**DOKUZ EYLÜL UNIVERSITY**
**GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES**

# ON NOETHERIAN RINGS

**by**
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**İZMİR**

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**M. Sc. THESIS EXAMINATION RESULT FORM**

We have read the thesis entitled **"ON NOETHERIAN RINGS"** completed by **EVREN ZIPLAR** under supervision of **YRD.DOÇ.DR. GÖKHAN BİLHAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Evren ZIPLAR

# ON NOETHERIAN RINGS


## ABSTRACT

In this thesis;Generally, we explain Noetherian rings.We investigates non-Noetherian rings for which each proper subring is Noetherian.And we prove the most important theorems on Noetherian rings.In addition, we looks over short modules and almost short modules in the last part.



**Keywords :** Group, ring, module, Noetherian ring, Noetherian module, locally Noetherian module, short module, almost short module, coatomic module, factor module.

# NOETHER HALKALARI ÜZERİNE

## ÖZ

Bu tezde; Her öz althalkası Noetherian olan, ama kendisi Noetherian olmayan halkaları açıkladık.Ve, Noetherian halkaları ile ilgili en önemli teoremleri ispat ediyoruz. Ayrıca, son kısımda kısa ve hemen hemen kısa modülleri detaylı olarak inceliyoruz.

**Anahtar sözcükler :** Group, halka, modül, Noetherian halka, Noetherian modül, yerel Noetherian modül, kısa modül, hemen hemen kısa modül, coatomic modül, bölüm modülü.

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# CHAPTER ONE

# INTRODUCTION

## 1.1 Definitions on group , ring , module

**Definition(Group) 1.1.1.** A set $G$ together with a binary operation $*$ is called a group, if the following statements are satisfied :

a-) The binary operation $*$ is associative.

b-) There is an element $e$ in $G$ such that

$$e*x=x*e=x \quad \text{for all} \quad x\in G.$$

This element is called identity element for $(G,*)$.

c-) For each $x\in G$, there is an element $x^{-1}\in G$ with the property

$$x*x^{-1}=x^{-1}*x=e$$

The element $x^{-1}$ is the inverse of $x$ in $(G,*)$.

**Definition(Abelian group) 1.1.2.** A group $(G,*)$ is called Abelian if it's binary operation $*$ is commutative.

**Definition(Subgroup) 1.1.3.** Let $G$ be a group. Let $A$ be a subset of $G$. If $A$ together with binary operation of $G$ is also a group, then $A$ is called a subgroup of $G$. Denoted by $A\leq G$.

**Definition(Proper and Trivial Subgroups) 1.1.4.** If $G$ is a group, then the subgroup consisting of $G$ itself is the improper subgroup of $G$. All other subgroups are proper subgroups. The subgroup $\{e\}$ is trivial subgroup of $G$. All other subgroups are nontrivial.

**Definition(Normal subgroup) 1.1.5.** Let $G$ be a group and $H$ be a subgroup of $G$. Then $H$ is called a normal subgroup of $G$ if $aH = Ha$ for every $a \in G$.

**Definition(Quotient group) 1.1.6.** Let $G$ be a group and $H$ be a normal subgroup of $G$. Then the set of all left cosets $G/H$ is a group and called quotient (or factor) group of $G$. The binary operation is defined as :

$$\bullet : G/H \times G/H \to G/H$$

where $(aH)\bullet(bH) = (ab)H$.

**Definition(Divisible group) 1.1.7.** A divisible group is an Abelian group $G$ such that for any positive integer $n$ and any $g$ in $G$, there exists $y$ in $G$ such that $ny = g$.

**Note 1.1.8.** $G$ is a divisible group $\Leftrightarrow$ $nG = G$, for every $n$ in $Z^+$.

**Definition(Generating set for group) 1.1.9.** Let $G$ be a group and $X$ be a subset of $G$. The subgroup generated by $X$ is defined by the intersection of all subgroups of $G$ containing $X$, and denoted by $\langle X \rangle$. That is;

$$\langle X \rangle = \bigcap_{\substack{H_i \leq G \\ X \subseteq H_i}} H_i$$

**Definition(Finitely generated group) 1.1.10.** If a group $G$ is generated by a finite set $\{x_1, x_2, ..., x_n\}$, then $G$ is called finitely generated.

**Definition(Order of a group)1.1.11.** The number of elements in a group $G$ is called the order of the group.If the group is infinite, then we say, the order of the group is infinite.The order of a group $G$ is denoted by $|G|$.

**Definition(Order of an element)1.1.12.** Let $G$ be a group. The smallest positive integer "$n$" with $g^n = e$ is called the order of the element $g$ of $G$.

**Definition(Torsion subgroup) 1.1.13.** Let $G$ be a group and $H$ be a subgroup of $G$. If every element of $H$ is of finite order, then we say $H$ is a torsion subgroup of $G$.

**Definition(Torsion-free group) 1.1.14.** If no element of a group is of finite order, then this group is called torsion free.

**Definition(Mixed group) 1.1.15.** Let $G$ be an Abelian group. If group $G$ consist of both torsion and non-torsion elements, then group $G$ is called mixed group.

**Definition(Short exact sequence)1.1.16.** A sequence of homomorphisms

$$... \to A_{n+1} \overset{\alpha_{n+1}}{\to} A_n \overset{\alpha_n}{\to} A_{n-1} \to ...$$

is said to be *exact* if $Ker\alpha_n = \mathrm{Im}\,\alpha_{n+1}$ for each $n$.

$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ is exact if and only if $\alpha$ is injective, $\beta$ is surjective, and $Ker\beta = \mathrm{Im}\,\alpha$, so $\beta$ induces an isomorphism $C \cong B/\mathrm{Im}\,\alpha$. This can be written $C \cong B/A$ if we think of $\alpha$ as an inclusion of $A$ as a subgroup of $B$.

The exact sequence $0 \to A \to B \to C \to 0$ is called *short exact sequence.*(Vick,1973).

**Definition(Group Extension) 1.1.17.** $G'$ is an extension of $G$ if there is an exact sequence

$$0 \to H \xrightarrow{\alpha} G' \xrightarrow{\beta} G \to 0$$

The situation is sometimes described by saying that $G'$ is an extension of $G$ by $H$.

In other words; an extension of a group $G$ by a group $H$ is a group $G'$ with a normal subgroup $M$ such that $M \cong H$ and $G'/M \cong G$.(Vick,1973).

**Definition(p-primary component) 1.1.18.** Let $G$ be a finite Abelian group and let $p \in N$ be a prime.The $p$-primary component of $G$ is the subgroup of all elements whose order is a power of $p$.

**Definition(p-quasicyclic group) 1.1.19.** Let $p$ be a prime number.The $p$-quasicyclic group(or Prüfer $p$-group) is the $p$- primary component of $Q/Z$. The $p$-quasicyclic group is denoted by $Z\left(p^{\infty}\right)$ or $Z\left[p^{\infty}\right]$ and $Z\left(p^{\infty}\right) = \left\{\frac{m}{p^{i}} + Z : p \text{ is a prime}, i \in N, m \in Z\right\}$.

**Definition(Ring) 1.1.20.** Let $R$ be a non-empty set with two binary operations "$+$" and "$.$" defined on it.We say that the set $R$ together with binary operations "$+$" and "$.$" is a ring and we denote it by $(R,+,.)$ if the following statements are satisfied :

a-) $(R,+)$ is an Abelian group.

b-) " . " is associative. In other words;

$$a(bc)=(ab)c$$

where for all $a,b,c$ in $R$.

c-) " . " is distributive over " + ". In other words;

$$a.(b+c)=a.b+a.c$$
$$(b+c).a=b.a+c.a$$

where for all $a,b,c$ in $R$.(Ribenboim, 1967).

**Definition( Subring) 1.1.21.** Let $R$ be a ring and $S$ be a subset of $R$. If $S$ is also a ring with operations of $R$ ( i.e if $(R,+,.)$ is the ring, $(S,+,.)$ is also ring.), then $S$ is called subring of $R$.(Ribenboim, 1967).

**Definition(Generating set for ring) 1.1.22.** Let $R$ be a ring and $X$ be a subset of $R$. The subring generated by $X$ is defined by the intersection of all subrings of $R$ containing $X$, and denoted by $\langle X\rangle$. That is;

$$\langle X\rangle=\bigcap_{\substack{H_i\le R\\ X\subseteq H_i}} H_i .$$

**Note 1.1.23.** If $R$ is a ring with unity and commutative, then

$$\langle X\rangle=\left\{\sum_{i=1}^{s} r_i a_i : r_i\in R, a_i\in X, s\in N, 1\le j\le s\right\}.$$

**Definition(Trivial multiplication) 1.1.24.** Let $R$ be a ring. If $xy=0$ for all $x, y$ in the ring $R$, we say that $R$ is the zero ring on $R^+$, the additive group of $R$; we will also say in this case that $R$ has the trivial multiplication.(Fuchs,Gilmer,1970).

**Definition(Ideal) 1.1.25.** Let $R$ be a ring and $\varnothing \neq \mathrm{I} \subset R$. If

(i) $a-b \in \mathrm{I}$ for every $a,b \in \mathrm{I}$

(ii) $ra \in \mathrm{I}$ $(ar \in \mathrm{I})$ for $\forall a \in \mathrm{I}$ and $\forall r \in R$,

$\mathrm{I}$ is called a left (right) ideal of $R$.

**Definition(Factor ring) 1.1.26.** Let $I$ be an ideal of a ring $R$ and let $(I,+)$ be a normal subgroup of the group $(R,+)$. Then the set $R/I = \{x+I : x \in R\}$ with the operations

$$\oplus : (x+I) \oplus (y+I) = (x+y)+I$$
$$\bullet : (x+I) \bullet (y+I) = xy+I$$

is a ring. The ring is called factor ring.

**Definition(Annihilator) 1.1.27.** Let $R$ be a ring and $x \in R$. The set $\{r \in R : rx = 0\}$ is called annihilator of $x$, and denoted by $Ann^{l}(x)$. At the same time, the set is left ideal of $R$.

**Definition (Module) 1.1.28.** Let $R$ be a ring with identity and let $M$ be an Abelian group. A left $R$-module is an Abelian group $M$ with a mapping defined as follows:
The mapping is called scalar multiplication defined;

$$R \times M \to M$$

where $(r,m) \to r.m$ and the
scalar multiplication has the following properties:

(i) $r_1(r_2 m) = (r_1 r_2)m$ where for all $r_1, r_2$ in $R$ and $m$ in $M$.

(ii) $r(m_1 + m_2) = rm_1 + rm_2$ where for all $r$ in $R$ and $m_1, m_2$ in $M$.

(iii) $(r_1 + r_2)m = r_1 m + r_2 m$ where for all $r_1, r_2$ in $M$ and $m$ in $M$.

If the following statement is satisfied too, then $M$ is said unitary.

(iv) $1_R.m = m$ for all $m$ in $M$.(Ribenboim, 1967).

**Definition (Finitely generated module) 1.1.29.** A left $R$-module $M$ is finitely generated if and only if there exist $a_1, a_2, ..., a_n$ in $M$ such that for all $x$ in $M$, there exist $r_1, r_2, ..., r_n$ in $R$ with $x = r_1 a_1 + r_2 a_2 + ... + r_n a_n$. The set $\{a_1, a_2, ..., a_n\}$ is referred to as a generating set for $M$ in this case.(Ribenboim, 1967).

**Definition(Vector space over a field)) 1.1.30.** Let $F$ be a field.Then a vector space over the field $F$ is a set $V$ of vectors together with two operations,

1) Vector addition: $V \times V \to V$ denoted $v + w$, where $v, w \in V$, and

2) Scalar multiplication: $F \times V \to V$ denoted $av$, where $a \in F$, $v \in V$,

such that some axioms are satisfied.The following is a list of the axioms:

1. Vector addition is associative:

For all $u, v, w \in V$, we have $u + (v + w) = (u + v) + w$

2. Vector addition is commutative:

For all $v, w \in V$, we have $v + w = w + v$.

3. Vector addition has an identity element:

There exists an element $0 \in V$, called the zero vector, such that $v + 0 = v$ for all $v \in V$.

4. Vector addition has inverse element:

For all $v \in V$, there exists an element $w \in V$, called the additive inverse of $v$, such that $v + w = 0$.

5. Distributivity holds for scalar multiplication over vector addition:

   For all $a \in F$ and $v, w \in V$, we have $a(v+w) = av + aw$ .

6. Distributivity holds for scalar multiplication over field addition:

   For all $a, b \in F$ and $v \in V$, we have $(a+b)v = av + bv$.

7. Scalar multiplication is associative:

   For all $a, b \in F$ and $v \in V$, we have $a(bv) = (ab)v$.

8. Scalar multiplication has an identity element:

   For all $v \in V$, we have $1v = v$, where $1$ denote the multiplicative identity in $F$.

**Note 1.1.31.** These are just the axioms of a module, so the axioms can be concisely described "a vector space is a module over a ring which is also a field.".In other words, vector spaces are $F$-modules.

**Definition(Submodule) 1.1.32.** Let $M$ be a left $R$-module. A subset $N \subseteq M$ is called a submodule of $M$ if $N$ is a left $R$-module too. (Ribenboim, 1967).

**Note 1.1.33.** Let $M$ be a left $R$-module. If $\varnothing \neq N \subseteq M$, then $N$ is a submodule of $M$ if and only if $N$ is a subgroup of the additive group of $M$ and $r.n + k \in N$ where for every $r$ in $R$ and $n, k$ in $N$.(Ribenboim, 1967).

**Definition (Factor module) 1.1.34.** Let $M$ be a left $R$-module and $N$ be a submodule of $M$. Define $M/N = \{m + N : m \in M\}$. Then we can make $M/N$ an $R$-module by defining

$$r.(m+N) = r.m + N$$

New $R$-module $M/N$ is called a quotient(or factor) module of $M$.
(Ribenboim, 1967).

**Definition(External direct sum) 1.1.35.** The external direct sum $\{a_1 \oplus a_2 : a_1 \in A_1, a_2 \in A_2\} = A_1 \oplus A_2$ of two $R$-modules $A_1$ and $A_2$ is the $R$-module consisting of all ordered pairs $(a_1, a_2)$ for $a_i \in A_i$, with module operations defined by

$$(a_1, a_2) + (a_1', a_2') = (a_1 + a_1', a_2 + a_2') \ , \ r(a_1, a_2) = (ra_1, ra_2)$$

where $r \in R$.

**Definition(Internal direct sum)1.1.36.** $M$ is called the internal direct sum of its submodules $\{B_i\}_{i \in I}$ if

(1) $M = \sum_{i \in I} B_i = \left\langle \bigcup_{i \in I} B_i \right\rangle = \left\{ \sum_{i=1}^{n} r_i b_i : r_i \in R, b_i \in \bigcup_{i \in I} B_i, n \in Z^+ \right\}$

(2) $B_j \cap \left( \sum_{i \neq j} B_i \right) = 0$

If $M$ is direct sum of its submodules $\{B_i\}_{i \in I}$, then we denote it by $M = \oplus_{i \in I} B_i$. And also, Let $\{B_i\}_{i \in I}$ be a family of submodules of $M$ .Then; $M = \oplus_{i \in I} B_i$ if and only if every $m \in M$ can be written $m = \sum_{i \in I_0} b_i$ uniquely for some finite $I_0 \subseteq I$.

**Definition( Direct summand)1.1.37.** Let $R$ be a ring and $B \subseteq A$ left $R$-modules.Then $B$ is called a direct summand of $A$ if there exist a left $R$-submodule $C$ such that $A = B \oplus C$.

**Definition (Indecomposable module)1.1.38.** A module is defined to be indecomposable if it is non-zero and cannot be written as a direct sum of two non-zero submodules. (Ribenboim, 1967).

**Definition(Noetherian module)1.1.39.** A module $M$ is Noetherian if it obeys the ascending chain condition with respect to inclusion, i.e. if every set of increasing sequences of submodules eventually becomes constant.In other words; if for every chain

$$M_1 \subseteq M_2 \subseteq ...$$

of submodules of $M$, there is an integer $n$ such that $M_i = M_n$ for all $i \geq n$.(or the sequence stops getting bigger at $M_n$.)

If a module $M$ is Noetherian, then the following are equivalent.

1. $M$ satisfies the ascending chain condition on submodules.
2. Every submodule of $M$ is finitely generated.
3. Every set of submodules of $M$ contains a maximal element.

## 1.2. Teorems and Notes on groups , rings , modules

**Theorem 1.2.1.** Let $G$ be a group and $X$ be a subset of $G$. Then;

$$\langle X \rangle = \left\{ \prod_{i=1}^{n} x_i^{n_i} : n_i \in Z, n \in Z^+, x_i \in X \right\} = A$$

**Proof.** Let us show $\langle X \rangle = A$. By definition $\langle X \rangle$ is intersection of all subgroups of $G$ containing $X$. If we show $A$ is a subgroup of $G$ containing $X$, then this means $\langle X \rangle \subseteq A$. Let us do it now:

Firstly $\langle X \rangle \subseteq A$, because any $x \in X$ can be written as $x = x^1 = \prod_{i=1}^{1} x^1 \in A$.

For any $\prod_{i=1}^{p} x_i^{n_i}, \prod_{j=1}^{k} y_j^{s_j} \in A$,

$$\prod_{i=1}^{p} x_i^{n_i} \left( \prod_{j=1}^{k} y_j^{n_j} \right)^{-1} = x_1^{n_1} ... x_p^{n_p} \left( y_1^{s_1} ... y_k^{s_k} \right)^{-1} = x_1^{n_1} ... x_p^{n_p} . y_k^{-s_k} ... y_1^{-s_1} \in A .$$

Because this is powers and multiples of elements of $X$. So, $A$ is a group containing $X$.

Conversely, let us show every member of $A$ is also a member of $\langle X \rangle$, i.e. every member of $A$ belongs to every subgroup of $G$ containing $X$.

Let $\prod_{i=1}^{n} x_i^{n_i} \in A$, since $x_i \in X, x_i^{n_i} \in H_i$ for any $H_i$ containing $X$, because $H_i$ is a subgroup.Further, $\prod_{i=1}^{n} x_i^{n_i} = x_1^{n_1} ... x_n^{n_n} \in H_i$ again because $H_i$ is a subgroup.Hence $\prod_{i=1}^{n} x_i^{n_i} \in \bigcap_{X \subseteq H_i} H_i = \langle X \rangle$.

By Fuchs [4,Theorem 23.1.] we can write the following Theorem.

**Theorem 1.2.2.** Any divisible group $D$ is a direct sum of quasicyclic and full rational groups.The cardinal numbers of the sets of components $Z(p^{\infty})$ (for every $p$) and $Q$ form a complete and independent system of invariants for $D$.(Fuchs,1970)

**Theorem 1.2.3.** Let $G$ be a group and $H$ be a normal subgroup of $G$. If $G$ is a finitely generated group, then $G/H$ is finitely generated.

**Proof.** Since $G$ is finitely generated, then each element of $G$ can be written as $g = a_1x_1 + a_2x_2 + ... + a_nx_n$, where $\{x_1, x_2, ..., x_n\}$ is generator set for $G$. If we take an element $gH$ from $G/H$, then,

$$gH = (a_1x_1 + a_2x_2 + ...a_nx_n)H = (a_1x_1)H + (a_2x_2)H + ... + (a_nx_n)H = a_1(x_1H) + a_2(x_2H) + ... + a_n(x_nH)$$

Therefore, $\{x_1H, x_2H, ..., x_nH\}$ becomes a generator for $G/H$. So, $G/H$ is finitely generated too.

**Lemma 1.2.4.** Let $0 \to K \xrightarrow{f} L \xrightarrow{g} N \to 0$ be an exact sequence of $R$-modules.If $K$ and $N$ are finitely generated, then $L$ is also finitely generated.

**Proof.** Let $K_0 \subset K, N_0 \subset N$ be finite generating sets of $K$ resp $N$. Then there is a finite subset $L_0 \subset L$ with $g(L_0) = N_0$, and $f(K_0) \cup L_0$ is a finite generating set of $L$ since:

For every $m \in L$, there exist $l_i \in L_0$ and $r_i \in R$ with $(m)g = \sum r_i g(l_i)$. The element $m - \sum r_i l_i \in Keg = \operatorname{Im} f$ is a linear combination of elements in $K_0$ and $m = \sum r_i l_i + (m - \sum r_i l_i)$.

**Lemma 1.2.5.** *Let $G$ be an Abelian group . Suppose that $G$ is not finitely generated, but each proper subgroup of $G$ is finitely generated .The following statements hold.*(Fuchs,Gilmer,1970).

(i) if $H$ is a proper subgroup of $G$, then $G/H$ is not finitely generated, but each proper subgroup of $G/H$ is finitely generated .

(ii) $G$ is not the sum of two of its proper subgroups; in particular, $G$ is indecomposable .

**Proof.** ( i ) Let us define the binary operation as

$$\phi : G/H \times G/H \to G/H$$

where $(aH)\phi(bH) = (ab)H$ . Then, $G/H$ is not finitely generated

where $a, b \in G$ since $G$ is not finitely generated.

Now, let $G'/H$ be a proper subgroup of $G/H$ .Then, $G'/H$ is finitely generated where $G' < G$ since each proper subgroup of $G$ is finitely generated .

(ii) If $A$ and $B$ are subgroups of $G$ such that $G = A + B$, then $A$ or $B$ is not finitely generated, since $G$ is not finitely generated . Hence $G = A$ or $G = B$.

**Theorem(First isomorphism theorem) 1.2.6.** If $G$ and $H$ are groups and $f$ is a homomorphism from $G$ to $H$, then the kernel $K$ of $f$ is a normal subgroup of $G$, and the quotient group $G/K$ is isomorphic to the image of $f$ .In other words; if

$G, H$ are groups and $f : G \to H$ is a homomorphism, then $Kerf$ is a normal subgroup of $G$ and $G/Kerf \cong \text{Im}\, f$ .

**Theorem(Second isomorphism theorem) 1.2.7.** Let $H$ be a subgroup of $G$ and let $N$ be a normal subgroup of $G$ . Then,

$$HN/N \cong H/H \cap N .$$

**Theorem(Third isomorphism theorem) 1.2.8.** Let $H$ and $K$ be normal subgroups of a group $G$ with $K \leq H$ . Then

$$G/H \cong \frac{G/K}{H/K} .$$

**Lemma 1.2.9.** If $N$ and $H$ are normal subgroups of $G$, with $N$ contained in $H$, then the map $\phi : G/N \to G/H$ which is defined by rule that $\phi(x+N) = x+H$ is an epimorphism.

**Proof.** First we will show that $\phi$ is well-defined.Note that if $x+N = y+N$ then $x^{-1} + y \in N \subseteq H$, so $x+H = y+H$ .So, $\phi$ is well-defined. Since $x$ is an arbitrary element of $G$, $\phi$ is surjective, and by definition of coset multiplication,

$$\phi(x+N)+\phi(y+N)=(x+H)+(y+H)=(x+y)+H=\phi((x+y)+N),$$

$\phi$ is homomorphism.As a result, the map $\phi$ is an epimorphism.

**Note 1.2.10.** $Z(p^{\infty})=\left\{\frac{m}{p^{i}}+Z : p \text{ is a prime }, i\in Z^{+}\cup\{0\}, m\in Z\right\}$ is a submodule of the quotient module $Q/Z$ .

Because for any $\alpha,\beta\in Z(p^{\infty})$, consider $\alpha-\beta$ .

Say, $\alpha=\frac{z_1}{p^{n_1}}+Z$ and $\beta=\frac{z_2}{p^{n_2}}+Z$ .

$$\alpha-\beta=\left(\frac{z_1}{p^{n_1}}+Z\right)-\left(\frac{z_2}{p^{n_2}}+Z\right)=\left(\frac{z_1}{p^{n_1}}-\frac{z_2}{p^{n_2}}\right)+Z=\frac{z_1p^{n_2}-z_2p^{n_1}}{p^{n_1+n_2}}+Z\in Z(p^{\infty}) .$$

So, it is a subgroup of $Q/Z$ and it's $Z$ - module property comes from

$z.\alpha=z.\left(\frac{z_1}{p^{n_1}}+Z\right)=\frac{zz_1}{p^{n_1}}+Z$ . So, $Z(p^{\infty})$ is a submodule of $Q/Z$ .

# CHAPTER TWO
# ALMOST NOETHERIAN RINGS

## 2.1 Definition of Noetherian Ring

**Definition 2.1.1.** A ring $A$ with identity $1_A$ is a left (right) *Noetherian ring* if for every ascending sequence of left (right) ideals of $A$,

$$a_1 \subset a_2 \subset a_3 \subset \ldots,$$

there is an integer $n$ such that $a_n = a_{n+1} = a_{n+2} = \ldots$.

Now, we want to give two important properties.

A ring $R$ has property (C1) if $R$ does not satisfy the ascending chain condition (a.c.c.) on two-sided ideals, but each proper subring of $R$ satisfies the a.c.c. on two-sided ideals; $R$ has property (C2) if $R$ does not satisfy the a.c.c on left ideals, but each proper left ideal of $R$ satisfies the a.c.c. on left ideals.

In other words; if $R$ is non-Noetherian , but each proper subring of $R$ is Noetherian; $R$ is said to have property (C1). And; if $R$ is not left Noetherian ring, but each proper left ideal of $R$ is left Noetherian ring; $R$ is said to have property (C2).(Fuchs,Gilmer,1970).

## 2.2 Noetherian Ring And Finitely Generated Ideal.

**Definition 2.2.1** A module $M$ is said to satisfy the *maximum(minumum) condition* on submodules if every non-empty set of submodules of $M$ contains a maximal (minimal) element with respect to set theoretic inclusion.

**Teorem 2.2.2.** *An $R$-module $M$ satisfies the ascending (descending) chain condition on submodules if and only if $M$ satisfies the maximal (minimal) condition on submodules.*

**Proof.** $(\Rightarrow)$: Let $\Gamma=\{M_i / i\in I\}$ be a set of submodules of $M$. Take any two elements $M_{i_1}$ and $M_{i_2}$ from $\Gamma$, then let's compare them with each other.If without loss of generality $M_{i_1}\subseteq M_{i_2}$, then take another $M_{i_3}$ from $\Gamma$ and compare with $M_{i_1}$, $M_{i_2}$. If in that case that the choosen one is not comparable (i.e. not contained in any one or not contains any one), then omit it from the discussion.By this way we obtain a sequence

$$M_{i_1}\subseteq M_{i_2}\subseteq\dots$$

By assumption, the sequence stops.Say the stopping point is $M_{i_n}$. Then $M_{i_n}$ becomes maximal element of $\Gamma$. Because otherwise, i.e. if something includes $M_{i_n}$, then it would be in the sequence and this is a contradiction.

$(\Leftarrow)$: Let $M_1\subseteq M_2\subseteq\dots$ be a sequence of submodules of $M$. Then consider the set $\Gamma=\{M_1,M_2,...\}$, since $M$ satisfies maximum condition , then $\Gamma$ will have a maximal element , say $M_0$.

Above sequence stops at $M_0$, because suppose $\exists M_n$ in the sequence such that $M_n\geq M_0$. But then $M_n$ must stay in $\Gamma$ and since $M_0$ is maximal in $\Gamma$, then $M_n\geq M_0$ can not be true.

**Proposition 2.2.3.** *Let $M$ be an $R$-module, then $M$ is Noetherian if and only if every submodule of $M$ is finitely generated.*

**Proof.** $(\Rightarrow)$: Let $N$ be a submodule of $M$. Suppose that $N$ is not finitely generated and look for a contradiction.

Let $\Gamma = \{X \leq N : X \text{ is finitely generated}\}$. Then $\Gamma \neq \varnothing$ since $0 \in \Gamma$, so; since $M$ is Noetherian, $\Gamma$ has a maximal member say $C$.

Since $C$ is finitely generated then let's say $\{c_1, c_2, ..., c_n\}$ generates $C$. Take any $n \in N$, then $\{n, c_1, c_2, ..., c_n\}$ generates another module say $N'$ and obviously $N' \in \Gamma$. Since "$n$" is arbitrary member of $N$, it implies that $N \subseteq N'$. Therefore $N$ must be finitely generated.

$(\Leftarrow)$: Given a chain of submodules of $M$

$$M_1 \subseteq M_2 \subseteq \ldots$$

then $\bigcup M_i$ is a submodule of $M$. Since every submodule of $M$ is finitely generated, then $\bigcup M_i = M_{i_0}$ is also finitely generated where $M_{i_0}$ is a module from the above chain.Let $\{a_1, a_2, ..., a_k\}$ be a generating set of $M_{i_0}$.Choose $N$ large enough to ensure that $a_i \in M_N$ for $i = 1, 2, ..., k$.Then $M_{i_0} \subset M_N$ and hence $M_N = M_i = M_{i_0}$ for all $i \geq N$.Thus $M$ must be satisfy the ascending chain condition.

**Corollary 2.2.4.** *A commutative ring $R$ is Noetherian if and only if every ideal of $R$ is finiteley generated. In other words, the a.c.c. on ideals of $R$ is equivalent to condition that each ideal of $R$ is finitely generated. In particular, if $R$ is a ring with propety $(C1)$ and $(C2)$, then $R$ is the only ideal of $R$ that is not finitely generated . That is; a ring $R$ is Noetherian if and only if every ideal of $R$ is finitely generated .*

## 2.3 The Most Important Teorems On Noetherian Rings

**Theorem 2.3.1.** *Let $G$ be an Abelian group satisfying the hypothesis of Llemma 1.2.5. Then $G \cong Z\left(p^{\infty}\right)$ for some prime $p$.* (Fuchs,Gilmer,1970).

**Proof.** First, we will prove that $G$ is divisible . We suppose that $G$ is not a divisible group .Then, since $G$ is not a divisible group, it must not be $pG = G$ by the definition of divisible group, then it must be $pG \subseteq G$ for some prime $p$ .Hence; If we consider $G/pG$ as a vector space over $Z/(p)$, then it is not finitely generated.In other words; since $G$ is not finitely generated, $G/pG$ must not be finitely generated. But each proper subspace of $G/pG$ is finitely generated by Lemma 1.2.5. But, this is impossible .As; If a vector space has not a basis, every subspace of it has not a basis.In other words;Since $G/pG$ is a vector space and has not a basis, each proper subspace of $G/pG$ must not have a basis.That is, each proper subgroup of $G/pG$ must not be finitely generated.So, It is a contradiction by assumption.. Thus, it must be $pG = G$ and $G$ is divisible . By Theorem 1.2.2., it follows that $G$ is the direct sum of quasicyclic groups and full rational groups. Since $G$ is indecomposable by Lemma 1.2.5., $G$ is either a quasicyclic group or a full rational group. But a full rational group does not have the property that each of its proper subgroups is finitely generated , and thus $G \cong Z\left(p^{\infty}\right)$ for some prime $p$ .

**Example 2.3.2.** The group $Q^{(p)} = \left\{ \frac{m}{p^k} : m \in Z, k \in Z \right\}$ is a full rational group.But, the proper subgroups of the group are not finitely generated.

**Theorem 2.3.3.** *Let $R$ be a ring without identity that satisfies property $(C1)$ or property $(C2)$. Then $R$ has the trivial multiplication.*(Fuchs,Gilmer,1970).

**Proof .** We show that $R$ has the trivial multiplication if $R$ satisfies $(C1)$**.** The proof for the case when $R$ satisfies $(C2)$ is essentially the same and we omit it .

If $\{A_i\}_{i=1}^{\infty}$ is any infinite ascending chain of ideals of $R$, and if $A=\bigcup_{i=1}^{\infty}A_i$, then $A$ is an ideal of $R$ and $\{A_i\}_{i=1}^{\infty}$ is an infinite ascending chain of ideals of $A$. Obviously; since $R$ has property $(C1)$, it must be $A=R$.

Now, we will show that $R$ has the trivial multiplication. In other words; we show that $Rx=0$ where $x\in R$. Since, $x\in R=\bigcup_{i=1}^{\infty}A_i$, we get $x\in A_m$ for some $m\in w$, hence $Rx\subseteq RA_m\subseteq A_m\subset R$. Therefore, $Rx=0$ or $Rx$ is a proper subring of $R$. If $Rx=0$, we are done; if $Rx$ is a proper subring of $R$, then by the hypothesis on $R$, it follows that $Rx$ is finitely generated as an ideal of $Rx$. Let $\{r_ix\}_{i=1}^{n}$ be a set of generators of $Rx$, considered as an ideal of $Rx$.

If $r\in R$, then

$$rx=\sum_{i=1}^{n}s_ir_ix+\sum_{i=1}^{n}r_ixt_i+\sum_{i=1}^{n}u_ir_ixv_i+\sum_{i=1}^{n}\lambda_ir_ix$$

where $s_i$, $t_i$, $u_i$, $v_i\in Rx$, $\lambda_i\in Z$.

Letting $t_i=t_i'x$ and $v_i=v_i'x$ for each $i$, we have that

$$rx=\sum_{i=1}^{n}s_ir_ix+\sum_{i=1}^{n}r_ixt_i'x+\sum_{i=1}^{n}u_ir_ixv_i'x+\sum_{i=1}^{n}\lambda_ir_ix,$$

and therefore,

$$\left[r-\sum_{i=1}^{n}\left(s_ir_i+r_ixt_i'+v_ir_ixv_i'+\lambda_ir_i\right)\right]x=ux=0\ .$$

Therefore, $u$ belongs to the left annihilator $L$ of $x$, where $Ann^{l}(x)=\{u\in R:ux=0\}=L$. And, so $r\in\left(\{r_i\}_{i=1}^{n}\right)+L$. Therefore, since $r$ was an arbitrary element of $R$, it follows that $R=\left(\{r_i\}_{i=1}^{n}\right)+L$. But, $L$ is a left ideal, hence a subring of $R$, and therefore cannot be finitely generated as an ideal of $L$, since $R$ is not finitely generated as an ideal of $R$. Thus $R=L$ so that $Rx=Lx=0$.

**Theorem 2.3.4.** *Let $R$ be a ring without identity. The following conditions are equivalent* .(Fuchs,Gilmer,1970).

a-) $R$ satisfies property $(C1)$.

b-) $R$ satisfies property $(C2)$.

c-) $R$ is the zero ring on a $p$-quasicyclic group .

**Proof.** A $p$-quasicyclic group, which we denote by $Z(p^{\infty})$, is an Abelian group generated by a set $\{c_i\}_{i\in Z^+}$ such that $c_i$ has order $p^i$ and $pc_{i+1} = c_i$ for each $i \in Z^+$. So that, the non-zero proper subgroups of $Z(p^{\infty})$ are precisely the finite cyclic groups generated by $c_i$'s. Thus , with the trivial multiplication, the proper subrings (proper ideals) of $Z(p^{\infty})$ are just the proper subgroups of $Z(p^{\infty})$. It then follows that the zero ring on $Z(p^{\infty})$ satisfies (C1) and (C2). Thus; we have just proved that $c \Rightarrow a$ and $c \Rightarrow b$ .

If $R$ satisfies $(C1)$ or $(C2)$, then $R$ has the trivial multiplication , by Theorem 2.3.3. Finally, the subrings (and ideals and left ideals ) of $R$ are precisely the subgroups of $R^+$. Therefore, $R^+$ is an Abelian group satisfying the hypothesis of Theorem 2.3.1., so that $R^+ \cong Z(p^{\infty})$ for some prime $p$, and $R$ is the zero ring on a quasicyclic group.

# CHAPTER THREE
# ALMOST SHORT MODULES

## 3.1 The Connections Between Short And Almost Short Modules

**Definition 3.1.1.** An $R$-module $M$ is called *almost Noetherian* if every proper submodule of $M$ is finitely generated.(Bilhan & Smith).

**Definition 3.1.2.** A module $M$ is called *short* if for any submodule $N$ of $M$, $N$ or $M/N$ is Noetherian .(Bilhan & Smith).

**Definition 3.1.3.** An $R$-module $M$ is called *almost short* if for any submodule $N$ of $M$, $N$ is finitely generated or $M/N$ is finitely generated.(Bilhan & Smith).

By definitions, clearly any finitely generated module is almost short and also any short module is almost short.However converses of these facts are not true.

**Example 3.1.4.** Prüfer $p-group$ $Z\left(p^{\infty}\right)$ is almost short module for any prime $p$. But it is not finitely generated.

Let $R$ be a non-Noetherian ring.Then obviously left $R$-module ${}_RR\oplus {}_RR$ is also non-Noetherian.Since it is finitely generated, it is almost short.However it is not short, because the submodule ${}_RR$ and factor module $\left({}_RR\oplus {}_RR\right)/{}_RR$ are not Noetherian.

Like short modules, direct sum of two almost short modules is not almost short.An example is $Z\left(p^{\infty}\right)\oplus Z\left(p^{\infty}\right)$ for any prime $p$.

**Proposition 3.1.5.** *If $R$ is left Noetherian then each finitely generated left $R$-module $A$ is Noetherian.* (Bilhan & Smith).

**Proof.** We must show that each submodule $B$ of $A$ is finitely generated.Let $x_1, x_2, ..., x_n$ be a system of generators for $A$. If $n=1$, then $A \cong R/I$ for some left ideal $I$. Therefore $B \cong J/I$ for some left ideal $J$ containing $I$. Since $J$ is finitely generated, so is $B$. We now proceed by induction and assume that the proposition is proved for modules $A$ generated by fewer than $n$ elements.Assume $n>1$ and let $A'$ denote the submodule of $A$ generated by $x_1$. There results an exact sequence $0 \to A' \to A \to A'' \to 0$ with both $A'$ and $A''$ generated by fewer than $n$ elements.This exact sequence induces an exact sequence $0 \to B' \to B \to B'' \to 0$ with $B' \subset A', B'' \subset A''$. Thus $B'$ and $B''$ are finitely generated, and therefore $B$ is also finitely generated.

**Proposition 3.1.6.** *The following condition holds for a module $M$ and any submodule $N$.*

$M$ is Noetherian if and only if $N$ and $M/N$ are Noetherian.

**Proof.** An ascending chain of submodules of $N$ consists of a chain of submodules of $M$; an ascending chain of submodules of $M/N$ corresponds to a chain of submodules of $M$ that contains $N$.It is clear that if $M$ is Noetherian, then so are $N$ and $M/N$.

Conversely, assume that $N$ and $M/N$ are Noetherian, and let $M_0$ be a submodule of $M$.Then $M_0 \cap N$ and $M_0/(M_0 \cap N) \cong (M_0 + N)/N$ are both finitely generated, so $M_0$ is finitely generated.

**Proposition 3.1.7.** Let $R$ be a left Noetherian ring. Then almost short $R$-modules are short .(Bilhan & Smith).

**Proof.** Let $M$ be an almost short $R$-module. Let $N$ be a submodule of $M$ that is not Noetherian.Then it has a submodule $X$ (say) which is not finitely generated. By assumption $M/X$ is finitely generated and hence $M/N$ is also finitely generated. Since $R$ is left Noetherian then $M/N$ is Noetherian.

**Proposition 3.1.8.** *Let $R$ be any ring. An $R$-module $M$ is short if and only if every submodule of $M$ is almost short.*(Bilhan & Smith).

**Proof.** Suppose that $M$ is a short $R$-module. Let $N$ be any submodule of $M$. Suppose that $K$ is a submodule of $N$ which is not finitely generated. Then $M/K$ is Noetherian so that any submodule is finitely generated. Conversely, let $N$ be any submodule of $M$. Suppose that $N$ is not Noetherian. Then there is a submodule $L$ of $N$ such that $L$ is not finitely generated. Let $H$ be any submodule of $M$ containing $N$. Then $H$ is almost short and its submodule $L$ is not finitely generated so that $H/L$ is finitely generated and hence $H/N$ is finitely generated. It follows that $M/N$ is Noetherian.

The following corollary is the direct consequence of Proposition 3.1.7. and Proposition 3.1.8 .

**Corollary 3.1.9.** *Over a left Noetherian ring $R$, an $R$-module $M$ is (almost) short if and only if every submodule of $M$ is (almost) short.*

### 3.2 Coatomic Module and Extensions.

**Definition 3.2.1.** A module $M$ is called *coatomic* if every proper submodule of $M$ is contained in a maximal submodule of .(Gungöroğlu,1998).

**Lemma 3.2.2.** *Let $R$ be any ring.Then any extension of a Noetherian $R$-module by a short $R$-module is short.* (Bilhan & Smith).

**Proof.** Let $K$ be a Noetherian submodule of an $R$-module $M$ such that $M/K$ is short.Let $N$ be any submodule of $M$. Then $N \cap K$ is Noetherian. If $N/(N \cap K)$ is Noetherian, then so too is $N$. Suppose that $N/(N \cap K)$ is not Noetherian.Then $(N+K)/K$ is a non-Noetherian submodule of the short module

$M/K$. It follows that $M/(N+K)$ is Noetherian.But, $(N+K)/N \cong K/(N \cap K)$ which is Noetherian. Thus, $M/N$ is Noetherian.

**Example 3.2.3.** For each prime $p$, the $Z$-module $M_p = \{m/p^n : m,n \in Z, n \geq 0\}$ is a short $Z$-module which is not almost Noetherian.

**Corollary 3.2.4.** *Let $R$ be any ring and let an $R$-module $M = M_1 \oplus M_2$ be a direct sum of a short submodule $M_1$ and a Noetherian submodule $M_2$.Then $M$ is short.*

**Example 3.2.5.** The $Z$-module $Z \oplus Z(p^\infty)$ is short but not almost Noetherian, for every prime $p$.

**Lemma 3.2.6.** *Let $R$ be any ring.Then any extension of a short $R$-module by a Noetherian $R$-module is short.* (Bilhan & Smith).

**Proof.** Let $K$ be a submodule of an $R$-module $M$ such that $K$ is short and $M/K$ is Noetherian. Let $N$ be any submodule of $M$.Then $(N+K)/K$ is Noetherian and hence so too is $N/(N \cap K)$.If $N \cap K$ is Noetherian, then so too is $N$.Suppose that $N \cap K$ is not Noetherian.By hypothesis, $K/(N \cap K)$ is Noetherian and hence $M/(N \cap K)$ is Noetherian.This implies that $M/N$ is Noetherian.

**Corollary 3.2.7.** Let $R$ be any ring and let $M$ be an $R$-module such that $M$ contains submodules $N \supseteq L$ with $N/L$ short and both $M/N$ and $L$ Noetherian.Then $M$ is short.

**Proposition 3.2.8.** Let $R$ be any ring and let $M$ be a non-zero short $R$-module.Then $M$ is almost Noetherian or $M$ contains maximal submodule.(Bilhan & Smith).

**Proof.** Suppose that $M$ is not almost Noetherian.Then there exist a proper submodule $L$ of $M$ such that $L$ is not Noetherian.Because $M$ is short, the module $M/L$ is Noetherian and hence $M/L$ (and also $M$ ) contains a maximal submodule.

**Note 3.2.9.** A module $M$ has *finite uniform dimension* if $M$ does not contain a direct sum of an infinite number of non-zero submodules. (Bilhan & Smith).

**Lemma 3.2.10.** *Let $R$ be any ring.Then any short $R$-module has finite uniform dimension.* (Bilhan & Smith).

**Proof.** Suppose that a module $M$ contains an infinite direct sum $N_1 \oplus N_2 \oplus N_3 \oplus ...$ of non-zero submodules $N_i \left(i \geq 1\right)$. Let $N = N_1 \oplus N_3 \oplus N_5 \oplus ....$ Clearly the submodule $N$ is not Noetherian. Morever, $N_2 \oplus N_4 \oplus N_6 \oplus ...$ embeds in $M/N$ so that the module $M/N$ is not Noetherian.Thus $M$ is not short.

**Lemma 3.2.11.** *Let $R$ be any ring and let $M$ be a short $R$-module.Then every submodule and every homomorphic image of $M$ is short.* (Bilhan & Smith).

**Theorem 3.2.12.** *Let $R$ be any ring. An $R$-module $M$ is short if and only if $M$ contains submodules $N \supseteq L$ such that $N/L$ is almost Noetherian and $M/N$ and $L$ are both Noetherian.*(Bilhan & Smith).

**Proof.** The sufficiency follows by corollary 3.2.7.Conversely, suppose that $M$ is a non-Noetherian short module.There exist submodule $H$ of $M$ such that $H$ is not finitely generated.By lemma 3.2.10. and 3.2.11 every factor module of $H$ has finite uniform dimension.Shock[9,Theorem 3.7.] proves that a non-finitely generated module $X$ , such that $Y/Z$ has a maximal submodule for all submodules $Y \supset Z$ of $X$ , contains a submodule $U$ such that the module $X/U$ has non-finitely generated socle.Hence there exist submodules $N \supset L$ of $H$ such that the non-zero module $N/L$ does not contain a maximal submodule.But Lemma 3.2.11.gives that $N/L$ is short.By Proposition 3.2.8. $N/L$ is almost Noetherian.Because $N/L$ is not

Noetherian we have $N$ is not Noetherian and hence $M/N$ is Noetherian.Finally, because $N/L$ is not Noetherian we have $M/L$ is not Noetherian and hence $L$ is Noetherian. This completes the proof.

**Lemma 3.2.13.** *Let $R$ be any ring and let $M$ be an almost short $R$-module. Then $M$ is coatomic or short.*(Bilhan & Smith).

**Proof.** Suppose that $M$ is not coatomic. Then there exists a proper submodule $L$ of $M$ such that $L$ is not contained in a maximal submodule of $M$. Then $M/L$ is not finitely generated. Suppose that $L$ has a non-finitely generated submodule $K$. Since $M$ is almost short then $M/K$ is finitely generated and so too $M/L$. Then $L$ is Noetherian. Moreover, for any proper submodule $N$ of $M$ containing $L$, the module $M/N$ is not finitely generated so that $N$ is Noetherian and hence $N/L$ is Noetherian. The result follows by Corollary 3.2.7. by choosing $N = M$.

**Proposition 3.2.14.** *Let R be any ring. Any extension of a Noetherian module by an almost short module is almost short.* (Bilhan & Smith).

**Proof.** Let $L$ be any submodule of $M$ and $N$ be a Noetherian submodule of $M$ such that $M/N$ is almost short. Then $L \cap N$ is Noetherian. If $L/(L \cap N)$ is finitely generated, then so too is $L$. If $L/(L \cap N)$ is not finitely generated, then $(L+N)/N$ is also not finitely generated and since $M/N$ is almost short then $M/(L+N)$ is finitely generated. Since $(L+N)/L \cong N/(L \cap N)$, then $(L+N)/N$ is finitely generated. Hence $M/L$ is finitely generated. Therefore $M$ is almost short.

Note that we can clearly say the following result by Proposition 3.2.14.

**Corollary 3.2.15.** *$M = N \oplus K$ be a direct sum of an almost short submodule and a Noetherian submodule, then $M$ is almost short.*(Bilhan & Smith).

**Proposition 3.2.16.** *Any extension of an almost short module by a Noetherian module is almost short.* (Bilhan & Smith).

**Proof.** Let $K$ be a submodule of an $R$-module $M$ such that $K$ is almost short and $M/K$ is Noetherian. Let $N$ be any submodule of $M$. Then $(N+K)/K$ is Noetherian and hence so too is $N/(N\cap K)$. If $N\cap K$ is finitely generated then so too is $N$. Suppose that $N\cap K$ is not finitely generated. By hypothesis, $K/(N\cap K)$ is finitely generated and hence $M/(N\cap K)$ is finitely generated. This implies $M/N$ is finitely generated.

**Proposition 3.2.17.** *Let $M$ be a non-zero almost short R-module then either it is almost Noetherian or contains maximal submodules.* (Bilhan & Smith).

**Proof.** If $M$ is not almost Noetherian then there exists a proper submodule $N$ of $M$ which is not finitely generated. Hence $M/N$ is finitely generated. Therefore $M/N$ has a maximal submodule $K/N$ so that $K$ is a maximal submodule of $M$.

### 3.3 Finitely Generated Almost Short Modules

**Proposition 3.3.1.** *Let $K$ and $L$ be submodules of an almost short module $M$ such that $M=K+L$. Then there exists a finitely generated submodule $P$ of $K$ such that $M=P+L$ or there exists a finitely generated submodule $S$ of $L$ such that $M=K+S$.*(Bilhan & Smith).

**Proof.** Suppose that $K$ is finitely generated then set $K=P$. If $K$ is not finitely generated, since $M$ is almost short then the module $M/K$ is finitely generated. But $M/K\cong L/(L\cap K)$ so that $L/(L\cap K)$ is finitely generated. It follows that, there exists a finitely generated submodule $S$ of $L$ such that $L=S+(L\cap K)$. In this case, $M=L+K=S+(L\cap K)+K=S+K$.

**Corollary 3.3.2.** *Let $M$ be an almost short module which is not finitely generated. Let $K$ and $L$ be two submodules of $M$. Then $K+L$ is finitely generated if and only if both $K$ and $L$ are finitely generated.*(Bilhan & Smith).

**Proposition 3.3.3.** *A module $M$ is almost short if and only if for every submodule $N$ of $M$, $N$ is Noetherian or $M/N$ is finitely generated.*(Bilhan & Smith).

**Proof.** Let $N$ be a submodule of $M$ such that $M/N$ is not finitely generated. Let $L$ be any submodule of $N$. Then obviously $M/L$ cannot be finitely generated. Then by assumption $L$ must be finitely generated. Therefore $N$ is Noetherian. Converse of the statement is clear.

**Proposition 3.3.4.** *An R-module $M$ is almost short if and only if $M$ is not the union of a chain of non-Noetherian proper submodules.*(Bilhan & Smith).

**Proof.** Suppose that $M=\bigcup_{i\in I} N_i$ is the union of a chain of non-Noetherian proper submodules $N_i$ with $i\in I$ for any index set $I$. Let $j\in I$, then $N_j$ is not Noetherian so that $M/N_j$ is finitely generated by Proposition 3.3.3. Let $J=\{i\in I : N_j \subseteq N_i\}$, then $M/N_j$ is the union of the submodules $N_i/N_j$ where $i\in J$. It follows that $M/N_j$ is not finitely generated, because $N_i/N_j$ is a proper submodule for all $i\in J$. Thus $M$ is not almost short.

Conversely suppose that $M$ is not almost short. Then there exists a submodule $N$ of $M$ such that $N$ is not Noetherian and $M/N$ is not finitely generated. Then $M/N$ is the union of a chain of proper submodules $H/N$ where each $H$ is a submodule of $M$ containing $N$. Clearly $M$ is the union of the proper submodules $H$ and each $H$ is non-Noetherian.

### 3.4 Locally Noetherian Module And Factor Module

**Definition 3.4.1.** A module $M$ is *locally Noetherian* if every finitely generated submodule of $M$ is Noetherian. (Bilhan & Smith).

**Proposition 3.4.2.** *An almost short module is either finitely generated or locally Noetherian.* (Bilhan & Smith).

**Proof.** Suppose that $M$ is an almost short module which is not finitely generated. Let $N$ be any finitely generated submodule of $M$. Then by Proposition 3.3.3. $N$ is Noetherian or $M/N$ is finitely generated. If $M/N$ was finitely generated then $M$ would also be finitely generated. Therefore $N$ must be Noetherian.

**Lemma 3.4.3.** *Any factor module of an almost short module is almost short.* (Bilhan & Smith).

**Proof.** Let $M$ be an almost short module and $K$ be a submodule of $M$. Let $N$ be any submodule of $M$ containing $K$. Suppose that $N/K$ is not finitely generated. Then $N$ is not finitely generated and hence by assumption $M/N$ is finitely generated. Result follows by the isomorphism $\dfrac{\dfrac{M}{K}}{\dfrac{N}{K}} \cong \dfrac{M}{N}$.

**Corollary 3.4.4.** *Let M be an almost short module and let N be a Noetherian submodule of M. Then $M/N$ is almost short if and only if $M$ is almost short.*

**Note 3.4.5.** Finally, together with the propositions given in chapter 3 , we can draw the following picture where the are arrows are strict.

$$\begin{array}{ccccc} \text{Noetherian} & \rightarrow & \text{almost Noetherian} & \rightarrow & \text{short} \\ \downarrow & & & & \downarrow \\ \text{Finitely generated} & & \rightarrow & & \text{almost short} \end{array}$$

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