DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

CONVERGENCE OF SEQUENCES AND NETS IN A SPECIAL TOPOLOGICAL SPACE

by
Mehmet Fatih ÖZDEMİR

August, 2007
İZMİR

CONVERGENCE OF SEQUENCES AND NETS IN A SPECIAL TOPOLOGICAL SPACE

A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of Dokuz Eylül University
In Partial Fulfillment of the Requirements for the Degree of Master of Science
in Mathematics

by
Mehmet Fatih ÖZDEMİR

August, 2007
İZMİR

M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"CONVERGENCE OF SEQUENCES AND NETS IN A SPECIAL TOPOLOGICAL SPACE"** completed by **MEHMET FATİH ÖZDEMİR** under supervision of **ASSİST. PROF. DR. AHMET ZEMCİ ÖZÇELİK** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assist. Prof. Dr. AHMET ZEMCİ ÖZÇELİK

Supervisor

Prof. Dr. ŞENNUR SOMALI

(Jury Member)

Assist. Prof. Dr. ADEM ÇELİK

(Jury Member)

Prof.Dr. Cahit HELVACI
Director
Graduate School of Natural and Applied Sciences

ACKNOWLEDGMENTS

I wish to express my gratitude to my advisor, Assist. Prof. Dr. Ahmet Zemci Özçelik for support and encouragement he gave me while I was writing my thesis.

I owe my thanks to my girl friend Sultan Yılmaz who shared same feelings with me and encouraged me in writing this thesis.

Finally, I am deeply grateful to my family who never gave up supporting and encouraging me in my difficult times and the great patience they showed me during my study. Of course, I am also grateful to my fiancée for morale support and understanding. All these encouragement and support kept me going even when everything seemed to collapse.

Mehmet Fatih ÖZDEMİR

CONVERGENCE OF SEQUENCES AND NETS
IN A SPECIAL TOPOLOGICAL SPACE

ABSTRACT

Convergence concept is one of the most important concepts of topology. In this thesis, firstly classic meaning of convergence concept is investigated and is tried to discuss this mean. The relation between accumulation point and limit is stated clearly. Here is given the proof of theorem, known as if a sequence has a limit then it's unique, is valid only in T_2 – space. A sequence can have infinitely many limits in different topological spaces.

Especially, the problem of convergence of sequence in bitopological spaces is explained. Convergence of sequences is discussed in the case of topological structures, which build the bitopological space, is comparable, the conditions that appeared are explained.

Keywords : Convergence, Sequences, Bitopological Spaces.

ÖZEL TOPOLOJİK UZAYLARDA DİZİLERİN VE AĞLARIN YAKINSAMASI

ÖZ

Yakınsama kavramı topolojinin en önemli kavramlarından biridir. Bu tezde öncelikle yakınsama kavramının klasik anlamı üzerinde durulmuş ve bu anlamın sorgulanmasına girişilmiştir. Yığılma noktası ve limit arasındaki ilişki net bir şekilde ifade edilmiştir. Bir dizinin limiti varsa tektir şeklinde bilinen teoremin aslında sadece T_2 uzayında geçerli olduğunun ispatı verilmiştir. Farklı topolojik uzaylarda bir dizinin sonsuz tane limitinin de olabileceği gösterilmiştir.

Özel olarak bitopolojik uzaylarda dizinin yakınsaması problemine bir yorum getirilmiştir. Bitopolojik uzayı oluşturan topolojik yapıların karşılaştırılabilir olması durumunda dizilerin yakınsaması tartışılmış ve ortaya çıkacak durumlar için yorumlar yapılmıştır.

Anahtar Kelimeler : Yakınsama, Diziler, Bitopolojik Uzaylar.

CONTENTS

Page

THESIS EXAMINATION RESULT FORM ……... ii

ACKNOWLEDGEMENTS …….. iii

ABSTRACT …… iv

ÖZ ……. v

CHAPTER ONE – INTRODUCTION ……... 1

1.1 Terminology and Notation ……..….. 1

1.2 Topological Definitions ……..…... 2

1.3 Finer and Coarser Topologies ……..…. 7

1.4 Nets in Topological Spaces……..... 10

CHAPTER TWO – CONVERGENCE OF SEQUENCES …….. 12

2.1 Definitions About Sequence ……. 12

2.2 Convergence of Sequences in Topological Meaning ……... 14

2.3 Failure of The Sequences ……..... 15

CHAPTER THREE – BITOPOLOGICAL SPACES ……. 20

3.1 Definitions About Bitopological Spaces ……..… 20

3.2 Convergence of Sequences in Bitopological Spaces …….. 21

REFERENCES ……..……..…… 25

CHAPTER ONE
INTRODUCTION

Convergence which is one of the most important concept for topology in general as for analysis in special, has been the main problem of this thesis. Discussions are done about convergence in three parts in this thesis.

At first part, firstly main definitions and theorems about topology are given. Applications about comparable topological spaces are done. Nets, which are more general concept from sequences, are explained.

At second part, main definitions of sequences concept and mean of convergence of sequences are studied. Convergence of a sequence is explained topologically. Relation between accumulation points and limit is discussed and then limit of a sequence not unique in diffirent topological spaces is expressed.

At last part, bitopological spaces concept and main definitions of that space are given. Resolution is presented for problem of convergence of sequences in bitopological spaces. Discussion are done about the resolution according to topologies are comparable.

1.1 Terminology and Notation

A topology τ on the set X is shown by (X, τ) as a topological space. A property is said to be topological if the property is preserved under the homeomorphisms. A subset Y of X is said to be a subspace of (X, τ) if it is denoted by the topology τ_Y which has the sets which occur by the intersection Y and the sets of τ. In a topological space (X, τ), the usual closure (interior) of a subset A of X is denoted by $(\overline{A})$, (Å).

1.2 Topological Definitions

First of all we will remind some topological definitions and theorems.

Definition 1.2.1. A metric on a nonempty set X is a function,

$d: XxX \to IR$

$(x, y) \to d(x, y)$

that satisfies the following conditions:

M_1 : $d(x, y) \geq 0$ for all x, y in X

M_2 : $d(x, y) = d(y, x)$ for all x, y in X

M_3 : $d(x, y) = 0$ if and only if $x = y$

M_4 : $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z in X.

Definition 1.2.2. Let (X, d) be metric space, let ε be a positive number and let $x_0 \in X$. The set $B_d(x_0, \varepsilon) = \{x : d(x_0, x) < \varepsilon, x \in X\}$ is called ε - ball (ε - neighborhood of x_0) with center at x_0.

Definition 1.2.3. A subset T of a metric space (X, d) is said to be open set (d- open set). If for each $x \in T$, there is an open ball $B(x, \varepsilon)$ such that $B(x, \varepsilon) \subseteq T$.

From the definition of open set, a closed set is defined as complement of open set.

$A \subset X$, A is a closed set iff A^C is an open set,

A is an open set iff A^C is a closed set.

Example 1.2.4. Let S be the plane, that is, the product of the set of the real numbers with itself. We define three metrics on S as follows.

For $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ two points in S,

$$d_1(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

$$d_2(P_1, P_2) = \max \{ |x_2 - x_1| , |y_2 - y_1| \},$$

$$d_3(P_1, P_2) = |x_2 - x_1| + |y_2 - y_1|.$$

The ball of radius a about the point O = (0, 0) relative to each of these metrics is indicated by the shaded areas in Figure 1.2. Note that a ball does not necessarily have a circular, or even a smooth, boundary.

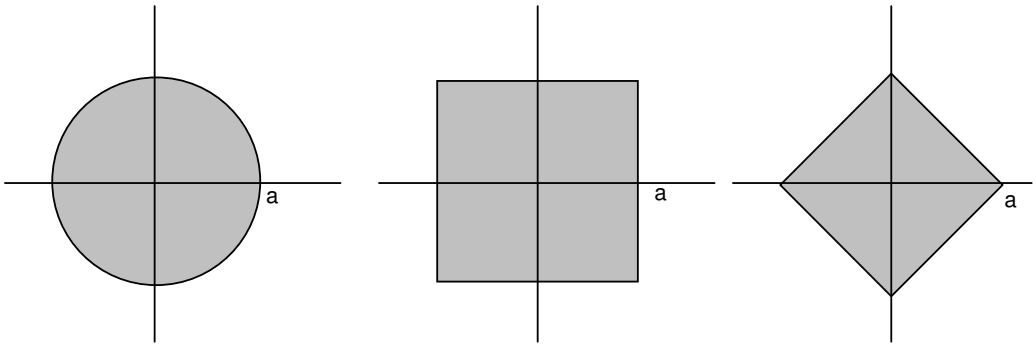


Figure 1.2.

Definition 1.2.5. A topology on nonempty set X is a class τ subsets X having the following axioms:

A_1) X and $\varnothing$ are in τ

A_2) The union of any number of sets in τ belongs to τ

A_3) The intersection of any finite number of sets in τ belongs to τ.

Definition 1.2.6. Let (X, τ) be any topological space and T is any subset of X. If T belongs to τ then T is called an open (τ - open) set. That is; T is an open iff $T \in \tau$.

Definition 1.2.7. Let X is a non empty set and $\tau = \beta(x)$ is the collection of all subsets of X. Then τ is a topology on X, this topology is called discrete topology and the space (X, τ) is called discrete space. In this space all subsets of X are open sets.

Definition 1.2.8. Let X is a non empty set and let $\tau = \{X, \varnothing\}$. Then τ is a indiscrete topology on X. In this space open sets are X and $\varnothing$.

Definition 1.2.9. Let (X, τ) be topological space. A subset F of X is called closed if the set F^C is open. So;F is a closed set iff F^C is an open set,

T is an open set iff T^C is a closed set.

Definition 1.2.10. Let X be a topological space and A is a subset of X. A point $x \in A$ is, called an interior point of A if x belongs to an open set T contained in A that is, $x \in T \subset A$ where T is an open set.

Definition 1.2.11. Let X be a topological space and $A \subset X$. The set of all interior points of A, denoted by int(A) or $\mathring{A}$, is called the interior of A.

Definition 1.2.12. Let X be a topological space and A is any subset of X. A point $x \in X$ is called an exterior point of A if it belong to the set $\{\mathring{A}^C\}$. The exterior of A, written ext (A), is the set of all exterior points of A.

$$x \in \text{ext}(A) \Leftrightarrow x \in \{\mathring{A}^C\} = \text{int}(A^c).$$

Definition 1.2.13. Let (S,τ) be a topological space. Let $A \subset S$. A point $s \in S$ is a limit point of A if for each $U \in \tau$ such that $s \in U$,

$$(U - \{s\}) \cap A \neq \varnothing.$$

Definition 1.2.14. The closure of a set $A \subset S$, denoted by $\overline{A}$, is the set,

$$\overline{A} = A \cup \{ s \in S; s \text{ is a limit point of } A\}.$$

Theorem 1.2.15. The closure $\overline{A}$ of a set A is closed.

Proof. We must show that $\overline{A}'$ is open. For this it suffices to show that for each $s \in \overline{A}'$ there exists an open set U_s with $s \in U_s \subset \overline{A}'$. Then $s \in U_s$ for each s implies $\overline{A}' \subset \bigcup_{s \in \overline{A}'} U_s$ and $U_s \subset \overline{A}'$ for each s implies $\bigcup_{s \in \overline{A}'} U_s \subset \overline{A}'$. Thus $\overline{A}' = \bigcup_{s \in \overline{A}'} U_s$ is a union of open sets and hence is open .

Now let $s \in \overline{A}'$. Then s is not a limit point of A, so there exists an open set U_s such that $s \in U_s$ and $(U_s - \{s\}) \cap A = \varnothing$. Furthermore, $s \notin A$ because $s \notin \overline{A}$ and hence, in fact, $U_s \cap A = \varnothing$. Since each element of U_s is contained in an open set, namely U_s

itself, whose intersection with A is $\varnothing$, it follows that U_s contains no limit points of A and $U_s \cap \overline{A} = \varnothing$; that is, $U_s \subset \overline{A}'$.

Theorem 1.2.16. A set A is closed if and only if $A = \overline{A}$.

Proof. Assume A is closed. Then A' is open. If $s \notin A$, then A' is an open set containing s such that $(A' - \{s\}) \cap A = \varnothing$. Thus s is not a limit point of A. Hence all limit points of A lie in A; that is, $A = \overline{A}$.
Conversely, if $A = \overline{A}$, then A is closed by the previous theorem.

Definition 1.2.17. Let (X, τ) be a topological space a subclass of β of τ (i.e. collection of open sets) is said to be a base for they topology τ, if each member of τ is the union of the members of β.

Definition 1.2.18. Let (X, τ) be a topological space and $x \in X$. A local base at x is a collection β_x of τ.
1) If $B \in \beta_x$ then $x \in B$
2) If $T \in \tau$ and $x \in T$, then $\exists B \in \beta_X \ni B \subseteq T$.

Corollary 1.2.19. A metric space has the structure of a topological space in which the open sets are unions of balls.

Definition 1.2.20. Let S be a set, and let β_1 and β_2 be bases for topologies τ. β_1 and β_2 are equivalent if they generate the same topology; that is, if $\tau_{\beta 1} = \tau_{\beta 2}$.

Corollary 1.2.21. The three metrics p_1, p_2, p_3 on the plane S described earlier all determine the same topology on S.(Example 1.2.4.)

Remark 1.2.22. It is mor true, however, that all metrics on a given space give the same topology. For example, consider the space R of real numbers with the following two metrics.

$$p_1(r_1, r_2) = |r_1 - r_2|$$

$$p_2(r_1, r_2) = \begin{cases} 0 , (r_1 = r_2) \\ 1 , (r_1 \neq r_2) \end{cases}$$

Then open sets in the topology defined by p_1 are the usual open sets (generated by open intervals) in R, whereas the collection of open sets defined by p_2 is the set 2^n of all subsets of R. For, in fact, relative to p_2, $B_r(\frac{1}{2}) = \{r\}$ for each $r \in R$, so each "point" is an open set; and, hence, so is each union of "points," that is, each subset of R.

Theorem 1.2.23. Let S be a topological space and let $A \subset S$, if A is open in S, then every open set in A is open in S. If A is closed in S, then every closed set in A is closed in S.

Proof. If A is open in S and B is open in A, then $B = U \cap A$ for some open set U in S; so B is open in S since both U and A are open in S.

If A is closed in S, and B is closed in A, then A – B is open in A;
so $A - B = A \cap U$ for some open set U in S. Thus,

$$B = A - (A - B) = A - A \cap U = A \cap U'.$$

Since A and U' are both closed in S, so is $A \cap U' = B$.

Definition 1.2.24. A topological space (X, τ) is said to be T_o if given any two distinct point x and y of X there exist an open neighborhood of at least one which does not contain the other. In other words,
if $x, y \in X$, there exist on open set $U \in \tau$ such that either $x \in U$ and $y \notin U$ or
$y \in U$ and $x \notin U$.

Definition 1.2.25. A topological space (X, τ) is said to be T_1 if for any two distinct points x and y of X there is an open neighborhood of x which does not contain y and any open neighborhood of y which does not contain x. In other words,

if x, y $\in$ X, there exist an open set U_x, $U_y \in \tau$ containing x and y respectively, such that $y \notin U_x$, and $x \notin U_y$.

Definition 1.2.26. A topological space (X, τ) is called a Hausdorff topological space (or T_2) if for any two different points x_1 , x_2 of the set X one can find sets $G_1 \in \tau$, $G_2 \in \tau$ (τ - open sets) such that $x_1 \in G_1$, $x_2 \in G_2$, $G_1 \cap G_2 = \varnothing$. In other words,

if x, y $\in$ X, there exists disjoint open sets U_x and U_y containing x and y respectively.

1.3 Finer and Coarser Topologies

We have already encountered the probem of determining if two topologies on a set X are equal, that is, if they contain the same open sets. This section is devoted a more complete discussion of this topic.

Definition 1.3.1. Let X be any set, and suppose that τ_1 and τ_2 are two topologies on X. Then τ_1 is said to be stronger (finer) than τ_2 if $\tau_2 \subset \tau_1$, that is, any τ_2 –open set is also a τ_1 – open set. τ_1 is said to be strictly stronger than τ_2 if τ_1 is finer than τ_2 but $\tau_2 \neq \tau_1$. If τ_1 is finer than τ_2, then we may say that τ_2 is (weaker) coarser than τ_1.

Proposition 1.3.2. Let X be any set. Then two topology τ_1 and τ_2 on X are equal if and only if τ_1 is finer than τ_2 and τ_2 is finer than τ_1.

Proof : Since $\tau_1 \subset \tau_2$ and $\tau_2 \subset \tau_1$ then $\tau_1 = \tau_2$.

Example 1.3.3. Let X be any set. Then the discrete topology on X is finer than any topology on X and it is strictly finer than any other topology on X. Indiscrete topology on X is coarser than any topology on X.

Example 1.3.4. Let X = {a, b, c, d} and,

τ_1 = {$\varnothing$, X, {a}, {d}, {a, d}, {b, d}, {a, b, d}},

$\tau_2 = \{\varnothing, X, \{a\}, \{a, d\}\}$, $\tau_3 = \{\varnothing, X, \{a\}\}$, $\tau_4 = \{\varnothing, X, \{b, c\}, \{a, d\}\}$,

$\tau_5 = \{\varnothing, X\}$, $\tau_6 = P(x)$ then,

$\tau_1 \subset \tau_6$, $\tau_2 \subset \tau_6$, $\tau_3 \subset \tau_6$, $\tau_4 \subset \tau_6$, $\tau_3 \subset \tau_6$

$\tau_5 \subset \tau_1$, $\tau_5 \subset \tau_2$, $\tau_5 \subset \tau_3$, $\tau_5 \subset \tau_4$, $\tau_5 \subset \tau_6$

$\tau_2 \subset \tau_1$, $\tau_3 \subset \tau_1$ ….

But τ_4 is not weaker nor stronger then τ_1 , τ_2 , τ_3.

Proposition 1.3.5. Let X be any set and suppose τ_1 and τ_2 are two topologies on X. $\tau_1 \subset \tau_2$ iff every τ_1 – closed set is also τ_2 closed set.

Proof. Let $\tau_1 \subset \tau_2$ and B is any τ_1 – closed subset of X, then $B^C \in \tau_1 \subset \tau_2$, since B^C is τ_2 – open, so B is a τ_2 – closed.

Suppose every τ_1 – closed set is τ_2 – closed set. Take any τ_1- open set T, then T^C is τ_1 – closed implies that T^C is τ_2 – closed, means that T is τ_2 – open, so $\tau_1 \subset \tau_2$.

Example 1.3.6. Consider the cofinite topology τ and the usual topology U on the real line R. Since every finite subset of IR is U – closed set; hence the complement of any finite subset of IR which is τ - open, is also U – open set. In other words, τ is weaker that U, i.e. $\tau \subset U$.

Example 1.3.7. Consider the usual topology U, left topology D^- and right topology D^+ on the real line IR. Since all the D^- - closed sets and al the D^+ - closed sets are the U – closed sets then both the topology D^- and D^+ are weaker then the topology U, that is $D^- \subset U$ and $D^+ \subset U$ but the topology D^- and the topology D^+ are not comparable.

Proposition 1.3.8. Let X be any set and τ_1 , τ_2 are two topologies on X. Then $\tau_1 \subset \tau_2$ if and only if for τ_2 – closure of A is contained by τ_1 – closure of A for all $A \subset X$ that is;

$$\tau_1 \subset \tau_2 \Leftrightarrow \overline{A}\,\tau_1 \supset \overline{A}\,\tau_2 , \quad \forall A \subset X.$$

Example 1.3.9. Let $X = \{a, b, c\}$ and $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau_2 = \{\emptyset, X, \{a\}\}$. Clearly τ_1 and τ_2 are topologies on X and $\tau_2 \subset \tau_1$. Take a subset $A = \{c\}$. Then, $\overline{A}\,\tau_1 = \{a\}$ and $\overline{A}\,\tau_2 = \{b, c\}$ and $\overline{A}\,\tau_1 \subset \overline{A}\,\tau_2$.

Theorem 1.3.10. Let τ_1 and τ_2 be two topologies on the set X. Then $\tau_1 \subset \tau_2$ if and only if for each point $x \in X$ and each τ_1 – neighborhood of x is also τ_2 – neighborhood of x, that is,

$$\tau_1 \subset \tau_2 \iff V\tau_1(x) \subset V\tau_2(x) \text{ for all } x \text{ in } X.$$

Proof. Let $V \in V\tau_1(x)$ then $\exists\, T \in \tau_1 \ni x \in T \subset V$ since $\tau_1 \subset \tau_2$ and $T \in \tau_1$ then $T \in \tau_2$ so $V \in V\tau_2(x)$.

Let $V\tau_1(x) \subset V\tau_2(x)$ for all $x \in X$ and let $T \in \tau_1 \ni x \in T$. Then $T \in V\tau_1(x)$ (because T is open contain X.) so $T \in V\tau_2(x)$ hence $T \in \tau_2$, this means that $\tau_1 \subset \tau_2$.

Theorem 1.3.11. Let τ_1 and τ_2 are two topologies on a set X and let β_1 and β_2 be two bases for τ_1 and τ_2 respectively. Then the following are equivalent;

a) τ_2 is finner than τ_1

b) for each $x \in X$ and each $B \in \beta_1 \ni x \in B$, there is a set $D \in \beta_2 \ni x \in D \subseteq B$.

Proof. (a $\Rightarrow$b) Suppose τ_2 is finer than τ_1 , let $x \in X$ and let $B \in \beta_1$, $x \in B$. Since $B \in \tau_1 \subset \tau_2$ then $B \subset \tau_2$. Since β_2 is a base for τ_2 then $\exists\, D \in \beta_2 \ni x \in D \subset B$.

(b $\Rightarrow$ a) let $U \in \tau_1 \ni x \in U$ since β_1 is a base for τ_1 then $\exists B_x \in \beta_1 \ni x \in B_x \subset U$. By condition b) $\exists\, D_x \in \beta_2 \ni x \in D_x \subset B_x \subset U$ so $x \in D_x \subset U$ means that $U \in \tau_2$ and $\tau_1 \subset \tau_2$.

Corrollary 1.3.12. Let τ_1 and τ_2 be two topologies on X and let β_1 and β_2 be bases for τ_1 and τ_2 respectively. Then $\tau_1 = \tau_2$ ($\beta_1 \cong \beta_2$) if and only if,

a) $\forall\, B \in \beta_1 \ni x \in B$, $\exists\, D \in \beta_2 \ni x \in D \subset B$ and

b) $\forall\, B \in \beta_2 \ni x \in C$, $\exists\, A \in \beta_1 \ni x \in A \subset C$.

Example 1.3.13. Use the above corollary to show that the lower (upper) limit topology on IR is strictly finer then the usual topology on IR.

Solution. $\beta_1 = \{[a, b) : a, b \in IR, a < b\}$ is a basis for U_{low}

$\beta_2 = \{(a, b) : a, b \in IR, a < b\}$ is a basis for U

take $[1, 2) \in \beta_1$ and $1 \in [1, 2)$. However if $(a, b) \in \beta_2$ and $1 \in (a, b)$ then, $(a, b) = (1 - \varepsilon, b)$ and $(1 - \varepsilon, b) \not\subset [1, 2)$ from this $U_{low} \not\subset U$.

But since every $(a, b) \in \beta_2$ is equal to the union of sets $[a + \frac{1}{n}, b)$ in β_1,

$$(a, b) = \bigcup_{n \in IN} [a + \frac{1}{n}, b)$$ then U is strictly weaker then U_{low}.

Theorem 1.3.14. If τ_i are topologies on X. Then $\tau = \bigcap_i \tau_i$ is a topology on X.

Proof : A_1) $\forall i \in I, \varnothing \in \tau_i$ and $x \in \tau_i$ then $\varnothing \in \tau$ and $x \in \tau$.

A_2) Let $T_k \in \tau$ then $T_k \in \tau_i$ for all $i \in I$ and $\cup T_k \in \tau_i$ therefore $\cup T_k \in \tau$.

A_3) Let $U, V \in \tau$ then $U, V \in \tau_i, \forall i \in I$ and $U \cap V \in \tau_i$ so for all $i \in I$ so $U \cap V \in \tau$.

Remark 1.3.15. From above theorem if $\tau = \cap \tau_i$ then $\tau \subset \tau_i$ for all $i \in I$.

Remark 1.3.16. It follows Theorem 1.3.14. that, given a family $\{\tau_i\}$ of topologies on a set X, there always exist a weakest topology τ on X $\ni \tau = \cap \tau_i$ and $\tau \subset \tau_i$, $\forall i \in I$.

1.4 Nets in Topological Spaces

If x_n is a sequence of points in a topological space R, x_n is said to be convergent to $x_0 \in R$ if for each neighbourhood U of x_0 there exists $n_0(U)$ such that $x_n \in U$ for all $n \geq n_0$ (U). x_0 is called limit of the sequence (x_n), and we write $(x_n) \to x_0$. If R is Hausdorff, a sequence can only have one limit. In this case we write $x_0 = \lim_n (x_n)$.

In a general topological space R an accumulation point of a set M need not be limit of a sequence of points of M. By using directed sets, however, the concept of limit is generalized in such a way that every accumulation point becomes a limit.

Let A be a directed set, and let M be a general set. If for each $\alpha \in A$ an $x_\alpha \in M$ is given, the x_α form a net in M. When A = 1, 2, … we obtain sequences special cases of nets.

The net x_α, $\alpha \in A$, is said to be convergent to $x_0 \in R$ if for each neighbourhood U of x_0 there exists $\beta(U) \in A$ such that $x_\alpha \in U$ for all $\gamma \geq \beta$ (U). x_0 is called limit of x_α and we write $x_\alpha \rightarrow x_0$.

The concept of subsequence can also be generalized: a subset B of a directed set A is cofinal if for each $\alpha \in A$ there exists $\beta \in B$ with $\beta \geq \alpha$. If a subset of a directed set is not cofinal, then its complement is. If x_α, $\alpha \in A$, is a net, the x_β form a cofinal subnet if the β form a cofinal subset of A.

If x_0 is limit of the net x_α, x_0 is also limit of every cofinal subnet. A point y_0 is called an adherent point of the net x_α if every neighbourhood of y_0 contains a cofinal subnet. Every adherent point the net x_α is a closure point of the subset of R consisting of the distinct x_γ, but not conversely. Every limit of x_α is an adherent point.

CHAPTER TWO
CONVERGENGE OF SEQUENCES

2.1. Definitions About Sequence

We will remind the properties that we have been introduced to the notions of the convergence of sequences, limits and continuity in calculus.

Definition 2.1.1. A sequence in a set X denoted by $\{a_n , n \in IN\}$ or $(a_n)_{n\in IN}$ is a function from the set IN to X.

Definition 2.1.2. A subset as $(a_n)_{n\in IN} = \{a_n \in X : n\in IN\}$ is called a sequence in X.

Definition 2.1.3. We say that a sequence $(a_n)_{n\in IN}$ converges to the limit L, and we write $\lim_{n\to\infty} a_n = L$ if for every positive real number ε there exists an integer N(which may depend on ε) such that : if $n \geq N \Rightarrow |a_n - L| < \varepsilon$.

Example 2.1.4. Show that $\lim_{n\to\infty} \frac{c}{n^p} = 0$ for any real number c and any $p > 0$.

Solution. Let c and n be any real number and $p > 0$, $\forall \varepsilon > 0$, if $\exists N = N(\varepsilon)$,

$\left|\frac{c}{n^p} - 0\right| < \varepsilon$ if $n^p > \frac{|c|}{\varepsilon}$.

That is, if $n \geq N$, the least integer greater than $\left(\frac{|c|}{\varepsilon}\right)^{\frac{1}{p}}$. By definition 2.1.3.,

$\lim_{n\to\infty} \frac{c}{n^p} = 0.$

Example 2.1.5. Let $(a_n)_{n\in IN}$ be any sequence in R.

$(a_n) \to a \in IR \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in IN \ni \forall n \geq n_0 \Rightarrow |a_n - a| < \varepsilon.$

This means $a - \varepsilon < a_n < a + \varepsilon$ or each term after the n_0 th lies inside interval $(a - \varepsilon , a + \varepsilon)$ only the terms before a_{n_0} and there are only a finite number of them can lie outside the interval $(a - \varepsilon, a + \varepsilon)$.

Example 2.1.6. In IR the following sequences,

$\{1, 1/2, 1/3, \dots \}$,

$\{1, 0, 1/2, 0, 1/3,0, 1/4, 0 \dots \}$,

$\{1, -1/2, 1/3, -1/4, \dots \}$ converge to 0,

since every $(-\varepsilon, \varepsilon)$ contains almost al of the elements of each of the sequences.

Example 2.1.7. Any constant sequence $\{x_0, x_0, \dots \}$ converges to x_0. Indeed;

$\forall \varepsilon > 0 \ , \exists N = N(\varepsilon) \ , \ n \geq N \ , \ |x - x_0| < \varepsilon.$

$\{1, 1, 1, \dots \} \to 1$

$\{-3, -3, -3, \dots\} \to -3$

Example 2.1.8. Consider the sequence $\{1/2, 1/4, 3/4, 1/8, 7/8, \dots \}$ that is the sequence,

$$a_n = \begin{cases} \dfrac{1}{2^{\frac{n}{2}+1}}, \text{ if } n \text{ is even} \\ \\ 1 - \dfrac{1}{2^{(n+1)/2}}, \text{ if } n \text{ is odd} \end{cases}$$

Since every neighborhood of 0 and every neighborhood of 1 contains infinitely many terms of the sequence (a_n). Neither 0 nor 1 is a limit of the sequence (a_n).

Proposition 2.1.9. The limit of a sequence is unique.

Proof : Assume that there exist two limits x and y for the sequence (a_n) and $x \neq y$. That is;

$\lim_{n \to \infty} a_n = x \quad , \quad \lim_{n \to \infty} a_n = y.$

We let $\varepsilon = |x - y| / 2$ then $|a_n - x| < \varepsilon$ for $n \geq N_1$, and similarly , $|a_n - y| < \varepsilon$ for $n \geq N_2$. Therefore if N_0 is the larger of N_1 and N_2, for $n \geq N_0$ we have both $|a_n - x| < \varepsilon$ and $|a_n - y| < \varepsilon$, so that,

$2\varepsilon = |x - y| = |x - a_n + a_n - y| \leq |x - a_n| + |a_n - y| < \varepsilon + \varepsilon = 2\varepsilon.$

And it is impossible, this is a contradiction. Hence $x = y$.

2.2 Convergence of Sequences in Topological Meaning

In this section we will try to construct topological structures with convergence of sequences.

Definition 2.2.1. (X, τ) topological space, a sequence $(x_n) \subset X$ and $x \in X$. If for every neighborhood of X there exist a natural number n_0, such that for every natural number $n \geq n_0$, x_n is an element of this neighborhood than x is called the limit point of this sequence, that is;

$\forall V \in V(x), \ \exists n_0 \in IN$, i.e., $\forall n \geq n_0$, $x_n \in V$.

This means the sequence (x_n) converges to x. It is shown as,

$(x_n) \rightarrow x$ or $\lim_{n \to \infty} x_n = x$.

Definition 2.2.2. (X, τ) topological space, $A \subset X$, $x \in X$. If for every neighborhood of x has an element y from A such that $x \neq y$, x is called accumulation point. The set of accumulation points of the set A is denoted by ' Ã'.

x is an accumulation point of $A \Leftrightarrow \forall V \in V(x)$, $A \cap V \neq \varnothing$ and $\exists y \in A \cap V$, $x \neq y$.

Remark 2.2.3. The limit point and the accumulation point are not same.

Example 2.2.4. Let (X, τ) be a topological space, $(x_n) \subset X$ is a sequence such that,

$x_n = x \in X, \forall n \in IN$.

That is;$(x_n) = (x, x, x, x, \ldots)$.

From the definition of limit $\forall\ V \in\ v(x)$, $\forall n \in$ IN, $x_n = x \in\ V$. Hence $(x_n) \to x$ so x is the limit point of (x_n).

However all of the terms of this sequence equal to x by definition 2.2.2, x is not accumulation point.

Example 2.2.5. (IR, U) usual topology,

The sequence $(x_n) = ((-1)^n + 1/n)$, $n\in$ IN, has no limit point.

Since, if n is odd $\lim_{n\to\infty} (x_n) = (1 - n)/n = -1$,

if n is even $\lim_{n\to\infty} (x_n) = (1 + n)/n = 1$

they are not equal so there is no limit. However -1 and $+1$ are accumulation points by definition 2.2.2.

Example 2.2.6. (IR, U) usual topology, the sequence $(x_n) = ((-1)^n)$ has nor limit point neither accumulation point.

Example 2.2.7. Let us show that 0 is the limit point of the sequence (1/n), $n\in$ IN.
The neighborhood $V \in\ V(0)$ has infinitely many elements from our sequence that is: $\exists n_0 \in$ IN such that $\forall n \geq n_0$, $x_n \in\ V$, $V \in\ V(0)$ so 0 is the limit point.

Moreover 0 is an accumulation point.

$\forall\ V \in\ V(0)$, $n \in$ IN , $(1/n) \cap V \neq \varnothing$ and $\exists\ y \in\ (1/n) \cap\ V$, $0 \neq y$.

Definition 2.2.8. $(X\ \tau)$ topological space, a sequence $(X_n) \subset X$ and $x \in\ X$. If every neighborhood of x contains infinitely many terms of (x_n) that is, $\forall V \in V(x)$, $\forall n_0 \in$ IN, $\exists n \geq n_0$, $x_n \in\ V$, then x is called the closure of (x_n).

2.3 Failure of The Sequences

Here is the most important part of our project. We will see that first countable space helps us to construct topological structures by convergence of sequences. Behind this we will show that the limit of a sequence is not unique as we say in chapter 2.2. In this topic we will need a special space or for uniqueness.

Theorem 2.3.1. (X, τ) is first countable space. A is a subset of X and $x \in X$. x is a closure point of A iff there exist a sequence (x_n) in A which converges to x.

Proof : Let $x \in \overline{A}$.

(X, τ) is first countable space so X has a decreasing countable neighborhood base $\{U_n\}_{n \in N.}$ Since $x \in \overline{A}$, $\forall n \in IN$, $\forall U_n \in G(x)$, $U_n \cap A \neq \emptyset$.

Choose $x_n \in U_n \cap A \neq \emptyset$, $\forall n \in IN$.

Then compose a sequence (x_n) by using these $x_{n'}$s . It is clear that this sequence converges to the point x. Let us take a sequence $(x_n) \subset A$. By definition of convergence, $\forall U \in v(x)$, $\exists n_0 \in IN$, $\forall n \geq n_0$, $x_n \in U$.

Hence $x_n \in U \cap A$, so $x \in \overline{A}$ by definition 1.2.14.

Definition 2.3.2. For a function $f : (X, \tau_1) \to (Y, \tau_2)$.

If for every convergent sequence (x_n) of X, x_n's converge to a point x then $f(x_n)$ converges to f(x), that is; $(x_n) \subset X$, $(x_n) \to x \Rightarrow f(x_n) \to f(x)$, then f is called sequential continuous on the point x.

Theorem 2.3.3. If the function $f: (X, \tau_1) \to (Y, \tau_2)$ is continuous on the point $x \in X$ then it is sequential continuous on the point x.

Proof : Let us take a sequence $(x_n) \subset X$, $(x_n) \to x$.

Since f is continuous on x we have, $\forall V \in v(x)$, $f^{-1}(V) \in v(x)$.

Let us say $U = f^{-1}(v)$, since $x_n \to x$, $\exists n_0 \in IN$ such that $\forall n \geq n_0$, $x_n \in U$.

By using this we have $f(x_n) \in V$ and by definition 2.2.1, $f(x_n) \to f(x)$.

By definition 2.3.1 f is sequential continuous. Generally the converse of the theorem is not true.

Example 2.3.4. Let τ be cofinite topology on R,

$\tau = \{ A \subset R \mid A = \emptyset$ or $X - A$ is finite$\}$.

$(x_n) \subset IR$, $x_n \to x \Leftrightarrow \exists n_0 \in IN$ such that $\forall n \geq n_0$, $x_n = x$;

that is $(x_n) = \{x_1, x_2, \ldots x_{n(0)}, x, x, x, \ldots\}$.

If we take $A = \{x_n \mid x_n \neq x\} \subset IR$ then it is clear that $x \in (X - A) \in \tau$.

The identity function $f:(IR, \tau) \to (IR , U)$ is sequential continuous.

However $f^{-1}(]0, 1[) = [0, 1] \subset IR$ is not an open set in τ.

Hence f is not continuous.

Theorem 2.3.5. (X, τ) is first countable space.

The function $f:(X, \tau_1) \to (Y, \tau_2)$ is convergent on point $x \in X$ iff f is sequential continuous on the point x.

Proof : By previous theorem.

$\Leftarrow$ Let f be sequential continuous but not continuous on the point x. Since (x, τ) is first countable space, X has a decreasing countable neighborhood base $\{U_n\}_{n\in N}$ by theorem 2.1.1.

f is not continuous on x,

$\exists V \in v(f(x)), \forall n \in IN$ and $\forall U(n) \in v(x)$, $U_n \not\subset f^{-1}(V)$.

By this property for every natural number n compose a sequence (x_n) by choosing $x_n \in U_n$ such that $x_n \notin f^{-1}(V)$.

It is clear that $x_n \to x$. Since $f(x_n), \notin V$, the sequence $(f(x_n))$ does not converge to f(x). This contradicts with definition 2.3.1. Finally f is continuous on $x \in X$.

Theorem 2.3.6. If (x , τ) is first countable space, then the following properties are hold.

(1) A subset $T \subset X$ is open iff T contains infinitely many points of every sequences which converge to any point of T that is:

$T \in \tau \Leftrightarrow (\forall (x_n) \to x \in T) \Rightarrow \exists n_0 \in IN \ni \forall n \geq n_0 , x_n \in T$.

(2) $F \subset X$ is closed iff for every convergent sequence (x_n) of F, such that (x_n) converges to the point x, x is an element of F.

$F^C \in \tau \Leftrightarrow \{(x_n) \subset F, (x_n) \to x\} \Rightarrow x \in F$.

Proof . (1) $\Rightarrow$ Let $T \in \tau$ and (x_n) be any sequence which converges to a point $x \in T$.

Since , $T \in v(x)$, $\exists n_0 \in IN \ni \forall n \geq n_0 , x_n \in T$.

$\Leftarrow$ Suppose $T \neq \overset{o}{T}$ then $\exists\, x \in T$ such that $x \notin \overset{o}{T}$.

Suppose $V_1 \supset V_2 \supset \ldots$ be a neighborhood base of a point x.

Then $V_n \cap T^c \neq \emptyset$, $\forall\, n \in IN$, because if $\exists n_0 \in IN$.

With $V_{n_0} \cap T^c = \emptyset$ this implies that,

$x \in V_n \subset T \Rightarrow x \in \overset{o}{T}$ this is a contradiction by $x \notin \overset{o}{T}$.

So we obtain a sequence (x_n) such that $x_n \in V_n \cap T^c$ this means that $x_n \notin T$.

This contradicts with $(x_n) \subset T$. Therefore $T = \overset{o}{T}$. So $T \in \tau$.

(2) $\Rightarrow$ Let $F \subset X$ is closed set and $(x_n) \subset F$ is a sequence such that $(x_n) \to x$.

Suppose $x \notin F$ then $x \in F^c$, that is; $\exists\, n_0 \in IN \ni \forall n \geq n_0$, $x_n \in F^c$.

This is a contradiction with $(x_n) \subset F$.

$\Leftarrow$ Let $x \in \overline{F} \ni \exists (x_n) \subset F$ and $(x_n) \to x$ so x must be an element of F that is $x \in F$.

This means that $\overline{F} \subset F$ so F is closed subset of X.

Example 2.3.7. The discrete space (X, P(X)) and a sequence (x_n),

$(x_n) \to x \Leftrightarrow \exists n > k$, $k \in IN$, $x_n = x$.

This means the sequence is $\{x_1, x_2, \ldots, x_k, x, x, \ldots\}$.

Example 2.3.8. (IR, U) usual topology , $(x_n) \subset IR$ and $x \in IR$.

$(x_n) \to x \Leftrightarrow \forall\, \varepsilon > 0, \exists\, n_0 \in IN \ni \forall n \geq n_0$, $|x_n - x| < \varepsilon$.

Solution**.** $(x_n) \to x \Leftrightarrow \forall\, \varepsilon > 0$ and $\forall\, V =]\, x - \varepsilon\, ,\, x + \varepsilon\, [\ \in v(x)$,

$\exists n_0 \in IN \ni \forall n \geq n_0$, $x_n \in\,]x - \varepsilon\, ,\, x + \varepsilon\, [\Leftrightarrow |x_n - x| < \varepsilon$.

Example 2.3.9. The indiscrete space $\{X, \{X, \phi\}\}$.

For every sequence $(x_n) \subset X$ converges to the every point of $x \in X$.

Because for every element of X there exist only one open set X which is an element of the neighborhood of x. This open set X contains finitely many terms of the sequence. By limit definition every point of x is a limit point.

A sequence can have lots of limits. It is about its space. Then lets think about the following question:

"When the the topology becomes finer is the possibility of having limit more than one increase or decrease?"

Theorem 2.3.10. Let (X, τ) be first countable space.

Every convergent sequence in X has only one limit iff the space is Hausdorff space.

Proof . $\Rightarrow$ Let (X, τ) be not T_2 . There exist x, y such that,

$U \in v(x)$ and $V \in v(y)$, $U \cap V \neq \varnothing$.

Since the space is first countable space for x and y there exist decreasing neighborhood base $(U_n)_{n \in IN}$ and $(V_n)_{n \in IN}$ respectively.

Since $U \cap V \neq \varnothing$, choose $x_n \in U_n \cap V_n$, n is natural number. Compose the sequence (x_n).

It is clear that $U_1 \cap V_1 \supset U_2 \cap V_2 \supset \ldots \supset U_n \cap V_n \supset \ldots$ and this sequence converges both of x and y.

But this is a contradiction so our assumption is false.

$\Leftarrow$ Let (X, τ) be a Hausdorff space and (x_n) is a convergent sequence in this space. Assume that this sequence has two different limit points x and y.

Since (X, τ) is T_2 ; $\exists\, U \in v(x)$ and $\exists\, V \in v(y)$ $\ni$ $U \cap V = \phi$.

Since the sequence converges to x,

$\forall\, U \in v(x)$ and $\exists n_0 \in IN$ such that $\forall n \geq n_0$, $x_n \in V$.

Since the sequence converges to y,

$\forall\, V \in v(y)$ and $\exists m_0 \in IN$ such that $\forall n \geq m_0$, $x_n \in V$.

Let $p_0 = \max \{n_0 , m_0\}$, $\forall n \geq p_0$, $x_n \in U \cap V$.

However this is contradiction for T_2 space.

Thus x = y.

The sequences is not enough to construct topogical structures and we learned an important thing .

CHAPTER THREE
BITOPOLOGICAL SPACES

3.1 Definition About Bitopological Spaces

In 1963, J.C. Kelly considered about the definition of Bitopological Spaces, for this purpose he use the definition of quasi–pseudo – metric and metrizable topologies which is defined by Monthane.

Any distance function d(,) on a set X (which is any non–negative real valued function d(,) defined on the product XxX and satisfying the triangle inequality) has the property that the open d-spheres (i.e.sets of the form $\{y : d(x, y) < \varepsilon\}$ form a base for a topology D for X. We say that D is the topology determined by d(,). The purpose of this note is to observe that, given on a set X any such (unsymetric)quasi – metric p(,) whatsoever, then associated with p(x, y) is the function:

q(x, y) = p(y, x) and that if one studies the two topologies determined for X by p(,) and q(,).

Definition 3.1.1. A quasi-pseudo – metric on a set X is a non – negative real – valued function p(,) on the product XxX such that,

i) $p(x, x) = 0$ $(x \in X)$

ii) $p(x, z) \leq p(x, y) + p(y, z)$ $(x, y, z \in X)$

If p(,) satisfies the condition,

i) $p(x, y) = 0$ if and only if $x = y$ $(x, y \in X)$

then p(,) is a quasi – metric of course if also $p(x, y) = p(y, x)$ $(x, y \in X)$ then p(,) is a metric.

Definition 3.1.2. Let p(,) be a quasi– pseudo – metric on X, and let q(,) be defined by $q(x, y) = p(y, x)$, $(x, y \in X)$ (where q(,) is also quasi – pseudo – metric which is easily verified). We say that p(,) and q(,) are conjugate, and we will denote the set X with this structure by (X, p, q).

Definition 3.1.3. If p(,) is a quasi – pseudo – metric on a set X, then $\{y : p(x, y) < k\}$ the open p – sphere with centre x and radius $k > 0$, is denoted by P(x, k). Just as in the classical case, the collection of all p – spheres forms a base for a topology. The topology determined in this way by p(,) will be denoted by P and will be called the quasi – pseudo – metric topology of p(,). Similarly q(,) determines a topology Q for X. Thus the natural topological structure associated with a quasi – pseudo – metric on a set X is that of the set X with two topologies.

Definition 3.1.4. In 1963, Kelly introduced the concept of bitopological spaces. A set X equipped with two topologies is called a bitopological space and denoted by (X, τ_1, τ_2).

Let (X, τ_1, τ_2) be a bitopological space, $A \subseteq X$, $int_i(A)$ will denote the interior of A with respect to τ_i ; i= 1,2 and $cl_i(A)$ will denote the closure of A with respect to τ_i; i = 1,2.

Definition 3.1.5. A subset T in (X, τ_1 , τ_2) is called open in X if $int_1(T)= int_2(T) = T$.

Definition 3.1.6. A subset F in (X, τ_1 , τ_2) is called closed in X if F^c (complement of F) is an open in X.

3.2 Convergence of Sequences in Bitopological Spaces

In this part, we will introduced and discuss conditions of convergence of a sequence (x_n), where was taken from bitopological space (X, τ_1, τ_2), to point $x \in X$.

First at all, lets give the definition of convergence of a sequence (x_n) to point $x \in X$ in bitopological space (X, τ).

$(x_n) \to x_0$ then, for $\exists n_0 \in N \ni \forall n \geq n_0$, since $\forall V \in V(x_0)$, then $x_n \in V$.

In other words, all neighborhoods of point x cover all members of sequence (x_n) following from definite index. Sequence (x_n) has finite members outside of this neighborhood.

How we can define convergence of sequence (x_n) to x in topological space (X, τ_1, τ_2)? This question can be answered in many different ways.

If $A \subset X$ in space (X, τ_1, τ_2), then, when $x \in$ Å,

$\exists T \in \tau_1$, $x \in T \subset A \subset X$ and $\exists T \in \tau_2$, $x \in U \subset A \subset X$ must hold.

$int_1 A = int_2 A = A \Rightarrow x \in intA$.

Because at that, there may be a relation between topologies τ_1 and τ_2.

This relation mean that topologies are comparable. We will explain following parts according to this idea.

Topologies must be comparable to discuss convergence. This result can also get in that way.

Let a sequences (x_n) be taken from spare (X, τ_1, τ_2). The mean of sequence (x_n) converge to point $x \in X$ in this space;

$(x_n)_{\tau_1} \to x$ and $(x_n)_{\tau_2} \to x$.

That is, $\exists n_1 \in IN \ni \forall n \geq n_1$, $\forall V \in V\tau_1(x)$ and

$\exists n_2 \in IN \ni \forall n \geq n_2$, $\forall V \in V\tau_2(x)$.

Let's write results that getting by this assumptions.

1) If $(x_n)\tau_1,\tau_2 \to x$, then $(x_n)_{\tau_1} \to x$ and $(x_n)_{\tau_2} \to x$.

That is, if a sequence is convergent in a bitopological space, then it is also convergent in topologies that build this topological space.

2) If $(x_n)_{\tau_1} \to x$ and $(x_n)_{\tau_2} \to y$, then a sequence (x_n) may not be convergent in space (X, τ_1, τ_2).$(x \neq y)$

3) Let $(x_n)_{\tau_2} \to x$. For $(x_n)\tau_1, \tau_2 \to x$, how kind of relation may be between τ_1 and τ_2. Lets try to find an answer for this question.

a) If $\tau_1 = \tau_2$, then,

hence $(x_n)_{\tau_2} \to x$ then $(x_n)_{\tau_1} \to x$ and $(x_n)\tau_1,\tau_2 \to x$.

b) If $\tau_1 \subset \tau_2$, then $V\tau_1(x) \subset V\tau_2(x)$ By the assumption $(x_n)_{\tau_2} \to x$,

$\exists n_2 \in IN \ni \forall n \geq n_2$, $\forall V_2 \in V_2(x)$ so $x_n \in V_2$.

more over, hence $\forall V_1 \in V_1(x) \subset V_2(x)$, $\forall V_1 \in V_2(x)$ then,

$\exists n_1 \in IN \ni \forall n \geq n_1$, $\forall V_1 \in V_1(x)$ so $x_n \in V_1$.

That is, $(x_n)_{\tau_1} \to x$ and $(x_n)\tau_1,\tau_2 \to x$.

c) If $\tau_2 \subset \tau_1$, then $V\tau_1(x) \subset V\tau_2(x)$. $(x_n)_{\tau_2} \to x$ was assumed.

Do you think in which conditions this convergence exist?

Let's summarize that results.

1) If sequence (x_n) is convergent to $x \in X$ that is taken from (X, τ_1, τ_2), then it must be convergent to point x in both (X, τ_1) and (X, τ_2).

With same mentality, if sequence (x_n) converge to both (X, τ_1) and (X, τ_2) then in space (X, τ_1, τ_2) also converge to x.

$$(x_n)\tau_1, \tau_2 \to x \Leftrightarrow \begin{cases} (x_n)_{\tau_1} \to x \\ (x_n)_{\tau_2} \to x \end{cases}$$

2) If sequence (x_n) is not convergent in neither (X, τ_1) nor (X, τ_2) (In one or both of them not convergent) then, it is also not convergent is space (X, τ_1, τ_2).

3) If sequence (x_n) is convergent to point $x \in X$ in space (X, τ_1) and is convergent to point $y \in X$ in space (x, τ_2), then it can't be convergent in space (X, τ_1, τ_2).$(x \neq y)$

4) If it is known that sequence (x_n) is convergent to point $x \in X$ in space (X, τ_1), then the necessary condition to say that sequence (x_n) is convergen to point x in space (X, τ_1,τ_2) is $\tau_2 \subseteq \tau_1$.

Do you think how can we explain convergence without any comparasion between topologies τ_1 and τ_2?

REFERENCES

Alexandroff, P. & Hoph, H.(1935). *Topologie I.* Berlin.

Armstrong, M.A. (1983). *Basic Topology.* New York: Springer – Verlag.

Borges, C.R. (2000). *Elementary Topology and Applications.* Singapore: World Scientific Publishing.

Bourbaki, N. (1989). *General Topology*. Paris : Springer – Verlag.

Cullen, H.F. (1968). *Introduction to General Topology.* Boston, MA: Heath.

El - Sheikh, S.A. (2001). *A New Approach to Fuzzy Bitopological Spaces.* Elsevier, Information Sciences 137 , 283 – 301.

Engelking, R. (1989). *General Topology.* Berlin: Heldermann Verlag.

Gemignani, M.C. (1971). *Elementary Topology*.New York: Dover Publications.

Hocking, J.G.& Young, G.S.(1988). *Topology*. New York: Dover Publications.

Ivanov, A.A. (1968). *Problems of The Theory of Bitopological Spaces.* Journal Mathematical Society no. 5, 509 – 616.

Kelly, J.C. (1963). *Bitopological Spaces*, Proc. London Math. Soc.(3), 71 – 89.

Kowalsky, H.J. (1964). *Topological Spaces*. New York : Academic Press.

Kuratowski, K. (1968). *Topology, vol.II.* New York: Academic Press.

Mendelson, B. (1963). *Introduction to Topology.* London, England: Blackie& Son.

Munkres, J.R. (1975). *Topology: A First Course*. New Jersey: Prentice – Hall, Englewood Cliffs.

Nandi, J.N. (2004). *Bitopological & Pergect Function and Near Compactness.* Southeast Asian Bulletin of Methematics no. 27 , 877 – 882.

Özçelik, A. Z (1999). *Introduction to Topology*. Lecture Note.

Patty, C.W.(1967).*Bitopological Spaces.* Duke Mathematical Journal no.3, 387-391.

Roseman, D. (1999). *Elementary Topology*. New Jersey : Prentice Hall.

Steen, L.A. & Seebach, J.A., Jr. (1970). *Counterexamples in Topology*. U.S.A. Holt, Rinehart & Winston.

Tallafha A. (1999). *Countable Dense Homogenous Bitopological Spaces*. Journal of Mathematics no. 23, 233 – 242.

Tietze, H. (1923). *Beitrage zur allgemeinen Topologie I.* Math. Ann. 290 – 312.

Wiweger. (1993). *Bitopological Space*s. J. Math. Anal. Appl., 27 – 41.