomorphism ring of it. By the above proposition, E is a domain. Since M is co-isosimple, then it is both Artinian and uniform module and so E is a local domain since, for an Artinian uniform module M, the endomorphism ring of M, $End(M)$ is a local ring. □

2.6 Nei/Ei Artinian and Noetherian Modules

Nei/Ei Artinian and Noetherian modules are defined in Chaturvedi & Prakash (2021) as a generalization of iso version of Artinian and Noetherian modules. Furthermore, they give some examples and some characterizations of these terms.

Definition 2.6.1. *An R-module M is called **nei-noetherian** if for all ascending chains $M_1 \leq M_2 \leq \cdots$ of non-essential submodules of M, there can be found an index n such that $M_i \cong M_n$, for each $i \geq n$.*

Nei-artinian modules are also defined as a dual version of nei-noetherian modules. By definitions of these notions, it can be said that for a nei-noetherian (nei-artinian) module M, all submodules of M are also nei-noetherian (nei-artinian).

Example 2.6.0.1. *(Chaturvedi & Prakash (2021), Example 2.3) (1) All isonoetherian (isoartinian) modules are nei-noetherian (nei-artinian), but the converse does not need to be held.*
(2) Uniform modules are both nei-artinian and nei-noetherian.

Theorem 2.6.0.1. *(Chaturvedi & Prakash (2021), Theorem 2.6) For an R-module M, M is nei-noetherian (nei-artinian) if and only if all non-essential submodules of M are isonoetherian (isoartinian).*

Proof. ($\Rightarrow$) Let M be a nei-noetherian module and N be a non-essential submodule of M. Let $N_1 \leq N_2 \leq \cdots$ be an ascending chain of submodules of N. Then, since N is non-essential, all N_i's are also non-essential submodules of M. Therefore, it can be found an index n satisfying $N_i \cong N_n$, for every $i \geq n$. Then, N is isonoetherian.
($\Leftarrow$) Let all non-essential submodules of M are isonoetherian and $M_1 \leq M_2 \leq \cdots$ be an ascending chain of non-essential submodules of M. Therefore, since M_1 is non-essential, then it is isonoetherian. So, it can be found an index n satisfying $M_i \cong M_n$, for every $i \geq n$. Therefore, M is nei-noetherian. □

Proposition 2.6.0.1. *(Chaturvedi & Prakash (2021), Proposition 2.13) All nei-artinian modules contain an isosimple module.*

Proof. Let M be a nei-artinian module with $M_1 \geq M_2 \geq \cdots$ descending chain of non-essential submodules of M. So, it can be found an index n satisfying $M_i \cong M_n$, for each $i \geq n$. Since each non-essential submodule of nei-artinian modules is isoartinian by Chaturvedi & Prakash (2021), M_n is isoartinian so that it has an isosimple submodule. □

Definition 2.6.2. *A right R-module M is called **ei-noetherian** if, for all ascending chains $M_1 \leq M_2 \leq \cdots$ of essential submodules of M, there can be found an index n such that $M_i \cong M_n$, for each $i \geq n$.*

Ei-artinian modules are also defined as a dual version of ei-noetherian modules.

Proposition 2.6.0.2. *(Chaturvedi & Prakash (2021), Remark 3.2) A uniform module is isonoetherian if and only if it is ei-noetherian.*

Proof. ($\Rightarrow$) Let M be an isonoetherian module. So, it is ei-noetherian just by the definitions.
($\Leftarrow$) Let M be a uniform and ei-noetherian module. Since every submodule of a uniform module is essential, then by the definitions, M is isonoetherian. □

Proposition 2.6.0.3. *(Chaturvedi & Prakash (2021), Proposition 3.7) An ei-noetherian (ei-artinian) and nei-noetherian (nei-artinian) module M is isonoetherian (isoartinian).*

Proof. Let $M_1 \leq M_2 \leq \cdots$ be an ascending chain of submodules of M. If each M_i in the chain is non-essential or each one is essential, then there is nothing to prove since M is already both nei-noetherian and ei-noetherian. So, suppose there is an index n where M_n is essential so that for each $i \geq n$, M_i is essential. Since M is ei-noetherian, there can be found an index k such that $M_k \cong M_j$ for each $j \geq k \geq n$. Thus, M is isonoetherian. □

Now, we collect related open problems, some of which are given in Facchini & Nazemian (2019):
(1) Characterize the rings with an isosimple faithful right module.
(2) Is a direct sum of isosimple modules an isoartinian module?
(3) Is a semiprime right isoartinian ring right Noetherian?
(4) Characterize the rings over which simple and isosimple modules coincide.
(5) Is the ring whose cyclic modules are all isoartinian right Noetherian?
(6) Is the endomorphism ring of an isosimple right module a $PRID$?

CHAPTER THREE
RADICAL AND SOCLE OF MODULES WITH RESPECT TO ISOMORPHISM

In this chapter, "iso" versions of the radical of a module and the socle of a module will be examined.

In Section 3.1, *isoradical* and *I-series* of modules will be defined. They will be characterized and connected to isoartinian, isonoetherian and, isosimple modules. In Section 3.2, the *isosocle* of modules will be introduced and be compared with the standard socle of modules. Finally, in Section 3.3, the *co-isoradical* of a module will be mentioned.

3.1 Isoradical of a Module

Definition 3.1.1. *The (right)* ***isoradical*** I *-* $rad(R_R)$ *of a ring* R *is the intersection of the annihilators of all isosimple right* R*-modules and, the* $isoradical$ I*-*$rad(M)$ *of a module* M *is defined as the intersection of the kernels of all homomorphisms* $h : M \to I$*, where* I *is an isosimple module.*

Lemma 3.1.0.1. *For any ring* R*,*

$$I\text{-}rad(R_R) = \bigcap\{K | K \ is \ a \ right \ ideal \ of \ R \ and \ R/K \ is \ an \ isosimple \ module\}.$$

Lemma 3.1.0.2. *(Facchini & Nazemian (2017), Lemma 3.2)*
(1) If $R = R_1 \times R_2 \times \cdots \times R_n$ *is a direct product of rings, then*

$$I\text{-}rad(R) = I\text{-}rad(R_1) \times I\text{-}rad(R_2) \times \cdots \times I\text{-}rad(R_n).$$

(2) For any ring R*,* $I\text{-}rad(R/(I\text{-}rad(R_R))) = 0$*.*

Lemma 3.1.0.3. *For any R-module M, $I\text{-}rad(M) \leq rad(M)$.*

Proof. Let M be an R-module and a non-zero $x \in I\text{-}rad(M)$ so that x annihilates all isosimple submodules of M. Since each simple module is also an isosimple module, then x annihilates all simple submodules of M. Therefore, $x \in rad(M)$. Thus, $I\text{-}rad(M) \leq rad(M)$. □

Definition 3.1.2. *The **prime radical** of a ring is defined to be the intersection of all prime ideals in the ring, denoted by $P(R)$.*

Proposition 3.1.0.1. *For any ring R, $P(R) \leq I\text{-}rad(R_R)$.*

Proof. Let us take a non-zero element $x \in P(R)$, then it means that for some $n \geq 1$, $x^n = 0$ since it is in prime radical of R. Therefore, the element x also annihilates all of the isosimple right R-modules. So, it is an element of the intersection of the annihilators of all isosimple right R-modules so that $x \in I\text{-}rad(R_R)$. Then, for any ring R, $P(R) \leq I\text{-}rad(R_R)$. □

Note that a one-sided ideal I of a ring R is left *T-nilpotent* if, for every sequence $a_1, a_2, \cdots$ of elements of I, there exists a positive integer n such that $a_1 a_2 \cdots a_n = 0$. So, every left T-nilpotent ideal is nil.

Theorem 3.1.0.1. *(Facchini & Nazemian (2019), Theorem 4.6) Let R be a ring whose cyclic modules are all isoartinian. Then,*
(1) $I\text{-}rad(R_R) = P(R)$.
(2) $P(R)$ is left T-nilpotent.
(3) If $P(R)$ is right T-nilpotent, then it is nilpotent.

Proof. (1) By the previous proposition, we can say that $P(R) \leq I\text{-}rad(R_R)$. So, it is enough to show that $I\text{-}rad(R_R) \leq P(R)$. Let P be a prime ideal of R so that R/P is an isoartinian R-module. Therefore, it has an isosimple R-module M, say $M = K/P$, so $P \leq ann(M)$. Now, let x be an element of $ann(M)$, then $KxR \leq P$. Then, $x \in P$

so that $ann(M) \leq P$ implying that $P = ann(M)$. Thus, each prime ideal of R is the annihilator of an isosimple R-module, that is, $I\text{-}rad(R_R) = P(R)$.

(2) Assume contrarily, let $P(R)$ is not left T-nilpotent. Let us take $a_1, a_2, a_3, \cdots \in P(R)$ satisfying that $a_1a_2a_3\cdots a_n \neq 0$, for each n. Now, let J be an ideal of R, maximal among the ideals I with $a_1a_2a_3\cdots a_n \notin I$, for each n. Let K/J be an isosimple ideal of R/J. Then, it can be found an index m satisfying $x = a_1a_2\cdots a_m \in K$. Therefore, $a_{m+1} \in ann(K/J)$ because $a_{m+1} \in P(R) = I\text{-}rad(R_R)$ by (1), but this implies that $xa_{m+1} \in J$ which leads to a contradiction. So, $P(R)$ is left T-$nilpotent$.

(3) Let us $J :=P(R)$. Consider the ascending chain $r.ann(J) \leq r.ann(J^2) \leq \cdots$ So we have the descending chain $l.ann(r.ann(J)) \geq l.ann(r.ann(J^2)) \geq \cdots$ of ideals of R. Then, it can be found an index n satisfying $l.ann(r.ann(I^n)) \cong l.ann(r.ann(I^i))$ as an R-module for each $i \geq n$. Therefore, for each $i \geq n$, $r.ann(I^n) = r.ann(I^i)$. Now, our claim is that $I^{2n} = 0$, but if not, for some $a_1 \in I^n$, then we would get that $I^n a_1 \neq 0$. So, $I^{2n}a_1 \neq 0$ and then, for some $a_2 \in I^n$, we get that $I^n a_2 a_1 \neq 0$. If we continue in this manner, it would contradicts with the fact that I is right T-nilpotent. Therefore, $P(R)$ is nilpotent if it is right T-nilpotent. □

Lemma 3.1.0.4. *(Facchini & Nazemian (2017), Lemma 4.2) For a semiartinian module M, $rad(M) = I\text{-}rad(M)$.*

Proof. Let M be a semiartinian module. Now, it is enough to show that $rad(M) \leq I\text{-}rad(M)$ since $I\text{-}rad(M) \leq rad(M)$. For an isosimple module I, let f be a non-zero homomorphism from M to I. Then, I is also semiartinian since $I \cong M/ker(f)$ implying that I is simple. Therefore, $rad(M) = I\text{-}rad(M)$. □

Definition 3.1.3. *A chain $0 = P_0 < P_1 < \cdots < P_n = M$ of submodules of M is called an **I-series** for M if $P_i \ncong P_{i+1}$ for every i and, for each submodule K of M with $P_i \leq K \leq P_{i+1}$, we have that either $K \cong P_{i+1}$ or $K \cong P_i$. If a module M has an I-series, then it is called a module **of finite I-length**. $0 = P_0 < P_1 < P_2 < \cdots < P_n = M$ is called the I-length series for M and the **I-length** of a module M is the least such*

n and is denoted by I-$length(M) = n$.

Proposition 3.1.0.2. *(Facchini & Nazemian (2017), Lemma 5.3) If a module M is both isoartinian and isonoetherian, then it has finite I-$length$.*

Proof. Let M be both isoartinian and isonoetherian module, so it has an isosimple submodule I_1. Let R_1 denotes the set of all submodules N of M such that $N > I_1$ and $N \ncong I_1$. R_1 must be non-empty since, if it is empty, then M has already I-$length$ 1. Now, it can be found $I_2 \in R_1$ which is minimal in R_1 up to isomorphism. In this way, we can find a chain $I_1 < I_2 < \cdots$ of submodules of M, where $I_n \ncong I_{n+1}$ for every index n, and for each $L \leq M$ such that $I_n \leq L \leq I_{n+1}$, either $L \cong I_{n+1}$ or $L \cong I_n$. So, the chain must stop since M is isonoetherian by hypothesis. Therefore, $I_n = M$ for some index n. Thus, M has finite I-$length$. □

Proposition 3.1.0.3. *(Facchini & Nazemian (2017), Remark 5.4) M is isosimple if and only if I-$length(M) = 1$*

Proof. ($\Rightarrow$) Let M be an isosimple module. So it is isomorphic to all its submodules which are non-zero. Let us consider the chain $0 = P_0 < P_1 < P_2 < \cdots$ of submodules of M. So since all submodules are isomorphic to M, the chain must stop at $n = 1$. Therefore, the I-$length$ of M is 1.
($\Leftarrow$) Let I-$length(M) = 1$. Now, assume by the contrary that M has a non-zero proper submodule which is not isomorphic to itself. So, there exists a non-zero submodule that is not isomorphic to M, say P_1. Then, by considering the chain above, there must be a submodule P_k such that $P_0 < P_k < P_1$ where either $P_k \cong P_0$ or $P_k \cong P_1$. So, the I-$length$ of M must be bigger than 1, which is a contradiction. Therefore, M must be isosimple. □

3.2 Isosocle of a Module

Definition 3.2.1. *For any R-module M, **I-soc(M)** is defined as*

$$\sum\{h(U)|h : U \to M \; is \; a \; homomorphism \; for \; some \; isosimple \; right \; R\text{-}module \; U\}.$$

Theorem 3.2.0.1. *(Facchini & Nazemian (2017), Remark 4.3) Let M be a right R-module, and M_1, M_2 be submodules of M. Then:*
(1) $soc(M) \leq I\text{-}soc(M)$. The equality does not necessarily hold.
(2) If $M = M_1 \oplus M_2$, then $I\text{-}soc(M) = I\text{-}soc(M_1) \oplus I\text{-}soc(M_2)$.
(3) If $I\text{-}soc(M) = M$ and M is projective, then M is a direct summand of a direct sum of isosimple modules. Therefore, the isoradical of M is 0.
(4) If $I\text{-}soc(M) = M$, then M is finitely generated if and only if it is Noetherian.

Proof. (1) As an explanation, it can be said that since every simple module is also an isosimple module, $soc(M) \leq I\text{-}soc(M)$. The equality does not need to hold in general. For example, $I\text{-}soc(\mathbb{Z}_{\mathbb{Z}}) = \mathbb{Z}$ and $soc(\mathbb{Z}_{\mathbb{Z}}) = 0$.

(2) As a proof for the second statement, since if $h : U \to M_i$ is a homomorphism with U isosimple, then there exists homomorphisms $h' : U \to M_i$ and $i : M_i \to M_1 \oplus M_2$, where i is the inclusion map. Then, $I\text{-}soc(M_1) \oplus I\text{-}soc(M_2) \leq I\text{-}soc(M)$. For the reverse implication, for each homomorphism $h : U \to M_1 \oplus M_2$ with U isosimple, then there exist homomorphisms $h' : U \to M_1 \oplus M_2$ and $P_i : M_1 \oplus M_2 \to M_i$, where P_i is the projection map. Then, $I\text{-}soc(M) \leq I\text{-}soc(M_1) \oplus I\text{-}soc(M_2)$. Therefore, $I\text{-}soc(M) = I\text{-}soc(M_1) \oplus I\text{-}soc(M_2)$.

(3) It does not need to hold for a non-projective module M, and as an example, for the non-projective module $\mathbb{Z}/4\mathbb{Z}$, $I\text{-}rad(\mathbb{Z}/4\mathbb{Z}) = 2\mathbb{Z}$ which is non-zero.

(4) As a proof for the forth statement, if M is Noetherian, then it is clearly finitely generated. So, it is enough to show that if M is finitely generated and $I\text{-}soc(M) = M$, then M is Noetherian. Then, by our hypothesis, $M = \sum\{h(U)|h : U \to M for \; some \; isosimple \; U\}$ is a finite sum. So, there exists

an epimorphism $\phi : \bigoplus_{i=1}^{n} U_i \to M$ given by $\phi(u_1, \cdots, u_n) = \sum_{i=1}^{n} h_i(u_i)$, where $h_i : U_i \to M$ are homomorphisms with U_i isosimple. Therefore, $M = Im(\phi)$ and since U_i's are isosimple, they are Noetherian and so $\bigoplus_{i=1}^{n} U_i$ is also Noetherian so that its quotient $Im(\phi)$ is Noetherian, i.e., M is Noetherian.

□

Proposition 3.2.0.1. *(Facchini & Nazemian (2017), Lemma 4.4) For a non-singular R-module M, M is the sum of its isosimple submodules if and only if $I\text{-}soc(M) = M$.*

Proof. ($\Rightarrow$) Assume that $I\text{-}soc(M)$ equals M. Since it is a non-singular module, then $h(U) \cong U$ for all U and then M is a sum of isosimple modules.
($\Leftarrow$) Now, let M be a non-singular module such that $M = I_1 + I_2 + \cdots$, where all $I_i's$ are isosimple modules of M. It is already known that $I\text{-}soc(M) \leq M$, we must show that $M \leq I\text{-}soc(M)$. Since it is non-singular, for all U, $h(U) \cong U$, so for any element in M is also in $I\text{-}soc(M)$ by definition of iso-socle of a module which implies that $M \leq I\text{-}soc(M)$. □

Proposition 3.2.0.2. *(Facchini & Nazemian (2017), Lemma 4.7) Suppose $M = S_1 + S_2 + \cdots + S_n$, where the modules S_i are isosimple and pairwise non-isomorphic. Then the sum is a direct sum.*

Proof. This will be proven by induction on n. Let us consider the base case $n = 1$, then $M = S_1$, so the base case is done. Now, assume the induction hypothesis for $n-1$. Assume by the contrary that the sum is not direct for $M = S_1 + S_2 + \cdots + S_n$. So, we can find $a_i \in S_i$, for $i \leq n$, not all zero, where the sum of $a_i's$ is equal to 0. Now we claim that it can be found an isomorphism between a_1R to a_2R for all $r \in R$, which maps a_1r to a_2r so that S_1 and S_2 are isomorphic which will give us a contradiction. Now, it is enough to show that for all r in R, $a_1r = 0$ if and only if $a_2r = 0$. Let $a_1r = 0$, then $(a_2)r + (a_3)r + \cdots + (a_n)r = 0$ since the sum of $a_i's$ are 0. So, because of the inductive hypothesis, a_2r must be 0. Similarly, if a_2r is 0, a_1r must equal 0 which implies that non-zero submodules a_1R of S_1 and a_2R of S_2 are isomorphic, which contradicts that all $S_i's$ are pairwise non-isomorphic. □

Theorem 3.2.0.2. *(Facchini & Nazemian (2017), Theorem 4.9) For any ring R, the following are equivalent:*
(1) $I\text{-}soc(R) = R$.
(2) R is a sum of isosimple right ideals.
(3) For every R-module M, $I\text{-}soc(M) = M$.

Proof. The implications $(1 \Rightarrow 3)$, $(3 \Rightarrow 1)$ and $(2 \Rightarrow 1)$ comes from just by the definition of isosocle of rings and modules. So, it is enough to show $(1 \Rightarrow 2)$. So, let $I\text{-}soc(R) = R$. Then, by the Theorem 3.2.0.1, $I\text{-}rad(R_R) = 0$ and R is right Noetherian. Then, by Proposition 3.1.0.1, $P(R) = 0$, that is, R is a semiprime ring. Therefore, by Goodearl (1976), Corollary 3.32, R is non-singular and so by Proposition 3.2.0.1, R is a sum of isosimple right ideals.

□

3.3 Co-isoradical of a Module

In this section, the co-isoradical of a module will be defined and compared with radical and isoradical of a module.

Definition 3.3.1. *For any non-zero module M and any proper submodule N of it, if M/N is co-isosimple, then N is called* ***isomaximal****.*

As an example, $\mathbb{Z}_{p^n}$ is an isomaximal submodule of $\mathbb{Z}_{p^\infty}$ but is not a maximal submodule, where p is a prime number and $n \geq 1$.

Definition 3.3.2. *For a module M, the intersection of all isomaximal submodules is called left* ***co-isoradical*** *of M and denoted by $CI\text{-}rad(M)$.*

Definition 3.3.3. *A ring R is called as a left* ***Max*** *ring if every non-zero left R-module has a maximal submodule.*

Note for a left Max ring R, then $CI\text{-}rad(M) = rad(M)$ for each left R-module M. This is not true, in general. As an example, over $\mathbb{Z}$, $CI\text{-}rad(\mathbb{Z}_{p^\infty}) = 0$. However, rad($\mathbb{Z}_{p^\infty}$) = $\mathbb{Z}_{p^\infty}$.

Remark 3.3.0.1. *(Alvarado-García et al. (2020), Remark 3.11) Let $\mathcal{G}$ be the class of all co-isosimple modules. Then for every R-module M:*
$CI\text{-}rad(M) = \bigcap\{ker(p) \mid p : M \to C, C \in \mathcal{G} \text{ and } p \text{ an epimorphism.}\}$

Proposition 3.3.0.1. *(Alvarado-García et al. (2020), Proposition 3.12) Let M and N be modules and $p : M \to N$ be an epimorphism. Then, $p(CI\text{-}rad(M)) \leq CI\text{-}rad(N)$.*

Proof. Let $p : M \to N$ be an epimorphism and assume by the contrary that there exists an element in $p(CI\text{-}rad(M))$ which is not contained in $CI\text{-}rad(N)$, say y, so that $y = p(x)$ for some $x \in CI\text{-}rad(M)$. Then, it can be found $L \leq N$ as an isomaximal module that does not contain the element y. So, by the isomaximality of L, N/L is co-isosimple, then $p^{-1}(L) = ker(\pi f)$ is an isomaximal submodule of M, where $\pi : N \to N/L$ is the natural epimorphism. However, $p^{-1}(L)$ does not contain the element x, which gives a contradiction. So, $p(CI\text{-}rad(M)) \leq CI\text{-}rad(N)$. □

Proposition 3.3.0.2. *(Alvarado-García et al. (2020), Proposition 3.13) For a finitely generated R-module M, $CI\text{-}rad(M) = rad(M)$.*

Proof. Let M be a finitely generated module and $f : M \to C$ be an epimorphism for a co-isosimple module C. Then, C is finitely generated so that it has a simple quotient. Therefore, it is simple. Since co-isoradical of a module M is the intersection of $ker(f)$, where $f : M \to S$, with S is a simple module, $CI\text{-}rad(M) = rad(M)$ □

Proposition 3.3.0.3. *(Alvarado-García et al. (2020), Proposition 3.14) The following conditions are equivalent:*
(1) All co-isosimple R-modules are simple,
(2) $CI\text{-}rad(M) = rad(M)$ for every R-module M.

Proof. (1 $\Rightarrow$ 2) Let each co-isosimple R-module M be simple. So, $CI\text{-}rad(M)$ is the intersection of $ker(p)$, where $p : M \to S$, with S a simple module, so that $CI\text{-}rad(M) = rad(M)$ for each R-module M.
(2 $\Rightarrow$ 1) Let $CI\text{-}rad(M) = rad(M)$ for every R-module M and let M be a co-isosimple R-module. Then, $CI\text{-}rad(M) = 0$ and since $CI\text{-}rad(M) \leq rad(M)$, $rad(M)$ is also zero implying the existence of a maximal submodule P of M. So, the quotient M/P is simple. Since M is co-isosimple, by the definition of a co-isosimple module, $M \cong M/P$, so M is simple. □

Definition 3.3.4. *A module M is called **co-cyclic** if there is a simple module that is essential in M.*

Theorem 3.3.0.1. *(Alvarado-García et al. (2020), Theorem 3.16) The following conditions are equivalent:*
(1) R is a left V- ring,
(2) $CI\text{-}rad(M) = \{0\}$ for every R-module M,
(3) $I\text{-}rad(M) = \{0\}$ for every R-module M,
(4) For all co-cyclic R-modules are co-isosimple R-modules.

Proof. (1 $\Rightarrow$ 4) Let R be a left V-ring and M be a co-cyclic R-module. So M has a simple essential submodule, say S. Since R is a V-ring, S is injective so that it is a summand of M. As no summand can be essential unless it is already the module itself, the submodule S is equal to the module M itself. Therefore, M is a simple module so that it is co-isosimple.

(4 $\Rightarrow$ 1) Let S be a simple module which is not injective. Then, there can be found a minimal submodule N of $E(S)$ containing S properly since $E(S)$ is co-isosimple. So, $N/S \cong N$ is simple, which is a contradiction. Therefore, $S = E(S)$, that is, it is injective.

(1 $\Rightarrow$ 3) Since R is a left V-ring, then for each R-module M, $rad(M) = 0$ by Lam (1999), Theorem 3.75. Therefore, $I\text{-}rad(M) = 0$ by Lemma 3.1.0.3.

(3 $\Rightarrow$ 2) Let M be a co-isosimple module. Then, by hypothesis, it can be found a non-zero morphism α from M to I, where I is isosimple. Therefore, $\alpha(M) \cong M$ is isosimple. So, each co-isosimple module is isosimple. Thus, $CI\text{-}rad(M) \leq I\text{-}rad(M)$ for every module M, so $CI\text{-}rad(M) = \{0\}$ for every R-module M.

(2 $\Rightarrow$ 1) Let I be a proper left ideal of R. Then, since R/I is finitely generated, $rad(R/I) = 0$ by Alvarado-García et al. (2020), Proposition 3.13 so that I is an intersection of maximal left ideals. Therefore, R is a left V-ring by Lam (1999), Theorem 3.75.

□

Finally, we give related open problems, some of which are from Facchini & Nazemian (2019):

(1) Is a module of finite $I\text{-}length$ an isoartinian and isonoetherian module?

(2) Characterize the rings R such that $I\text{-}soc(R_R) = R$.

(3) Let R be a ring with $I\text{-}soc(R_R) = R$. Is R a direct sum of isosimple right ideals?

(4) Characterize the rings over which all isosimple modules are injective.

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APPENDICES

A.1: List of Symbols

R	an associative ring with the unit unless otherwise stated
$R[x]$	the ring of polynomials with indeterminate x
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$	the field of rational numbers
R-module or module	a right R-module
$\cong$	isomorphic
$\ll$	small submodule
$\leq$	submodule
$\leq_e$	essential submodule
$ker(f)$	the kernel of the homomorphism f
$Im(f)$	the image of the homomorphism f
$E(M)$	the injective hull of a module M
$End(M)$	the endomorphism ring of a module M
$soc(M)$	the socle of a module M
$rad(M)$	the radical of a module M
$Z(M)$	the singular submodule of a module M
$ann_R(M)$	the annihilator of a module M
$I-soc(M)$	the isosocle of a module M
$I-rad(M)$	the isoradical of a module M
$CI-rad(M)$	the co-isoradical of a module M