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Developing pre-service mathematics teachers' didactic reference points for the use of dynamic technology through micro-teaching

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Abstract
This study examines micro-teaching experiences of two pre-service mathematics teachers (PMTs) as they teach the concept of a cylinder using GeoGebra, focusing on their potential learning during instruction. The study was conducted within a practicum course in the final year of a mathematics teacher education programme in Türkiye. The notions of professional instrumental genesis and didactic reference points are adopted as the theoretical framework of the study. Data sources included PMTs' task planning, video recordings of their micro-teaching sessions, semi-structured individual interviews, group discussions, and their revised lesson plans. The analysis identified four key didactic reference points: 2D-3D screen coordination, use of sliders, GeoGebra's dynamic affordances, and unwrapping representations. By identifying the didactic reference points that teachers initially lacked, these results contribute to both research and teacher education. They indicate the relevance of a theoretical tool tested in a new context, expand research knowledge on the difficulties mathematics teachers face when integrating technology, particularly GeoGebra, and shed light on the phenomena involved in the birth of instrumented practices. Furthermore, the study underscores the value of micro-teaching as an effective professional development approach for supporting PMTs in developing didactic reference points for technology-based mathematics instruction, particularly with GeoGebra.

Keywords Didactic reference points · Micro-teaching · Pre-service mathematics teachers · The concept of a cylinder · Dynamic technology

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1 Introduction

Despite decades of research on technology integration in mathematics education, its widespread adoption in classrooms remains limited (Clark-Wilson et al., 2014, 2023). Over the past 20 years, the field has increasingly shifted its focus from student learning and tool potential to the "teacher factor" (Lagrange et al., 2003). While earlier studies emphasised students' use of digital tools, more recent research examines teachers' roles, challenges, and the development of professional knowledge in technology-enhanced teaching (Drijvers et al., 2010; Sinclair et al., 2023). Teaching mathematics with digital tools requires adapting instructional practices, managing new classroom dynamics, and designing innovative tasks (Healy & Lagrange, 2010). Integration involves more than tool use—it entails rethinking pedagogy and classroom management, necessitating time, training, and structured support (Clark-Wilson & Noss, 2015; Lagrange & Ozdemir, 2009). Effective professional development (PD) is thus essential for equipping teachers with the skills needed for teaching mathematics with technology.

These challenges are amplified for novice teachers, who face greater uncertainty in practice. As such, the PD of pre-service mathematics teachers (PMTs) is critical. PMTs must acquire both technological proficiency and pedagogical expertise to effectively integrate digital tools. Structured opportunities to engage with technology, reflect on its didactical implications, and design technology-based tasks are therefore essential. In this regard, practice-based training is a key component of effective PD for PMTs (Santagata & Yeh, 2014; Zbiek & Hollebrands, 2008).

To support PMTs in developing these competencies—particularly in using GeoGebra—the first and second authors restructured a compulsory practicum course, conceptualised as a PD programme. Building on their earlier research (Bozkurt & Yiğit Koyunkaya, 2022, 2023a), the course focused on designing and implementing tasks *using dynamic geometry software*, emphasising the use of dragging and sliders to *support generalisation* of mathematical concepts. Their prior findings highlighted the importance of leveraging such dynamic features as teaching instruments (Haspekian, 2011) to achieve a higher level of mathematical depth (Smith & Stein, 1998; Trocki & Hollebrands, 2018). In response, micro-teaching was integrated as a core element to provide hands-on experience and support PMTs in lesson planning and instructional refinement—based on prior evidence of its value in supporting technology integration (Bozkurt & Yiğit Koyunkaya, 2022, 2023a). Originating at Stanford University in the 1960s, micro-teaching involves short, peer-taught lessons followed by targeted feedback and iterative revision (Allen, 1967; Allen & Clarke, 1967). Its core features include brief instructional sessions, focused feedback, structured reflection, and opportunities for re-teaching (Cheng, 2017; Ledger & Fischetti, 2019). Recognised as an effective practice-based training model (Mukuka & Alex, 2024), micro-teaching in this study was conducted in a university classroom, where PMTs taught their technology-based tasks to peers, received feedback, and revised their plans.

The objective of this paper is to examine PMTs' experiences with designing and implementing technology-based mathematical tasks in the context of the PD course. Specifically, it explores how PMTs engage with micro-teaching to reflect on and revise their instructional plans for technology integration. This examination enables a deeper understanding of how micro-teaching functions as a practice-based training tool and how it supports the development of practical, topic- and technology-specific teaching knowledge. By shedding light on PMTs' evolving technological and pedagogical expertise through micro-teaching, the study contributes to our understanding of teacher

professional learning and offers practical implications for the design of teacher education programmes that aim to foster meaningful and effective technology integration in mathematics classrooms.

2 Theoretical tools and research literature that framed our work

2.1 Literature review

The complexity that technology introduces into mathematics teaching has been well documented (Clark-Wilson et al., 2014). This complexity has been studied from various perspectives. For instance, some studies show that the introduction of calculators leads to a proliferation of solving techniques for the same mathematical task compared to traditional paper-and-pencil methods (Artigue et al., 2003). The use of technology in classrooms increases uncertainty and improvisation, sometimes resulting in disorderly episodes that help explain teachers' reluctance to integrate technology.

However, research also highlights that such unexpected moments can serve as opportunities for PD. For example, Lagrange and Ozdemir (2009) identify "emergent goals" arising in sessions integrating new tools—goals that develop through how teachers manage disruptions and which can foster professional growth if accompanied by reflection:

> *While strategies based on the transmission of "good practices" taking little care of the complexity of the integration fail to engage teachers, reflecting upon actual more or less unsatisfactory classroom activities might help them to identify possible evolution.* (ibid., p. 83).

Similarly, Clark-Wilson and Noss (2015) study these unexpected situations and highlight what Clark-Wilson (2010) terms "hiccups": "perturbations experienced by teachers during lessons stimulated by their use of technology, which illuminate discontinuities within teachers' knowledge" (p. 138). These authors emphasise the potential of such unforeseen events in technology-rich classrooms to act as catalysts for teachers' professional learning. By disrupting habitual practices, these moments can promote the epistemological development of mathematics teachers.

This research connects to more specific case studies, such as Haspekian's work on spreadsheets (2005) and Scratch (Haspekian, 2017), Bakos' (2023) study of TouchTime software, and Giberti et al.'s (2024) research on Padlets, all of which explore the evolution of teaching practices through the process of *professional instrumental genesis* (Haspekian, 2014). These studies also observe that this professional learning involves teachers acquiring *didactic reference points* integral to the process. We explain these notions further in the next section, which presents the instrumental approach used to analyse our data.

2.2 Theoretical tools and framework

The *professional instrumental genesis* and the *didactic reference points* are part of the so-called Instrumental Approach to didactics of mathematics (Artigue, 2002; Guin et al., 2005; Lagrange, 1999).

2.2.1 Instrumental approach to didactics of mathematics

This approach is inspired by Vérillon and Rabardel's theory of instrumentation in psycho-ergonomics (Rabardel, 2002; Vérillon & Rabardel, 1995). This theory distinguishes between the artefact—the material object with its technical features—and the instrument, which refers to the cognitive construction that supports instrumented action, developed progressively through a process called *instrumental genesis* (IG). This process is not neutral with respect to the subject's conceptualisations. For instance, learning to represent functions graphically using a calculator can transform students' conceptualisations of the notions of function and curve.

Consequently, a teacher's activity becomes more complex due to the addition of new tasks. Trouche (2003) theorises this complexity by introducing the concept of *instrumental orchestration*, which refers to the teacher's role in managing students' IG, thinking about the links with the conceptualisations of paper-and-pencil environment and coordinating the systems of instruments in the case of multiple artefacts being used (computer, software, calculator etc.). Subsequent research has extended the notion of IG to teachers themselves. In this context, one must consider a phenomenon of *double instrumental genesis*, which involves the simultaneous development of both a *personal* and a *professional* instrument (Haspekian, 2011, 2014).

2.2.2 Double instrumental genesis of the teacher

In this approach, integrating a new tool, such as GeoGebra, into mathematics teaching requires the teacher to develop two distinct *instruments*, as defined by Rabardel (2002): a *personal instrument* and a *professional instrument*. The personal instrument emerges when the teacher uses GeoGebra for their own mathematical activities, such as solving geometry tasks, graphing functions, or formulating conjectures. This process corresponds to a personal instrumental genesis (IGpe), in which the teacher adapts GeoGebra to support personal exploration and problem solving. In contrast, the professional instrument is formed when GeoGebra is used for didactic purposes, such as designing and implementing classroom activities to teach geometry. This requires a process of professional instrumental genesis (IGpro), wherein the teacher appropriates the tool for instructional use and student engagement. Although the personal and professional instruments serve different goals, the development of the professional instrument is not independent of the personal one. Moreover, IGpro necessarily involves anticipating and supporting students' instrumental geneses, as discussed in Haspekian (2014).

2.2.3 Didactic reference points

In the continuum of her work, Haspekian examined teachers' IGpro across various contexts of novelty integration: spreadsheets (Haspekian, 2014), algorithmics teaching (Haspekian, 2017), Scratch and robots (Haspekian & Gélis, 2021), and the emerging discipline of computer science (Haspekian, 2023). Across these cases, she observed that IGpro development accompanies the acquisition of specific knowledge essential for session success, which she terms *didactic reference points* (French: *repères didactiques*). Defined as "*a professional knowledge, whether conscious or not, acquired by the teacher and which guides him/her in his/her teaching*" (Haspekian, 2023, p. 120),

these points constitute core elements feeding the schemes of teachers' IGpro (Haspekian, 2017, 2024).

Studying teachers' IGpro in diverse settings reveals that didactic reference points serve as navigational tools in daily practice, helping teachers manage sessions effectively. Difficulties in integrating new technology often stem from lacking these reference points, which are crucial to restoring the disrupted local balance caused by the introduction of the new technological environment.

The expression "*reference point*" translates the French word *repères*, which carries rich nuances. It encompasses both the idea of a static landmark as a referential point and a dynamic guiding mark. This guiding function is essential—it is not merely static knowledge. Calling it a *repère* highlights the dynamic and active role of this knowledge as an operative landmark, guiding the teaching activity, whereas not all teachers' knowledge has this function. For example, a teacher might know the didactic potential of student manipulation but never implement such sessions, thus not using that knowledge as a teaching guide.

The word *repères* also appears in expressions like *perdre ses repères* ("feeling lost") and its opposite, *trouver ses repères*, which goes beyond "knowing where we are" to imply "feeling at home". This meaning is important for our context: integrating a new teaching tool removes the teacher from familiar landmarks, explaining the difficulties in adopting new technologies as a need to find or build new didactic reference points. When introducing new technological environments in mathematics teaching, teachers' usual didactic reference points are disrupted, requiring them to create new ones—either through training and professional resources or via trial and error, often accompanied by hiccups or emergent goals.

As shown in Fig. 1, the micro-teaching sessions in this PD course, combined with reflective moments, facilitated the evolution of PMTs' IGpro by helping them develop new didactic reference points.

Theoretically, one way to identify a teacher's didactic reference point in the data is by observing changes made during repetitions of the same session, possibly cross-referenced with pre- and post-session interviews. In our study, we consider that a PMT has conceptualised a new didactic reference point if the knowledge meets three conditions: (1) it is missing in the initial plan and/or disrupts the micro-teaching session, (2) the PMT expresses new awareness of this knowledge during group discussions or individual interviews, and (3) they apply it in their revised plan. We use these criteria as indicators of the development of didactic reference points. Differences between the initial and revised plans, triangulated with teachers' comments in group discussions and interviews with the educator, provide evidence that PMTs adopt and reinvest these reference points. The reinvestment of such knowledge in revised plans signals that these points have become genuine didactic marks, guiding future teaching activity.

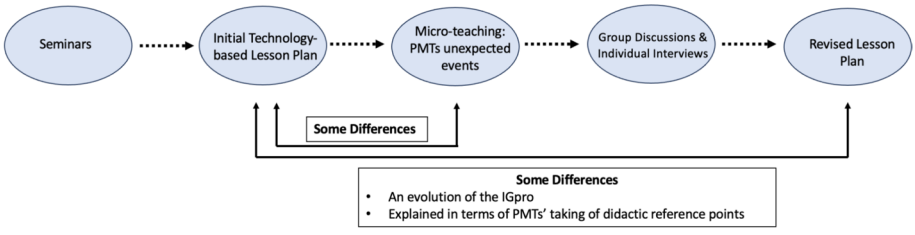


Fig. 1 Overview of the research-based PD course as the context for this study

The stages of the PD course allowed us, as researchers, to triangulate data sources, as illustrated in Fig. 1. This process demonstrates how the theoretical tools informed our analysis, supporting the identification of didactic reference points and the progression of PMTs' IGpro with GeoGebra.

Given the theoretical framework of the study, our research questions can be specified more precisely as follows:

What challenges do PMTs encounter while implementing their designed technology-based tasks during micro-teaching, and how do these challenges reflect the presence or absence of didactic reference points?

How do PMTs' experiences during micro-teaching, along with subsequent lesson revisions, contribute to the development of their IGpro, particularly through the emergence and reinvestment of didactic reference points?

3 Methods

3.1 The context and participants of the study

This study was conducted as part of a compulsory practicum course in the final year of a four-year mathematics teacher education programme at a Turkish university. The course spanned two semesters (14 weeks each). During the fall semester, it was delivered face to face, while the spring semester shifted online due to the COVID-19 pandemic. The data presented in this paper were collected during the fall semester and focus on identifying and supporting the development of didactic reference points as novice teachers engaged in micro-teaching 3D geometry concepts—specifically the cylinder—using GeoGebra. While direct follow-up observations of PMTs' classroom teaching would have strengthened our ability to evaluate how they refined and implemented their revised lesson plans, such observations were not feasible due to pandemic-related restrictions. We acknowledge this limitation and therefore relied on video recordings of the micro-teaching sessions, revised lesson plans, and both group and individual interviews. This triangulation of data sources allowed us to examine how PMTs reflected on their teaching experiences and reinvested new insights into their lesson planning, offering a robust basis for tracing the development of their IGpro and the emergence of didactic reference points.

Four volunteer PMTs participated in the research. They had completed coursework in both mathematics (e.g., calculus, abstract algebra, linear algebra) and mathematics education (e.g., teaching methods, research methods, curriculum and standards), including a technology-based course on GeoGebra that covered its features and task design. Participants were selected through a purposive sampling method (Merriam & Tisdell, 2016), based on their interest and proficiency in using GeoGebra, as well as the first author's observations of their pedagogical reasoning with technology. This paper focuses on two PMTs, *Ada* and *Mira* (pseudonyms), who both designed and micro-taught tasks involving the cylinder concept. Their teaching sessions, conducted in a scaled-down format for peers, revealed challenges in enacting their technology-based plans and provided insight into the emergence of didactic reference points.

The PMTs in this study were trained to teach mathematics to students in Grades 5–8. For this research, they were guided to focus on learning objectives related to solids from the Turkish mathematics curriculum for these grade levels (Turkish Ministry of National

Education, 2018). In Grades 5 and 6 in the curriculum, students learn to draw nets of cubes, square and rectangular prisms, and calculate their surface areas and volumes. In Grade 7, they build on this by drawing solids from different perspectives. In Grade 8, students are introduced to right prisms, right circular cylinders, pyramids, and cones, learning to construct these shapes, draw their nets, and derive surface area and volume formulas—particularly for the right circular cylinder. In this study, the PMTs designed and micro-taught technology-based tasks aimed at helping 8th-grade students (aged 13–14) understand and explore the surface area and volume formulas for the right circular cylinder using GeoGebra.

3.2 Description of the different stages of the PD course

The PD course was developed by restructuring an existing practicum course within a 4-year mathematics teacher education programme, which traditionally lacked emphasis on digital technology integration. To address this, the first and second authors redesigned the course as a PD opportunity, drawing on their prior research (Bozkurt & Yiğit Koyunkaya, 2022, 2023a) and experience as teacher educators. The aim was to support PMTs' IGpro development, specifically in using GeoGebra as a teaching instrument. A key focus was on leveraging GeoGebra's dynamic features—such as dragging and sliders (Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010; Laborde & Laborde, 1995)—to design higher level cognitive demand tasks (Smith & Stein, 1998).

As shown in Fig. 2, the course began with seminars, followed by PMTs designing a technology-based task on a selected 3D concept from the Turkish curriculum. The 3D focus was intentional, addressing PMTs' tendency to favour 2D geometry tasks, despite recognising the potential of dynamic software for teaching solids. PMTs then implemented their tasks in micro-teaching sessions conducted in an environment resembling the typical layout of Turkish classrooms, where an interactive whiteboard placed at the front alongside a traditional board. While this conventional setup tends to support teacher-led instruction and whole-class teaching, PMTs were explicitly encouraged to orchestrate productive mathematical discussions (Stein et al., 2008), adopt student-centred approaches and

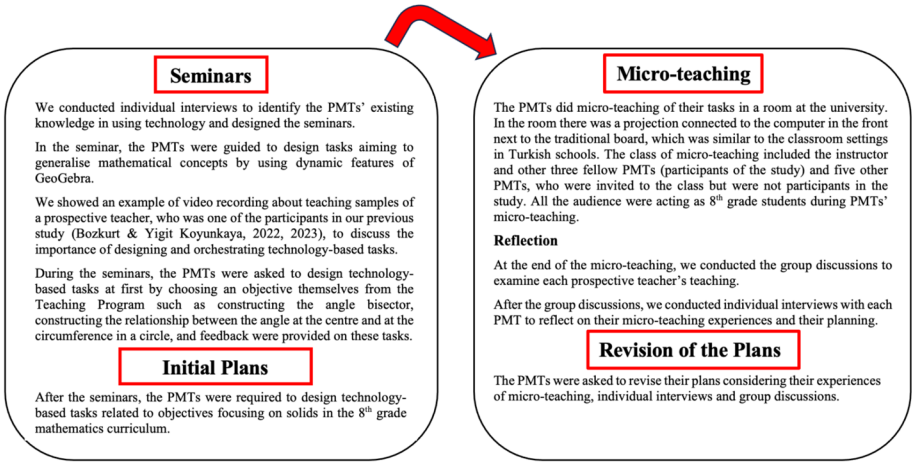


Fig. 2 The content of the PD course

consider how technology could facilitate active student engagement. Subsequently, the instructor (first author) facilitated group discussions to reflect on the tasks and teaching experiences and conducted individual interviews. Based on these reflections and feedback, PMTs revised their lesson plans, enabling an analysis of how micro-teaching contributed to refining their technology-based instructional design.

3.3 Data collection

To address the research questions, we collected multiple data sources: the PMTs' initial lesson plans, video recordings of their micro-teaching sessions, group discussions, individual semi-structured interviews (see Table 1), and their revised plans. The initial plans and micro-teaching videos provided insights into the challenges PMTs faced, while group discussions, interviews, and revised plans illustrated how they used these experiences to refine their planning. We analysed the initial plans to identify the mathematical goals, instrumental actions, and intended classroom orchestrations. These plans were then compared to actual implementations to examine how the PMTs enacted or diverged from their intended didactic reference points—that is, the guiding ideas or strategies they aimed to use in their teaching—and to identify what aspects were missed during teaching.

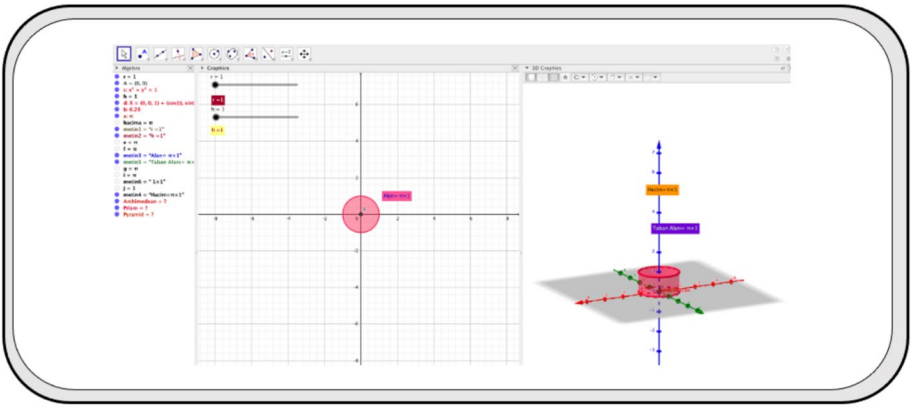
Both PMTs initially aimed to support student generalisation of surface area or volume formulas using GeoGebra. Ada focused on the volume formula of a right circular cylinder. Her task employed two sliders (representing radius and height) and coordinated the 2D and 3D views (see Fig. 3). The sliders were intended to help students observe how changes in dimensions affect volume, highlighting proportional relationships (see the task: https://www.geogebra.org/classic/mduumar8).

Mira designed a task for generalising the surface area formula of a right circular cylinder. She used four sliders to represent the cylinder's height, the opening of the bases, moving a black line segment on the cylinder, and the netting of the rectangle around the cylinder (see Fig. 4). Her plan involved drawing students' attention to dependencies among visual elements—for example, how segment IP on the 2D screen relates to the circular base's circumference, and how segment PO corresponds to the cylinder's height. Coordination between 2D and 3D views was intended to help students connect these geometric elements with the surface area formula (see the task: https://www.geogebra.org/classic/nkzbnwut).

Video recordings of the micro-teaching sessions were a primary data source for identifying instructional challenges, such as difficulties responding to student input, guiding discussion, addressing technical issues, or forgetting planned actions. Group discussions and individual interviews further explored these difficulties and how they informed plan revisions. In the group discussion, peers and the instructor analysed each micro-teaching session, focusing on teaching challenges and interpreting prompts/questions from a student perspective.

Table 1 The lengths of the video records

Participants	Micro- teaching	Group discussion	Individual interview
Ada	18 min	75 min in total [in which approximately 15 min of discussion was about Ada's micro-teaching]	40 min
Mira	30 min	75 min in total [in which approximately 15 min of discussion was about Mira's micro-teaching]	60 min



Mathematical Ideas

Students may derive the volume formula for a cylinder by generalising the multiplicative relationship between the height and the radius.

Technological Actions

Use of sliders attached to both the radius and the height of the cylinder: to identify relationships between different invariants.

Use of rotation: to raise awareness of the dependency between objects (e.g., the circle in 2D and the base area in 3D).

Use of dynamic text displaying the cylinder's measurements: to verify the equality between the area of the circle and the base area of the cylinder, and to observe how changes in radius and height affect the volume.

Classroom Orchestrations

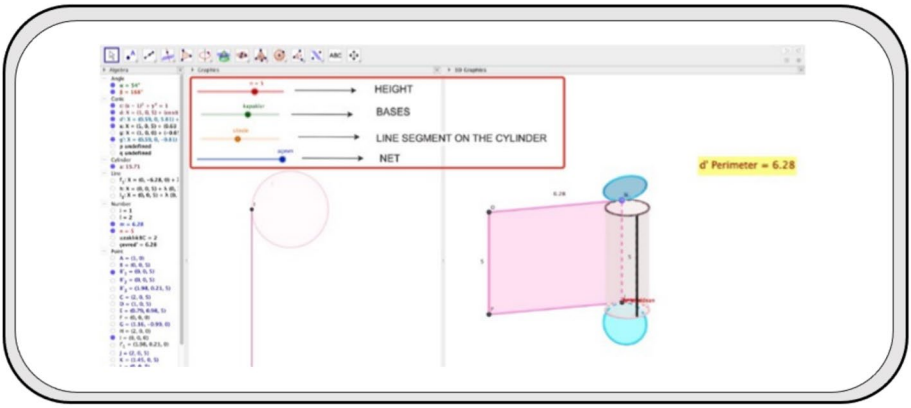
Technical-demo: Demonstration of 2D and 3D representations, including the use of sliders.

Discuss-the-screen: Ada planned to facilitate a discussion based on the screen, encouraging students to relate the 2D and 3D objects and identify the multiplicative relationship needed for the volume formula.

Link-screen-board: Ada intended for students to connect the relationships they observed between different values of height and radius on the screen with the area formula of a circle written on the board, guiding them toward deriving the volume formula of a cylinder.

Fig. 3 Ada's instructional plan for teaching the volume formula of a cylinder

Individual interviews targeted PMTs' recognition and application of didactic reference points. Although the term itself was not introduced, prompts elicited reflections aligned with this concept. PMTs indirectly identified reference points by discussing moments where these were missing or emerged during instruction. Revised lesson plans were analysed to examine how micro-teaching and reflection informed instructional changes. We paid particular attention to adjustments that addressed previously missing reference points noted in interviews. This multifaceted approach—combining planning, micro-teaching, reflection, and revision—provided a comprehensive view of how PMTs developed their instructional practices and use of digital tools through iterative engagement.



Mathematical Ideas

Students may identify the relationship between the line segment representing the circumference of the circle in 2D and the horizontal side of the rectangle formed in 3D, as well as between the height of the cylinder and the vertical side of the rectangle—ultimately leading them to derive the area formula of the cylinder.

Technological Actions

Three sliders are used to represent the height, the opening of the bases, and moving the black line segment on the cylinder:

- to illustrate the relationships between objects on the 2D and 3D screens.
- to "unwrap" the cylinder into a rectangular net, highlighting how the elements on both screens correspond to each other.

Classroom Orchestrations

Technical-demo: Introducing the role of the slider attached to the point.

Discuss-the-screen: By animating the sliders, Mira planned to guide a discussion on the relationship between the line segment near the circle on the 2D screen and the vertical side of the rectangle on the 3D screen. This was intended to support the wrapping and unwrapping of the cylinder and to use the area of the resulting rectangle to generalise the area formula of the cylinder.

Link-screen-board: Mira planned to draw the net of the cylinder on the board and connect it to the cylinder displayed on the screen during her explanation.

Fig. 4 Mira's instructional plan for teaching the surface area formula of a cylinder

3.4 Data analysis

Qualitative data analysis was guided by the theoretical constructs of IGpro and didactic reference points to identify challenges PMTs faced during the micro-teaching of technology-based cylinder tasks. We also examined how their IGpro with GeoGebra evolved through the teaching-reflection-revision cycle. This involved tracing how PMTs

anticipated challenges during planning, encountered them in practice, reflected on them in discussions and interviews, and adjusted their approaches in revised lesson plans—indicating the development of didactic reference points.

Our structured analysis began with lesson plans, which were examined to uncover PMTs' intentions, expectations, and instructional strategies for teaching cylinder concepts using GeoGebra. We then analysed micro-teaching videos to assess how plans were enacted, using a whole-to-part approach with a focus on the emergence of didactic reference points. These observations were triangulated with data from group discussions and interviews. Group discussions provided peer and researcher interpretations of didactic reference points, while interviews captured PMTs' own reflections on their learning.

To identify didactic reference points, we transcribed key moments where PMTs encountered unanticipated challenges. For instance, Ada developed a reference point coded as

Table 2 An example of the data analysis concerning Ada's didactic reference point on the use of sliders

Observations in micro-teaching: experiences/ challenges	**Our observation notes regarding Ada's micro-teaching:** • Ada intentionally manipulated the height (as well as the radius) slider to help construct the general formula for volume • At one point, she animated both sliders simultaneously and, after some observation, asked, "Can we obtain a formula from these relationships?" • Because the sliders were set at different speeds, the radius increased faster than the height. Ada then adjusted the speed of the height slider
Group discussions	**A suggestion from the group discussion related to Ada's teaching**: During the group discussion, both the instructor and the other PMTs noted that animating both sliders at the same time led to confusion. They emphasised the importance of adjusting or animating the sliders sequentially—one at a time—to maintain clarity
Individual interviews	**A direct quote from Ada's individual interview:** *"I did not anticipate that the speeds of the sliders would not be the same. During my micro-teaching, I decided to change the speed of the slider representing the height. While I was changing the speed, one of the students asked me a question, but my focus was on adjusting the slider settings at that moment."*
Revised plans	**In Ada's revised plan:** Ada revised the section concerning the manipulation of sliders and how she would guide the classroom discussion: *"Afterwards, based on what we have discussed and written on the board, I will ask how we can formulate the volume of the cylinder. I will give them time to discuss among themselves. As a hint, I will remind them that the volume depends on the height and the base area. Then, I will ask what operation might connect these two, expecting answers like addition or multiplication. I will again allow time for them to decide between the two. Finally, to support their reasoning, I will first animate the radius slider, then the height slider, stopping at random values to let students explore. After this exploration, I will wait for them to propose the multiplicative relationship. This way, they can investigate and derive the volume formula for the cylinder."*
Didactic reference points gained through the PD course	**Didactic reference point:** *Knowledge about how to use sliders advantageously for the didactic purpose of teaching mathematical generalisations* This didactic reference point emerged from Ada's initial challenge with mismatched slider speeds, which caused confusion among students. Although unplanned, her in-the-moment adjustment during micro-teaching, combined with feedback from the group discussion, led her to revise the lesson to better support student understanding

"*Knowledge about how to use sliders advantageously for the didactic purpose of teaching mathematical generalisations*". This emerged from an artefactual challenge during micro-teaching: mismatched slider speeds caused distorted visualisations. Although unplanned, Ada adjusted the speed of the slider during micro-teaching. Group discussion later highlighted the confusion caused by the animation, which she addressed in her revised plan. Table 2 illustrates an example of our analysis of Ada's mentioned didactic reference point. Rows detail observed challenges, discussion insights, interview quotes, and changes in her revised plan. The final row names the didactic reference point and summarises our inferences, based on comparing initial and revised plans and associated reflections.

To enhance the validity of the data, we triangulated multiple sources, including PMTs' initial and revised lesson plans, video recordings of micro-teaching, group discussions, and individual interviews. This triangulation strengthened the trustworthiness of the findings and enabled a more robust identification of PMTs' didactic reference points. Additionally, to ensure inter-coder reliability, we conducted independent analyses of the data, comparing our interpretations of the didactic reference points. Any discrepancies were resolved through discussions until we reached a consensus.

4 Results

In this section, we present our results on two PMTs and the evolution of their IGpro with GeoGebra through their micro-teaching experiences, which allowed them to gain four didactic reference points.

4.1 Didactic reference point 1: knowledge of the need to strengthen coordination between the 2D and 3D representations on screen

The first didactic reference point in teaching cylinder concepts relates to supporting students in understanding the volume and surface area formulas. From the planning stage, both Mira and Ada recognised the importance of coordinating GeoGebra's 2D and 3D screens to achieve this goal. However, doing so effectively required identifying appropriate instructional strategies for meaningful screen coordination.

During micro-teaching, both PMTs relied on slider tool, zoom-in/out, rotation, and other dynamic GeoGebra tools to manage the coordination of both screens. Ada began her lesson by demonstrating the 2D and 3D interfaces and used a slider linked to a line segment to show dynamic changes. Her aim was to help students understand key elements of the cylinder—particularly the base and height—by showing how manipulating the 2D circle affected the 3D representation. According to her plan, this was intended to clarify the relationship between the area of the circle and the base area of the cylinder. She reinforced this by rotating the 3D view to align with the 2D top-down perspective. A key challenge arose when adjusting the radius slider: the figures occasionally moved out of view on both screens (see Fig. 5), limiting students' ability to observe changes and respond to prompts—thereby weakening the intended visual connections.

In the individual interview, Ada explained she had not anticipated the figure moving off-screen during slider animation, as it had remained within frame on her own computer. When asked why she did not zoom in or out, she admitted feeling nervous:

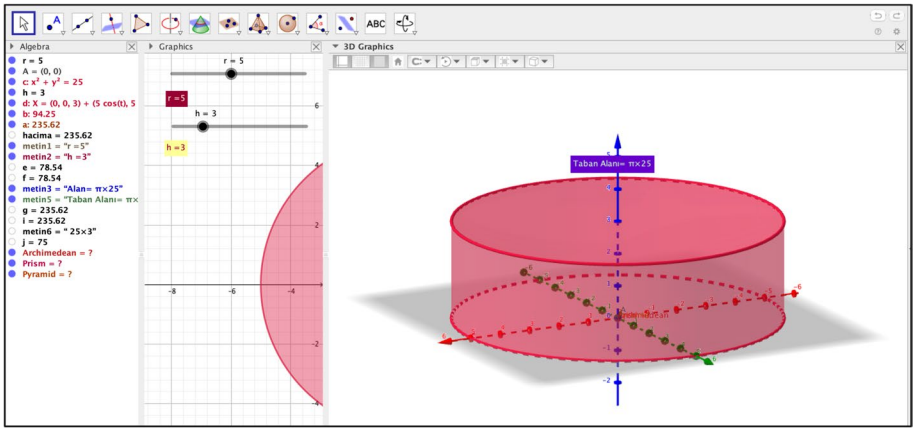


Fig. 5 Redrawn illustration of Ada's coordination of the two screens during micro-teaching

> I thought that the figure would get very small, and students could not see the dynamic text, so I was very nervous… I didn't know that boxes of the texts could be enlarged. I didn't want to try it during my first time because it could have been more difficult on the spot.

In her revised plan, Ada set the slider's maximum value to 5, stating: "If the radius exceeds 10, the cylinder won't fit within the 3D view, which may cause confusion". This practical adjustment reflected a new didactic reference point—an improved understanding of GeoGebra's limitations and how to structure tasks to maintain visual coherence. Although Ada could not resolve this issue during micro-teaching, her awareness of the problem shaped by her artefactual difficulties with GeoGebra's features and its pedagogical implications informed her revised instructional adjustments.

Mira also aimed to coordinate 2D and 3D screens to teach the surface area of a cylinder. She intended for students to connect the circle on the 2D screen to the horizontal side of the unwrapped rectangle in 3D. Using two sliders—one for the circumference and one for the unwrapping—she attempted to guide students' attention. However, they struggled to follow the intended relationships.

To guide their attention, Mira directed them to observe the black line segment on the cylinder in the 3D screen and the shape she had constructed on the 2D screen (see Fig. 6). Despite her efforts, students had difficulty coordinating the objects across both screens. While her primary goal was to help them relate the line segment on the 2D screen to the vertical side of the rectangle and recognise their simultaneous transformation during the unwrapping process, her instructions unintentionally led to confusion. By emphasising the black line segment on the 3D screen, she inadvertently made it unclear which elements students should focus on, hindering their understanding of the intended relationships.

After students identified the black line segment as the cylinder's height, Mira directed attention to the line segment near the circle on the 2D screen. However, students struggled to coordinate its meaning and asked which perspective to adopt. In response, Mira rotated the 3D cylinder to clarify their confusion—an unanticipated challenge during planning (see Bozkurt & Yiğit Koyunkaya, 2023b). In her interview, she reflected on how this moment revealed new artefactual and mathematical insights:

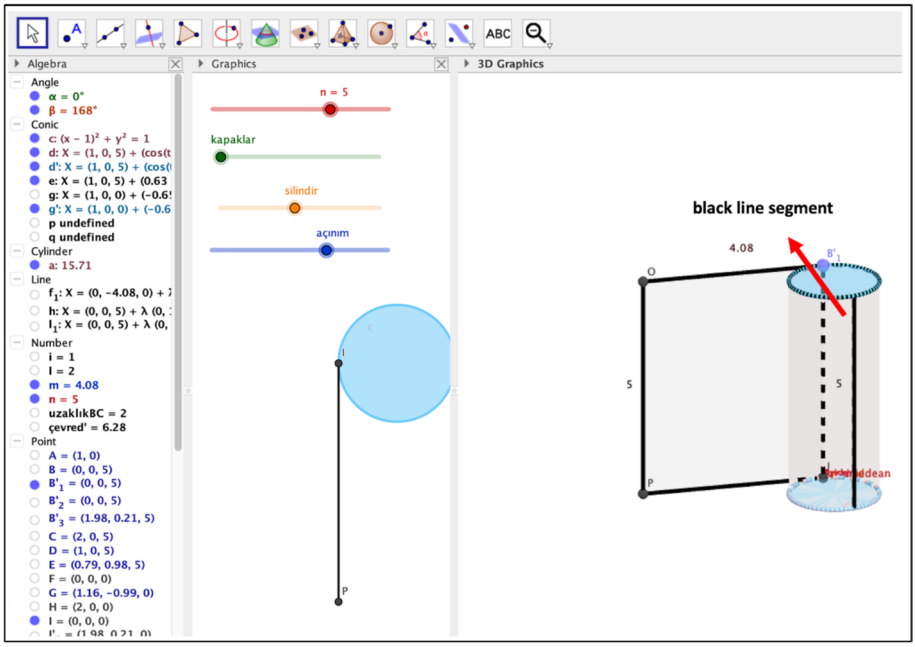


Fig. 6 Redrawn illustration of Mira's coordination of the two screens during micro-teaching

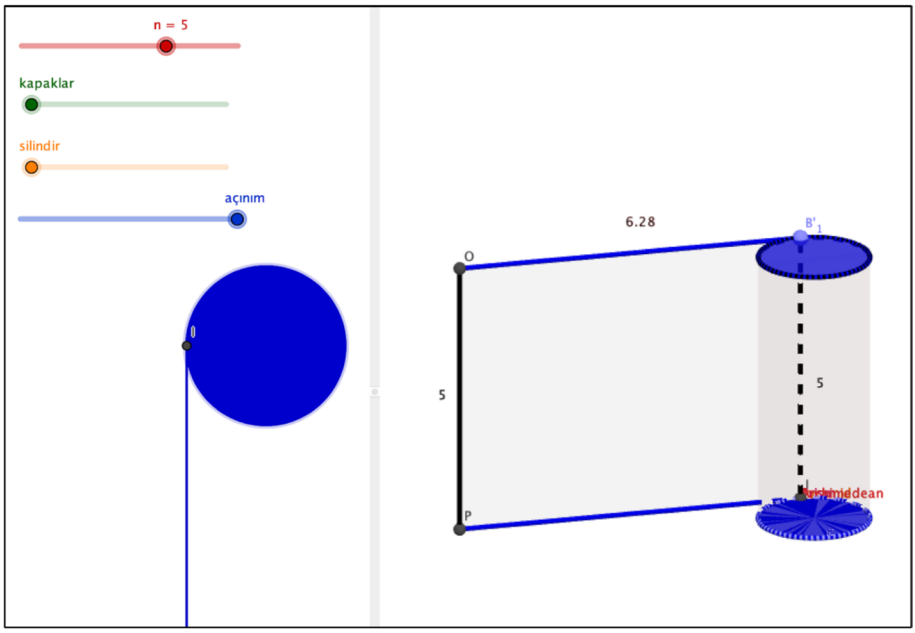


Fig. 7 Redrawn illustration of Mira's use of colour to highlight objects across both screens during micro-teaching

> I was not expecting that they would focus and question the 2D screen so much. That's why I did not examine the 2D part before the lesson. To be honest, I started considering the 2D part in micro-teaching… I answered students' questions by focusing on the 3D screen and rotated the top and the bottom of the cylinder whose appearance was in 2D, which I realised during the class.

To guide students, she used visual features in GeoGebra—colouring corresponding shapes on both screens and emphasising that the 2D circle matched the 3D base. She highlighted that their circumferences equalled the rectangle's longer side (see Fig. 7), explaining that unequal lengths would prevent the rectangle from wrapping the cylinder properly.

Initially, Mira had not fully considered how to coordinate 2D and 3D representations. Through micro-teaching, she recognised this gap—prompted by student questions—and developed a didactic reference point that deepened her mathematical and artefactual understanding regarding the relationships between the circle in 2D and the cylinder in 3D. This realisation became a didactic reference point, enhancing her instructional approach.

Table 3 PMTs' challenges and development of IGpro regarding coordination of 2D and 3D GeoGebra screens

PMT	Challenge during micro-teaching	Emerging didactic reference point	Evidence of IGpro development
Ada	The cylinder figures occasionally moved out of view during slider animation, limiting students' ability to observe changes. She felt nervous about zooming or resizing dynamic text during teaching	Awareness of GeoGebra's zoom and view limitations and the need to structure slider ranges to maintain visual coherence	Adjusted the slider's maximum value in her revised plan to keep the figure fully visible; acquired new artefactual knowledge about zooming and text resizing; integrated this knowledge into improved instructional design
	Initial unawareness of the impact of zoom and view settings on student comprehension during dynamic manipulation	Understanding the pedagogical importance of maintaining consistent visual frames to support student interpretation	Developed a more stable and coherent visualisation strategy in her revised lesson, reflecting professional and personal instrumental genesis
Mira	Students struggled to coordinate elements across the 2D and 3D screens; instructions emphasising the black line segment on 3D caused confusion about what to focus on	Recognition that clear instructional guidance and careful selection of visual focus are crucial when coordinating multiple representations	Developed the use of colour coding and aligning views between 2D and 3D to clarify relationships; reflected on unanticipated student focus and adapted teaching accordingly
	Difficulty anticipating students' attention to the 2D screen and managing perspectives; rotated 3D cylinder in response to confusion	Realisation of the importance of anticipating student perspectives and adjusting dynamic representations accordingly	Enhanced mathematical and artefactual understanding of the relationships between 2D and 3D representations, which was reinvested in task design and teaching practice

In conclusion, our observations and inferences indicate that both PMTs progressed in their IGpro, with GeoGebra becoming an effective instrument for teaching cylinder volume and surface area formulas. Ada advanced in both professional and personal instrumental genesis with GeoGebra. Initially unfamiliar with GeoGebra's zoom features, the micro-teaching experience prompted her to acquire artefactual knowledge, which helped her improve her IGpro for using these functions with didactic aims, critical for maintaining visual coherence during slider-based animations—an insight she reinvested in her revised plan.

Similarly, Mira recognised the need to improve coordination between 2D and 3D screens. In response, she revised her task by adjusting the 3D view to align with the 2D view and using colour to clarify relationships between representations. This refinement enhanced students' focus and reinforced her didactic intentions. For both PMTs, micro-teaching served as a catalyst for developing didactic reference points grounded in practical challenges. These experiences not only improved their technical use of GeoGebra but also informed their pedagogical strategies for teaching with dynamic visual tools. Table 3 summarises their challenges and IGpro development related to this first didactic reference point.

4.2 Didactic reference point 2: knowledge about how to use sliders advantageously for the didactic purpose of teaching mathematical generalisations

The second didactic reference point concerns technical knowledge about using sliders as a teaching tool for supporting mathematical generalisation. Both PMTs aimed to help students generalise relationships between height, surface area, and volume in a cylinder. Although their planning included slider use as a central technological action, their personal instrumental genesis with GeoGebra was not yet advanced enough to manage these features smoothly during teaching. In particular, they encountered challenges with slider speed and simultaneous animations—technical aspects they discovered in practice.

Both PMTs used GeoGebra's animation feature (activated by right-clicking and selecting "Animation On"), which enables automatic, continuous movement of slider values. This function supports continuous variation without requiring active user control, enabling students to focus on observing dynamic changes in the figure. Ada first animated the slider controlling the radius of the circular base and prompted students to observe how changes in base area affected volume of the cylinder, leading them to identify a proportional relationship. She then repeated this with the height slider and finally animated both simultaneously to show how volume depends on both dimensions. However, the sliders initially moved at different speeds, causing the radius to increase faster than the height. This discrepancy unintentionally added complexity to the representation, as the visual disproportion made it difficult for students to interpret a consistent multiplicative relationship. Recognising the issue, Ada paused the animation, adjusted the slider speeds, and resumed the animation with matched rates. This clearer representation, along with dynamic text showing volume values, helped students relate visual changes to symbolic generalisation. She concluded by asking, "Can we obtain a formula from these relationships?", prompting students to articulate the volume formula based on their observations.

At this stage, Ada manipulated the height slider while keeping the radius deliberately fixed at 1, using dynamic text to observe corresponding volume values. She then reversed the process—setting height to 1 and varying the radius—to explore how each variable

individually affects volume. To link visual representations with symbolic reasoning, she turned to the board and linked the screen activity to show how changes in height and radius relate multiplicatively to volume. By selecting various values and calculating volume, she guided students towards generalising the volume formula and introduced the concept of direct proportionality among the variables.

In her revised plan, rather than animating both sliders simultaneously, Ada chose to animate them consecutively. She also refined her instructional prompt:

> I will ask what operation connects height and base area—expecting responses like addition or multiplication. I'll give them time to choose, then animate the radius and height sliders separately, stopping at random values for exploration. This will allow them to investigate and decide on the appropriate multiplicative operation, leading them to derive the volume formula.

In Mira's case, sliders were the central technological feature in both planning and implementation. From the start of her micro-teaching, she introduced their functionality and designed her GeoGebra task with four sliders, intending to animate them simultaneously alongside guided questioning to explore geometric relationships. During the session, she attempted to animate three sliders simultaneously: (1) opening the cylinder's bases, (2) moving a black line segment along the height, and (3) unwrapping the rectangle around the cylinder, which is also related to the line segment IP on 2D screen. Her aim was to help students understand how the cylinder's net relates to surface area. However, technical difficulties arose during the session. The bottom base failed to open as intended, prompting her to restart the file. Additionally, the sliders did not animate in synchrony, despite her expectation based on prior rehearsal. This lack of synchronisation disrupted the intended flow of the visualisation.

Mira attempted to manually align the slider animations by dragging the sliders back and forth, trying to approximate simultaneous motion. She reflected on this challenge during the individual interview:

> When I dragged the sliders at home, they moved simultaneously. At home when I animated only the top one, the others moved at the same time. I didn't understand how it happened, but it was a good thing; they all animated together, starting at the same time, and ending in the same place. But in the class, for some reason this did not happen, I couldn't do it. They didn't start at the same time, so I had to constantly intervene. I tried to drag one forward, drag another one back because they needed to align simultaneously. I first stopped the one that started first, then the one that started later. If I did it in a real classroom, it could have been a problem for students to understand. Students might say, "These things don't move simultaneously, you are manually doing this".

Mira's reflection reveals a key moment of discrepancy between her planned use of digital technology and the realities of classroom implementation. Her experience illustrates a partial understanding of the animation function in GeoGebra, particularly the technical requirements for achieving synchronised animation across multiple sliders. While she believed the sliders had previously moved together because of some implicit system behaviour, her inability to replicate this during her micro-teaching highlights that her earlier success was likely coincidental rather than intentional or well understood. In relation to her use of sliders in GeoGebra, Mira developed a new understanding of how dynamic variation—specifically varying the height of a cylinder—can support students in grasping the formula for volume. Through her experience, she began to see the didactic value of using well-constructed visual representations to reveal multiplicative relationships in three-dimensional geometry. Despite

Table 4 PMTs' challenges and development of IGpro regarding slider animation and mathematical generalisation

PMT	Challenge during micro-teaching	Emerging didactic reference point	Evidence of IGpro development
Ada	The radius and height sliders moved at different speeds when animated simultaneously, leading to distorted visualisation of the volume relationship	Awareness of the importance of synchronizing slider speeds to maintain visual coherence when illustrating multiplicative relationships	Paused the animation during teaching to adjust slider speeds; later revised the lesson to animate sliders consecutively instead of simultaneously. Incorporated improved prompts to guide students toward generalising the volume formula
	Difficulty in isolating the effect of each variable (height or radius) on volume during simultaneous animation	Understanding the didactic value of varying one parameter at a time to help students perceive individual variable effects	Revised her instructional sequence to fix one variable (e.g., radius = 1) while varying the other, and supported symbolic reasoning by linking dynamic changes to volume formulas on the board
Mira	Attempted simultaneous animation of multiple sliders (base opening, height movement, unwrapping), but animations failed to synchronise, disrupting the intended visualisation	Recognition of the technical limitations of GeoGebra's animation function and the need for explicit control over slider coordination	Reflected on the unpredictability of prior success, acknowledged lack of control in class, and expressed concern about student misinterpretation. Gained awareness of the need to intentionally design and test slider synchronisation to support clear mathematical communication
	Repeated manual attempts to align sliders during teaching, creating cognitive and instructional overload	Realisation that imprecise animation can undermine the clarity of mathematical ideas being taught	Reflected in interview that uncoordinated animations could lead students to question the mathematical integrity of the representation. Developed awareness of the relationship between technical control and didactic clarity

technical challenges during the lesson, the micro-teaching provided a critical space for her to reflect on how the precision and coordination of slider animations are essential to conveying mathematical generalisations effectively. Table 4 summarises the PMTs' challenges and development of IGpro regarding the second reference point.

4.3 Didactic reference point 3: knowledge of how to use appropriate representations when employing dynamic variation of the height

The third didactic reference point relates to didactic knowledge that enables teachers to fully benefit from dynamic representations in GeoGebra. Lacking reference to this, Mira encountered challenges in fully exploiting the dynamic and visual affordances of GeoGebra. Her IGpro was not advanced enough to recognise that it is important to focus students' attention on relevant objects while using GeoGebra's dynamic features.

Indeed, during the planning stage, Mira had included a fourth slider representing the height and had discussed the role of dragging it. However, during the lesson, she forgot to use the slider representing the height to help students understand its role in the surface area formula and did not realise this at the time. Later, in the individual interview, she reflected on the height slider and acknowledged its crucial role:

Table 5 Mira's challenges and IGpro development regarding use of dynamic height variation in GeoGebra

Aspect	Description
Challenge during micro-teaching	Mira included a slider for height in her planned GeoGebra task but forgot to use it during the lesson, missing an opportunity to help students understand how height affects the cylinder's area. She also did not hide the segment representing height during unwrapping, causing confusion. The lack of coordinated use of the slider and instructional prompts prevented students from focusing on the key mathematical relationships
Emerging didactic reference point	Recognition of the importance of actively using all dynamic sliders, especially the height slider, to help students visually and conceptually grasp how height variation affects geometric properties. Acknowledgement of the need to hide or trace certain elements (e.g., the height segment) to focus student attention appropriately during dynamic representations
Evidence of IGpro development	In the individual interview and group discussions, Mira reflected on the missed opportunity and the pedagogical role of the height slider. She developed a new didactic reference point emphasising the coordination of dynamic technological tools and targeted instructional prompts. In her revised plan, she incorporated the use of a trace for the vertical side of the rectangle representing the cylinder's height, enhancing students' focus and conceptual understanding

> I forgot to change the height. There were four sliders there, and I never used the fourth one, and I always discussed it on the cylinder with the same height… If I changed the height of the cylinder, students could interpret what would also change in the cylinder. For example, the surface area of the rectangle would change, and naturally, the volume of the cylinder would change. Students could have constructed that relation if I hadn't forgotten to use the slider representing the height.

Additionally, while unwrapping the cylinder, Mira forgot to hide the segment representing the height of the cylinder. Her lack of slider uses and coordination between technological actions and prompts/questions prevented students from focusing on the roles of the vertical and horizontal sides of the rectangle. This caused confusion among the students. In the lesson she planned, this focus was crucial for students to generalise the surface area formula of the cylinder. Realising this issue and discussing the role of the sliders during the group discussion and interview prompted the development of a new didactic reference point (see Table 5). In her revised plan, she acknowledged the importance of using a trace for the vertical side of the rectangle, which also represents the height of the cylinder.

4.4 Didactic reference point 4: knowledge of the need for effective questioning and directing students' attention to support the use of dynamic unwrapping of the cylinder

The last didactic reference point concerns the teacher's role in interacting with students. In teacher-led activity using an interactive whiteboard to explore geometry with GeoGebra, such interactions are crucial. This professional knowledge was essential for enabling the PMT to make full use of GeoGebra's dynamic features when unwrapping the cylinder. This approach clearly demonstrated how the volume of a cylinder depends on its base area and height.

For this reference point, we draw only on Mira's planning and teaching, as Ada did not include cylinder unwrapping due to her lesson focus. For Mira, unwrapping the cylinder was a key strategy for teaching the surface area of cylinders with GeoGebra. She designed her task accordingly. As GeoGebra lacks a built-in macro for this, Mira invested significant time creating the construction from scratch, which proved more challenging than expected. She reflected on these difficulties during the interview:

> I want to start with how I couldn't do it. I tried every tool in GeoGebra—first rotating a rectangle to create a cylinder, but it didn't suit my goal. I looked at ready-made files, asked other teachers, and tested their suggestions, but none worked. I then decided to just show a cylinder and place a rectangle behind it. At first, the cylinder was centred on the rectangle [she explains how using a paper model]. I kept working for days and asked everyone for input. A friend suggested moving the rectangle to one side—and that's when I started to get the idea. If I'd used a ready-made file, I wouldn't have come up with this. I discovered a new way of unwrapping the cylinder and realised there's not just one way—there are many.

In the initial planning stage, Mira constructed a rectangle wrapped around the cylinder to support the surface area formula, using its vertical and horizontal sides as a basis. Her lesson plan centred on the wrapping and unwrapping process. However, difficulties with slider use and a lack of guiding questions hindered students from grasping the relationships between the objects. For example, she forgot to hide the cylinder's height during unwrapping, leading to confusion about which parts to focus on. She also directed students' attention to the black line segment involved in the unwrapping, but did not clearly guide them to see that the circle's perimeter on the 2D screen matched the rectangle's vertical side. In her revised plan, Mira addressed these issues by using GeoGebra's trace feature (see Table 6) and updated her instructional approach accordingly.

Table 6 Mira's challenges and IGpro development regarding effective questioning and directing attention in dynamic unwrapping

Aspect	Description
Challenge during micro-teaching	Mira invested considerable effort constructing the unwrapping task from scratch due to lack of ready-made GeoGebra macros, discovering new approaches but encountering technical and pedagogical difficulties. During the lesson, she struggled with slider coordination and guiding student attention, forgetting to hide the cylinder's height segment, which caused confusion. Her verbal prompts were insufficiently clear to help students relate the 2D circle's perimeter to the rectangle's sides, reducing conceptual clarity
Emerging didactic reference point	Recognition of the critical role of effective questioning and directing student focus when using dynamic GeoGebra tools. Understanding the need to use technological features (e.g., trace) to visually link 2D and 3D representations and to scaffold students' geometric reasoning during unwrapping
Evidence of IGpro development	Mira's reflection in interviews and revised planning showed enhanced professional instrumental genesis by integrating traces for key segments and points to support student visualisation. She updated her instructional strategy to animate sliders in reverse, use trace features, and deliver clearer verbal guidance, facilitating student understanding of the wrapping process and the geometric relationships involved

> Next, I will animate the sliders in reverse—the one for wrapping the cylinder will move backward, and the height slider will reset to its initial value. This time, I will use the trace for the line segment representing the cylinder's height so students can observe how the rectangle wraps around it. Simultaneously, I'll enable the trace of the points where the bases intersect with the rectangle, helping students see that the rectangle wraps precisely around the cylinder without overlapping.

5 Conclusions and discussion

This study examined two PMTs, Ada and Mira, as they designed, micro-taught, and revised GeoGebra-based tasks on cylinders. The findings highlight micro-teaching's effectiveness in supporting PMTs' IGpro by helping them develop key didactic reference points. Cross-analysis of their planning, teaching, and revisions identified four essential reference points: (1) coordination of 2D and 3D screens, (2) use of sliders, (3) utilising GeoGebra's dynamic affordances, and (4) employing visual affordances for unwrapping representations. While both PMTs showed initial awareness of these aspects, a gap emerged between their knowledge and their ability to apply it effectively.

Both PMTs aimed to coordinate objects across 2D and 3D screens but faced challenges due to gaps in (i) artefactual knowledge of GeoGebra's tools and (ii) mathematical understanding of the relationship between a circle in 2D and a cylinder in 3D. These difficulties prompted their IGpro development as they gained key didactic reference points, refining their ability to coordinate 2D-3D screens. Both also struggled to effectively integrate sliders due to technical knowledge gaps, realising the need for better instructional use of sliders. Mira alone acquired the final two reference points, overcoming confusion caused by limited coordination of technological actions and instructional prompts. This helped her to refine her IGpro in the importance of leveraging GeoGebra's dynamic affordances and structuring prompts to improve visualisation of mathematical concepts.

From a practical perspective, this study suggests that the structure of the PD course, particularly its micro-teaching component, effectively supported the development of PMTs' IGpro by fostering practice-based learning experiences (Niess, 2005, 2012). Previous research has established micro-teaching as a valuable tool for supporting PMTs' initial classroom experiences (Allen & Clark, 1967; Ledger & Fischetti, 2019). Our findings extend this by showing micro-teaching enabled PMTs to identify gaps in their artefactual and mathematical knowledge and to develop concrete didactic reference points, which helped refine their lesson planning and instructional practices (Bozkurt and Yiğit Koyunkaya 2022, 2023a; Zbiek & Hollebrands, 2008). The course's design, which emphasised collaborative discussions and feedback from peers and instructors, further supported reflective practice and the meaningful integration of digital tools. These results reinforce the importance of incorporating structured, reflective, and practice-oriented experiences into PD programmes (McCulloch et al., 2020). In line with the Association of Mathematics Teacher Educators (AMTE) Standards for Preparing Teachers of Mathematics (SPTM) (AMTE, 2017), this study highlights the value of engaging PMTs with research-based tools and pedagogical frameworks that support the development of core teaching practices. Together, these insights offer practical guidance for designing PD courses that promote PMTs' technological and pedagogical growth through sustained, hands-on engagement with digital technologies.

From a theoretical perspective, didactic reference points provide a useful lens to clarify the difficulties PMTs face when using technology in the classroom and to understand how their IGpro develops through such experiences. This study broadens the application of didactic reference points to encompass new educational contexts. Although primarily used to analyse in-service teachers' development in France (Haspekian, 2023) and Canada (Bakos, 2023), our findings show its relevance for pre-service teachers in the Turkish context, suggesting a broader role in shaping IGpro across diverse settings. In line with Goos' (2014) work on teacher proficiency trajectories, future research could explore how these reference points evolve over time in both pre-service and in-service programmes. This is particularly relevant given prior research showing that both PMTs and experienced teachers often hesitate to engage in open-ended, exploratory uses of technology, especially when outcomes are unpredictable (e.g. Lagrange & Caliskan-Dedeoglu, 2009). Such reluctance indicates that acquiring initial reference points is only the first step. Therefore, PD programmes should scaffold teachers' progression—beginning with simpler tasks and gradually introducing more complex scenarios—to build their confidence and capacity for managing technological complexity in the classroom. This study also identifies potential connections between didactic reference points and other constructs, particularly craft knowledge (Bozkurt & Ruthven, 2017; Ruthven, 2009). Our findings show that PMTs' micro-teaching experiences helped them recognise gaps in their knowledge and develop new didactic reference points, aligning with the experiential, practice-based nature of craft knowledge. Furthermore, these insights have implications for the design of classroom simulators (Emprin, 2022), which are developed to model anticipated didactic challenges. Given that our study identifies specific reference points related to teaching cylinders with GeoGebra, these could inform simulator scenarios aimed at supporting teachers' IGpro. Investigating how engagement with such simulators contributes to the formation and refinement of didactic reference points offers a promising direction for future research.

To acknowledge the generalisability of the results, we note that although this study focused on teaching cylinders using GeoGebra, the didactic reference points developed by the PMTs—such as coordinating 2D and 3D representations, using sliders effectively, leveraging dynamic and visual affordances, and aligning technological actions with instructional prompts—reflect broader pedagogical challenges and affordances common across many dynamic geometry systems (DGS). These findings suggest that the mathematical and artefactual knowledge gained is not confined to this specific topic or tool but has potential relevance across various mathematical content areas and DGS platforms. Thus, the reference points identified here offer a promising framework for supporting PMTs' IGpro development more broadly. Future research should examine the applicability of these reference points in diverse DGS environments and educational contexts to further assess and validate their generalisability.

To address a key limitation of this study, we acknowledge the inability to observe PMTs teaching in real classrooms due to the pandemic. Such observations, especially after lesson plan revisions, would have offered deeper insight into their IGpro development. Future research should overcome this gap by examining PMTs' instructional practices in real classroom settings and including post-teaching reflections to track their evolving knowledge. Moreover, further studies could investigate how mathematics teachers design and implement technology-based tasks across various curriculum goals, particularly in environments where students actively use digital tools. This would enhance our understanding of didactic reference points across diverse instructional contexts.

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Data availability The datasets generated and analysed during the current study are available from the corresponding author on request.

Declarations

Ethics approval In this study we followed the guidelines of the British Educational Research Association (BERA). All participants volunteered to participate in the data collection. To preserve confidentiality and anonymity, we ensured a secure data storage and changed names of the students to pseudonyms so that they cannot be identified in any publications.

Competing interests The authors declare no competing interests.

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