DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SOME MATHEMATICAL PRINCIPLES ON SOLUTIONS OF ELASTICITY OF QUASICRYSTALS

by
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İZMİR

SOME MATHEMATICAL PRINCIPLES ON SOLUTIONS OF ELASTICITY OF QUASICRYSTALS

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **“SOME MATHEMATICAL PRINCIPLES ON SOLUTIONS OF ELASTICITY OF QUASICRYSTALS”** completed by **MELİKE DERYA DALYAN** under supervision of **ASSIST. PROF. DR. ALİ SEVİMLİCAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Melike Derya DALYAN

SOME MATHEMATICAL PRINCIPLES ON SOLUTIONS OF ELASTICITY OF QUASICRYSTALS

ABSTRACT

An initial value problem for two dimensional quasicrystals is the main study of the thesis. The coefficients appearing in the generalized Hooke's law are taken in such a way that they are inconstant. The studied initial value problem is reduced to system of integral equations.

Keywords: Linear elastic theory, quasicrystals

YARI KRİSTALLERİN ELASTİĞİNİN ÇÖZÜMLERİ ÜZERİNE BAZI MATEMATİKSEL İLKELER

ÖZ

Bu tezin temel çalışma konusu iki boyutlu yarı kristaller için bir başlangıç değer problemidir. Genelleştirilmiş Hooke yasasında yer alan katsayılar sabit olmayacak şekilde alınmıştır. Çalışılan başlangıç değer problemi integral denklem sistemine indirgenmiştir.

Anahtar kelimeler: Doğrusal esneklik teorisi, yarı kristaller

CONTENTS

CHAPTER ONE
INTRODUCTION

Crystal is a solid with a geometric structure in which atoms, molecules or ions are arranged in a regular order. In solids, atoms are arranged in a repetitive sequence. Crystals have their own characteristical shapes and sizes. Crystal systems are separated from each other according to the symmetry of the repetitive simplest geometry. Most rotational symmetries, including five-, seven- and higher fold symmetries in two dimensions, and icosahedral symmetry in three dimensions Levine & Steinhardt (1984).

Quasicrystals (QCs) are different from periodic crystals. It is one of the kinds of aperiodic crystals by Levine & Steinhardt (1984). The unusual characteristical properties of quasicrystals come from special atomic structures with a certain symmetry and diffraction patterns. With the help of these diffraction patterns, the researchers discovered the differences between crystals and quasicrystals.

Quasicrystals were first observed by Shechtman et al. (1984). They observed from the electronic microscopy that a rapid cooled Al-Mn alloys. Moreover in 2011, D. Shechtman was awarded with the Nobel Prize in Chemistry with his discovery. Quasicrystal's observation was a striking event and an important discovery. The unusual structure of quasicrystals grabbed the attention of the researchers who work in the areas such as, physics, crystallography, chemistry, applied mathematics, etc. Quasicrystals affected the scientific world due to their structure and properties. These properties are; elasticity, brittle, low friction coefficient, high harness, low level of porosity and high wear resistance. Many of these features are effectively combined to provide technological point of view interesting applications which are under protection by patents recently (Dubois, 2005), Royer & Dieulesaint (2000). For example, motor vehicle engines, coatings, surgical applications, thin metal films. (Yang et al., 2014)

So far about 200 kinds of QC appearead in the book (Fan, 2016). One, two, and three dimensional quasicrystals are defined as three dimensional body with the special atom arrangements. The atom arrangement of one dimensional quasicrystal is quasi-periodic in direction, while two dimensional is quasi-periodic in a plane, and one dimensional quasicrystal is periodic in the plane which is orthogonal to this direction while two dimensional quasicrystal is periodic in the orthogonal direction. Different from one and two dimensional atom arrangements, three dimensional quasicrystal is quasi-periodic three dimensions without periodic direction (Çerdik Yaslan, 2019), which are important for the QC study.

Many problems for various quasicrystals have been studied, for example:

- Green's functions of Fourier images of the plane elasticity problems of two dimensional quasicrystals for dodecagonal, pentagonal and decagonal systems were obtained by using symbolic calculations by Akmaz & Akıncı (2009).
- The physical properties of one dimensional quasicrystals studied by Wang et al. (1997) .
- The time-dependent differential equations of elasticity for two dimensional quasicrystals with general structure of anisotropy for dodecagonal, octagonal, decagonal, pentagonal, hexagonal, triclinic systems were studied by Yakhno & Çerdik Yaslan (2011).
- The fundamental solution of the time-dependent differential equations of anisotropic elasticity in three dimensional quasicrystals were obtained by Çerdik Yaslan (2019).

The thesis has been studied depending on following structure: Chapter 1 contains some notations, generalized Hooke's law, the basic equations of the elasticity theory of quasicrystals that we actively used in this thesis.

The chapter 2 of the thesis contains the initial value problem for two dimensional quasicrystals which is the main problem of our study.

Fourier transform, d'Alembert formula, some information from matrix theory and linear algebra were actively used in this thesis. The method used in this thesis was also applied for the electric field equations by Sevimlican (2007).

CHAPTER TWO
EQUATIONS OF QUASICRYSTALS

In this chapter, we first give some notations used in thesis. Preliminary knowledge about quasicrystals and generalized Hooke's law will be given.

2.1 Notations

$u_i(x,t)$	phonon displacement field
$w_\alpha(x,t)$	phason displacement field
ε_{ij}	phonon strain
$w_{\alpha j}$	phason strain
σ_{ij}	phonon stress
$H_{\beta j}$	phason stress
C_{ijkl}	phonon elastic coefficient tensor
$K_{\beta j\alpha l}$	phason elastic coefficient tensor
R_{ijkl}	constitutive tensor of phanon-phason coupling
ρ	mass density
$\widetilde{V}$	change variable in V
$\mathcal{F}_{x_1x_2}$	Fourier operator with respect to the variables x_1 and x_2
$\widehat{\boldsymbol{V}}$	Fourier image of **V**
$\mathbf{A}^T$	Transpose of the $m \times n$ matrix $\mathbf{A} = [a_{ij}]_{m\times n}$

2.2 Basic Equations

The basic equations of the elasticity theory of quasicrystals will be given under this topic. The strain displacement relations of the elasticity theory of quasicrystals will be given.

Let $u = (u_1, u_2, u_3)$ and $w = (w_1, w_2, w_3)$ denote the phonon and phason displacement vectors, where $u_i = u_i(x,t)$ and $w_i = w_i(x,t)$, $i = 1,2,3$ are the functions of the space variable $x = (x_1, x_2, x_3)$ and the time variable t. ε_{ij} and $w_{\alpha l}$ are phonon and phason strains, respectively Akmaz & Akıncı (2009).

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,2,3 \tag{2.1}$$

$$w_{\alpha l} = \frac{\partial w_\alpha}{\partial x_l}, \qquad l = 1,2,3 \tag{2.2}$$

Here α signifies the number of the dimension. If α runs 1 to 3, it means that the QC has 3 dimensions, α runs 1 to 2, the QC has 2 dimensions. When α is 3, it means that the QC has 1 dimension.

The main problem of the this thesis is initial value problem of 2D quasicrystals which will be given in the following chapter.

2.3 Generalized Hooke's Law

In this section Hooke's Law and Generalized Hooke's Law will be defined. Symmetry properties between elastic tensors will be given.

Hooke's law allows us to find the tension that must be applied to elastically require a substance and the relation between stresses σ_{ij} and strains ε_{ij}. When referring to Hooke's Law for unidirectional tension, there is the following relationship between elastic tension and strain by Fan (2016).

$$\sigma_{xx} = E\,\varepsilon_{xx}. \tag{2.3}$$

Hooke's Original Law is generalized in situations where tension is two or three-dimensional.

$$\sigma_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}} = C_{ijkl}\,\varepsilon_{kl} + R_{ij\alpha l}\,w_{\alpha l}, \tag{2.4}$$

$$H_{\beta j} = \frac{\partial F}{\partial w_{ij}} = R_{kl\beta j}\,\varepsilon_{kl} + K_{\beta j\alpha l}\,w_{\alpha l}, \tag{2.5}$$

in which F denotes the free energy, or the strain energy density in the book of Dikici (1993).

C_{ijkl} is the elastic constant tensor, which consists of 81 components. Because of the symmetry of σ_{ij} and ε_{ij}, each of them has 6 independent components only, such that the independent components of C_{ijkl} reduce to 36. F is a quadratic function of ε_{ij}, considering the symmetry of ε_{ij}, then we have:

$$C_{ijkl} = C_{klij}, \tag{2.6}$$

so the independent components 36 reduce to 21.

$K_{\beta j\alpha l}$ and R_{ijkl} are constitutive tensor corresponding to the phason field and R and the phanon and phason coupling. The tensors $K_{\beta j\alpha l}$ and R_{ijkl} have to the following symmetries:

$$K_{\beta j\alpha l} = K_{\alpha l\beta j}, \tag{2.7}$$

$$R_{kl\beta j} = R_{lk\beta j}, \tag{2.8}$$

21 the relationship with the independent elastic constant is known to be Generalized Hooke's law. Generalized Hooke's law describes anisotropic elastic bodies containing crystals. Stress and strain tensors can also be expressed by corresponding vectors with 6 independent elements (Fan, 2016).

Equations (2.4) and (2.5) can be written in a shorter way, such as (3.8) using the summation symbol (Dikici, 1993, p. 5).

Indices can be changed to make the equations easier and to prevent repetitions. It is shown below (Dikici, 1993, p. 131):

Let $\alpha = ij$ and $\beta = kl$. To describe C_{ijkl} one can shortly use the notation $C_{\alpha\beta}$.

ij or kl	$\rightarrow$	α or β
11	$\rightarrow$	1
22	$\rightarrow$	2
33	$\rightarrow$	3
23 , 32	$\rightarrow$	4
13 , 31	$\rightarrow$	5
12 , 21	$\rightarrow$	6

This correspondence is possible due to symmetry properties in Çamlıdağ (2003) and Çerdik Yaslan (2011). By the help of the above notations instead of using four subindices, two subindices may also be used. For instance; $C_{1231} = C_{65}$.

CHAPTER THREE
THE BASIC EQUATIONS FOR 2D QUASICRYSTALS

In this chapter, we will define the main problem of the thesis. The studied problem is reduced to the system of vector integral equations.

3.1 Statement of the Problem and Assumptions

Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ be a space variable, $t \in \mathbb{R}$ be a time variable. According to the linear elastic theory of 2D QCs the strain-displacement relations are given by the following equations: (see Çerdik Yaslan (2011))

$$\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), \quad i, j = 1, 2, 3, \tag{3.1}$$

$$w_{\alpha l} = \frac{\partial w_\alpha}{\partial x_l}, \quad \alpha = 1, 2 \quad l = 1, 2, 3, \tag{3.2}$$

where $u_i(x, t)$ is phonon displacement field and $w_\alpha(x, t)$ is phason displacement field. ε_{ij} and $w_{\alpha j}$ are the phonon and phason strains respectively. The generalized Hooke's law of the elasticity problem of 2D QCs is given by Çerdik Yaslan (2019):

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + R_{ij\alpha l}w_{\alpha l}, \tag{3.3}$$

$$H_{\beta j} = R_{kl\beta j}\varepsilon_{kl} + K_{\beta j\alpha l}w_{\alpha l}, \quad i, j, k, l = 1, 2, 3, \quad \alpha, \beta = 1, 2, \tag{3.4}$$

Here σ_{ij} and $H_{\beta j}$ are phonon and phason stresses, respectively. C_{ijkl} and $K_{\beta j\alpha l}$ are the constitutive tensors corresponding to the phanon and phason fields and R_{ijkl} are the constitutive tensors of the phanon-phason coupling. Similar to Yang et al. (2014) we assume that the following symmetric properties are satisfied:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \tag{3.5}$$

$$K_{\beta j\alpha l} = K_{\alpha l\beta j}, \tag{3.6}$$

$$R_{ij\alpha l} = R_{ji\alpha l}, \tag{3.7}$$

The positivity of the elastic strain energy density requires that the elastic constant tensors C_{ijkl}, $R_{ij\alpha l}$, $K_{\beta j\alpha l}$ must be positive definite. In other words, when the strain tensors $\varepsilon_{ij}, w_{\alpha l}$ are not zero totally, the elastic constant tensors satisfy the following inequality:

$$\sum_{j,l,i,k=1}^{3} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} > 0, \quad \sum_{j,l=1}^{3}\sum_{\alpha,\beta=1}^{2} K_{\beta j\alpha l}w_{\beta j}w_{\alpha l} > 0, \quad \sum_{j,l,i=1}^{3}\sum_{\alpha=1}^{2} R_{ij\alpha l}\varepsilon_{ij}w_{\alpha l} > 0. \tag{3.8}$$

The dynamic equilibrium equations can be written in the following form (Yang et al., 2014), (Yakhno & Çerdik Yaslan, 2011)

$$\rho\frac{\partial^2 u_i(x,t)}{\partial t^2} = \sum_{j=1}^{3}\frac{\partial\sigma_{ij}(x,t)}{\partial x_j} + f_i(x,t), \quad i = 1,2,3, \tag{3.9}$$

$$\rho\frac{\partial^2 w_\beta(x,t)}{\partial t^2} = \sum_{j=1}^{3}\frac{\partial H_{\beta j}(x,t)}{\partial x_j} + g_\beta(x,t), \quad \beta = 1,2, \tag{3.10}$$

where $\rho > 0$ is the density; $f_i(x,t), i = 1,2,3$ and $g_\beta(x,t), \beta = 1,2$ are body forces densities for the phonon and phason displacements, respectively. Initial value problem for (3.9) and (3.10) is considered with the conditions

$$u_i(x,t)|_{t=0} = 0, \quad w_\beta(x,t)|_{t=0} = 0. \tag{3.11}$$

We assume that ρ and all coefficients appearing in the generalized Hooke's law, C_{ijkl}, $R_{ij\alpha l}$ and $K_{\beta j\alpha l}$ depend on x_3 variable.

Under these assumptions and notations, the system (3.9)- (3.11) can be written as follows:

For i=1 from (3.9) - (3.11) we have

$$\rho\frac{\partial^2 u_1}{\partial t^2} = \sum_{j=1}^{3}\frac{\partial \sigma_{1j}}{\partial x_j} + f_1(x,t), \quad i=1,2,3,$$
$$= \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1(x,t), \tag{3.12}$$

where

$$\sigma_{11} = C_{1111}\varepsilon_{11} + C_{1112}\varepsilon_{12} + C_{1113}\varepsilon_{13} + C_{1121}\varepsilon_{21} + C_{1122}\varepsilon_{22} + C_{1123}\varepsilon_{23}$$
$$+ C_{1131}\varepsilon_{31} + C_{1132}\varepsilon_{32} + C_{1133}\varepsilon_{33} + R_{1111}w_{11} + R_{1112}w_{12} + R_{1113}w_{13} \tag{3.13}$$
$$+ R_{1121}w_{21} + R_{1122}w_{22} + R_{1123}w_{23}.$$

$$\sigma_{12} = C_{1211}\varepsilon_{11} + C_{1212}\varepsilon_{12} + C_{1213}\varepsilon_{13} + C_{1221}\varepsilon_{21} + C_{1222}\varepsilon_{22} + C_{1223}\varepsilon_{23}$$
$$+ C_{1231}\varepsilon_{31} + C_{1232}\varepsilon_{32} + C_{1233}\varepsilon_{33} + R_{1211}w_{11} + R_{1212}w_{12} + R_{1213}w_{13} \tag{3.14}$$
$$+ R_{1221}w_{21} + R_{1222}w_{22} + R_{1223}w_{23}.$$

$$\sigma_{13} = C_{1311}\varepsilon_{11} + C_{1312}\varepsilon_{12} + C_{1313}\varepsilon_{13} + C_{1321}\varepsilon_{21} + C_{1322}\varepsilon_{22} + C_{1323}\varepsilon_{23}$$
$$+ C_{1331}\varepsilon_{31} + C_{1332}\varepsilon_{32} + C_{1333}\varepsilon_{33} + R_{1311}w_{11} + R_{1312}w_{12} + R_{1313}w_{13} \tag{3.15}$$
$$+ R_{1321}w_{21} + R_{1322}w_{22} + R_{1323}w_{23}.$$

Substituting $\sigma_{1j},\ j=1,2,3$ into the equation (3.12) we obtain

$$
\begin{aligned}
\rho(x_3)\frac{\partial^2 u_1}{\partial t^2} \quad = \quad & \frac{1}{2}\Big(C_{1111}\frac{\partial^2 u_1}{\partial x_1^2}+C_{1111}\frac{\partial^2 u_1}{\partial x_1^2}+C_{1112}\frac{\partial^2 u_1}{\partial x_1 \partial x_2}+C_{1112}\frac{\partial^2 u_2}{\partial x_1^2}\\
&+C_{1113}\frac{\partial^2 u_1}{\partial x_1 \partial x_3}+C_{1113}\frac{\partial^2 u_3}{\partial x_1^2}+C_{1121}\frac{\partial^2 u_2}{\partial x_1^2}+C_{1121}\frac{\partial^2 u_1}{\partial x_1 \partial x_2}\\
&+C_{1122}\frac{\partial^2 u_2}{\partial x_1 \partial x_2}+C_{1122}\frac{\partial^2 u_2}{\partial x_1 \partial x_2}+C_{1123}\frac{\partial^2 u_2}{\partial x_1 \partial x_3}+C_{1123}\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\\
&+C_{1131}\frac{\partial^2 u_3}{\partial^2 x_1}+C_{1131}\frac{\partial^2 u_1}{\partial x_1 \partial x_3}+C_{1132}\frac{\partial^2 u_3}{\partial x_1 \partial x_2}+C_{1132}\frac{\partial^2 u_2}{\partial x_1 \partial x_3}\\
&+C_{1133}\frac{\partial^2 u_3}{\partial x_1 \partial x_3}+C_{1133}\frac{\partial^2 u_3}{\partial x_1 \partial x_3}\Big)+R_{1111}\frac{\partial^2 w_1}{\partial x_1^2}+R_{1112}\frac{\partial^2 w_1}{\partial x_1 \partial x_2}\\
&+R_{1113}\frac{\partial^2 w_1}{\partial x_1 \partial x_3}+R_{1121}\frac{\partial^2 w_2}{\partial x_1^2}+R_{1122}\frac{\partial^2 w_2}{\partial x_1 \partial x_2}+R_{1123}\frac{\partial^2 w_2}{\partial x_1 \partial x_3}\\
&+\frac{1}{2}\Big(C_{1211}\frac{\partial^2 u_1}{\partial x_2 \partial x_1}+C_{1211}\frac{\partial^2 u_1}{\partial x_2 \partial x_1}+C_{1212}\frac{\partial^2 u_1}{\partial x_2^2}+C_{1212}\frac{\partial^2 u_2}{\partial x_2 \partial x_1}\\
&+C_{1213}\frac{\partial^2 u_1}{\partial x_2 \partial x_3}+C_{1213}\frac{\partial^2 u_3}{\partial x_2 \partial x_1}+C_{1221}\frac{\partial^2 u_2}{\partial x_2 \partial x_1}+C_{1221}\frac{\partial^2 u_1}{\partial x_2^2 \partial x_2}\\
&+C_{1222}\frac{\partial^2 u_2}{\partial x_2^2}+C_{1222}\frac{\partial^2 u_2}{\partial x_2^2}+C_{1223}\frac{\partial^2 u_2}{\partial x_2 \partial x_3}+C_{1223}\frac{\partial^2 u_3}{\partial x_2^2}\\
&+C_{1231}\frac{\partial^2 u_3}{\partial x_2 \partial x_1}+C_{1231}\frac{\partial^2 u_1}{\partial x_2 \partial x_3}+C_{1232}\frac{\partial^2 u_3}{\partial x_2^2}+C_{1232}\frac{\partial^2 u_2}{\partial x_2 \partial x_3}\\
&+C_{1233}\frac{\partial^2 u_3}{\partial x_2 \partial x_3}+C_{1233}\frac{\partial^2 u_3}{\partial x_2 \partial x_3}\Big)+R_{1211}\frac{\partial^2 w_1}{\partial x_2 \partial x_1}+R_{1212}\frac{\partial^2 w_1}{\partial x_2^2}\\
&+R_{1213}\frac{\partial^2 w_1}{\partial x_2 \partial x_3}+R_{1221}\frac{\partial^2 w_2}{\partial x_2 \partial x_1}+R_{1222}\frac{\partial^2 w_2}{\partial x_2^2}+R_{1223}\frac{\partial^2 w_2}{\partial x_2 \partial x_3}\\
&+\frac{1}{2}\Big(C_{1311}\frac{\partial^2 u_1}{\partial x_3 \partial x_1}+C_{1311}\frac{\partial^2 u_1}{\partial x_3 \partial x_1}+C_{1312}\frac{\partial^2 u_1}{\partial x_3 \partial x_2}+C_{1312}\frac{\partial^2 u_2}{\partial x_3 \partial x_1}\\
&+C_{1313}\frac{\partial^2 u_1}{\partial x_3^2}+C_{1313}\frac{\partial^2 u_3}{\partial x_3 \partial x_1}+C_{1321}\frac{\partial^2 u_2}{\partial x_3 \partial x_1}+C_{1321}\frac{\partial^2 u_1}{\partial x_3 \partial x_2}\\
&+C_{1322}\frac{\partial^2 u_2}{\partial x_3 \partial x_2}+C_{1322}\frac{\partial^2 u_2}{\partial x_3 \partial x_3}+C_{1323}\frac{\partial^2 u_2}{\partial x_3^2}+C_{1323}\frac{\partial^2 u_3}{\partial x_3 \partial x_2}\\
&+C_{1331}\frac{\partial^2 u_3}{\partial x_3 \partial x_1}+C_{1331}\frac{\partial^2 u_1}{\partial x_3^2}+C_{1332}\frac{\partial^2 u_3}{\partial x_3 \partial x_2}+C_{1332}\frac{\partial^2 u_2}{\partial x_3^2}\\
&+C_{1333}\frac{\partial^2 u_3}{\partial x_3^2}+C_{1333}\frac{\partial^2 u_3}{\partial x_3^2}\Big)+R_{1311}\frac{\partial^2 w_1}{\partial x_3 \partial x_1}+R_{1312}\frac{\partial^2 w_1}{\partial x_3 \partial x_2}\\
&+R_{1313}\frac{\partial^2 w_1}{\partial x_3^2}+R_{1321}\frac{\partial^2 w_2}{\partial x_3 \partial x_1}+R_{1322}\frac{\partial^2 w_2}{\partial x_3 \partial x_2}+R_{1323}\frac{\partial^2 w_2}{\partial x_3^2}
\end{aligned}
$$

$$
\begin{aligned}
&+c'(x_3)\frac{1}{2}\Big(C_{1311}\frac{\partial u_1}{\partial x_1}+C_{1311}\frac{\partial u_1}{\partial x_1}+C_{1312}\frac{\partial u_1}{\partial x_2}+C_{1312}\frac{\partial u_2}{\partial x_1}\\
&+C_{1313}\frac{\partial u_1}{\partial x_3}+C_{1313}\frac{\partial u_3}{\partial x_1}+C_{1321}\frac{\partial u_2}{\partial x_1}+C_{1321}\frac{\partial u_1}{\partial x_2}+C_{1322}\frac{\partial u_2}{\partial x_2}\\
&+C_{1322}\frac{\partial u_2}{\partial x_2}+C_{1323}\frac{\partial u_2}{\partial x_3}+C_{1323}\frac{\partial u_3}{\partial x_2}+C_{1331}\frac{\partial u_3}{\partial x_1}+C_{1331}\frac{\partial u_1}{\partial x_3}\\
&+C_{1332}\frac{\partial u_3}{\partial x_2}+C_{1332}\frac{\partial u_2}{\partial x_3}+C_{1333}\frac{\partial u_3}{\partial x_3}+C_{1333}\frac{\partial u_3}{\partial x_3}\Big)\\
&+r'(x_3)\Big(R_{1311}\frac{\partial w_1}{\partial x_1}+R_{1312}\frac{\partial w_1}{\partial x_2}+R_{1313}\frac{\partial w_1}{\partial x_3}\\
&+R_{1321}\frac{\partial w_2}{\partial x_1}+R_{1322}\frac{\partial w_2}{\partial x_2}+R_{1323}\frac{\partial w_2}{\partial x_3}\Big)+f_1(x,t).
\end{aligned}
$$

For i=2, from the equation (3.9) we have

$$
\begin{aligned}
\rho\frac{\partial^2 u_2}{\partial t^2} &=\sum_{j=1}^{3}\frac{\partial \sigma_{2j}}{\partial x_j}+f_2(x,t)\\
&=\frac{\partial \sigma_{21}}{\partial x_1}+\frac{\partial \sigma_{22}}{\partial x_2}+\frac{\partial \sigma_{23}}{\partial x_3}+f_2(x,t).
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
\sigma_{21} &= C_{2111}\varepsilon_{11}+C_{2112}\varepsilon_{12}+C_{2113}\varepsilon_{13}+C_{2121}\varepsilon_{21}+C_{2122}\varepsilon_{22}+C_{2123}\varepsilon_{23}\\
&+C_{2131}\varepsilon_{31}+C_{2132}\varepsilon_{32}+C_{2133}\varepsilon_{33}+R_{2111}w_{11}+R_{2112}w_{12}+R_{2113}w_{13}\\
&+R_{2121}w_{21}+R_{2122}w_{22}+R_{2123}w_{23}.
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
\sigma_{22} &= C_{2211}\varepsilon_{11}+C_{2212}\varepsilon_{12}+C_{2213}\varepsilon_{13}+C_{2221}\varepsilon_{21}+C_{2222}\varepsilon_{22}+C_{2223}\varepsilon_{23}\\
&+C_{2231}\varepsilon_{31}+C_{2232}\varepsilon_{32}+C_{2233}\varepsilon_{33}+R_{2211}w_{11}+R_{2212}w_{12}+R_{2213}w_{13}\\
&+R_{2221}w_{21}+R_{2222}w_{22}+R_{2223}w_{23}.
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
\sigma_{23} &= C_{2311}\varepsilon_{11}+C_{2312}\varepsilon_{12}+C_{2313}\varepsilon_{13}+C_{2321}\varepsilon_{21}+C_{2322}\varepsilon_{22}+C_{2323}\varepsilon_{23}\\
&+C_{2331}\varepsilon_{31}+C_{2332}\varepsilon_{32}+C_{2333}\varepsilon_{33}+R_{2311}w_{11}+R_{2312}w_{12}+R_{2313}w_{13}\\
&+R_{2321}w_{21}+R_{2322}w_{22}+R_{2323}w_{23}.
\end{aligned}
\tag{3.19}
$$

Substituting σ_{2j}, $j = 1, 2, 3$ into the equation (3.16), we obtain

$$
\begin{aligned}
\rho(x_3)\frac{\partial^2 u_2}{\partial t^2} = \frac{1}{2}\Big(& C_{2111}\frac{\partial^2 u_1}{\partial x_1^2} + C_{2111}\frac{\partial^2 u_1}{\partial x_1^2} + C_{2112}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + C_{2112}\frac{\partial^2 u_2}{\partial x_1^2} \\
& + C_{2113}\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + C_{2113}\frac{\partial^2 u_3}{\partial x_1^2} + C_{2121}\frac{\partial^2 u_2}{\partial x_1^2} + C_{2121}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} \\
& + C_{2122}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + C_{2122}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + C_{2123}\frac{\partial^2 u_2}{\partial x_1 \partial x_3} + C_{2123}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} \\
& + C_{2131}\frac{\partial^2 u_3}{\partial x_1^2} + C_{2131}\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + C_{2132}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} + C_{2132}\frac{\partial^2 u_2}{\partial x_1 \partial x_3} \\
& + C_{2133}\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + C_{2133}\frac{\partial^2 u_3}{\partial x_1 \partial x_3}\Big) + R_{2111}\frac{\partial^2 w_1}{\partial x_1^2} + R_{2112}\frac{\partial^2 w_1}{\partial x_1 \partial x_2} \\
& + R_{2113}\frac{\partial^2 w_1}{\partial x_1 \partial x_3} + R_{2121}\frac{\partial^2 w_2}{\partial x_1^2} + R_{2122}\frac{\partial^2 w_2}{\partial x_1 \partial x_2} + R_{2123}\frac{\partial^2 w_2}{\partial x_1 \partial x_3} \\
& + \frac{1}{2}\Big(C_{2211}\frac{\partial^2 u_1}{\partial x_2 \partial x_1} + C_{2211}\frac{\partial^2 u_1}{\partial x_2 \partial x_1} + C_{2212}\frac{\partial^2 u_1}{\partial x_2^2} + C_{2212}\frac{\partial^2 u_2}{\partial x_2 \partial x_1} \\
& + C_{2213}\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + C_{2213}\frac{\partial^2 u_3}{\partial x_2 \partial x_1} + C_{2221}\frac{\partial^2 u_2}{\partial x_2 \partial x_1} + C_{2221}\frac{\partial^2 u_1}{\partial x_2^2} \\
& + C_{2222}\frac{\partial^2 u_2}{\partial x_2^2} + C_{2222}\frac{\partial^2 u_2}{\partial x_2^2} + C_{2223}\frac{\partial^2 u_2}{\partial x_2 \partial x_3} + C_{2223}\frac{\partial^2 u_3}{\partial x_2^2} \\
& + C_{2231}\frac{\partial^2 u_3}{\partial x_2 \partial x_1} + C_{2231}\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + C_{2232}\frac{\partial^2 u_3}{\partial x_2^2} + C_{2232}\frac{\partial^2 u_2}{\partial x_2 \partial x_3} \\
& + C_{2233}\frac{\partial^2 u_3}{\partial x_2 \partial x_3} + C_{2233}\frac{\partial^2 u_3}{\partial x_2 \partial x_3}\Big) + R_{2211}\frac{\partial^2 w_1}{\partial x_2 \partial x_1} + R_{2212}\frac{\partial^2 w_1}{\partial x_2^2} \\
& + R_{2213}\frac{\partial^2 w_1}{\partial x_2 \partial x_3} + R_{2221}\frac{\partial^2 w_2}{\partial x_2 \partial x_1} + R_{2222}\frac{\partial^2 w_2}{\partial x_2^2} + R_{2223}\frac{\partial^2 w_2}{\partial x_2 \partial x_3} \\
& + \frac{1}{2}\Big(C_{2311}\frac{\partial^2 u_1}{\partial x_3 \partial x_1} + C_{2311}\frac{\partial^2 u_1}{\partial x_3 \partial x_1} + C_{2312}\frac{\partial^2 u_1}{\partial x_3 \partial x_2} + C_{2312}\frac{\partial^2 u_2}{\partial x_3 \partial x_1} \\
& + C_{2313}\frac{\partial^2 u_1}{\partial x_3^2} + C_{2313}\frac{\partial^2 u_3}{\partial x_3 \partial x_1} + C_{2321}\frac{\partial^2 u_2}{\partial x_3 \partial x_1} + C_{2321}\frac{\partial^2 u_1}{\partial x_3 \partial x_2} \\
& + C_{2322}\frac{\partial^2 u_2}{\partial x_3 \partial x_2} + C_{2322}\frac{\partial^2 u_2}{\partial x_3 \partial x_2} + C_{2323}\frac{\partial^2 u_2}{\partial x_3^2} + C_{2323}\frac{\partial^2 u_3}{\partial x_3 \partial x_2} \\
& + C_{2331}\frac{\partial^2 u_3}{\partial x_3 \partial x_1} + C_{2331}\frac{\partial^2 u_1}{\partial x_3^2} + C_{2332}\frac{\partial^2 u_3}{\partial x_3 \partial x_2} + C_{2332}\frac{\partial^2 u_2}{\partial x_3^2} \\
& + C_{2333}\frac{\partial^2 u_3}{\partial x_3^2} + C_{2333}\frac{\partial^2 u_3}{\partial x_3^2}\Big) + R_{2311}\frac{\partial^2 w_1}{\partial x_3 \partial x_1} + R_{2312}\frac{\partial^2 w_1}{\partial x_3 \partial x_2} \\
& + R_{2313}\frac{\partial^2 w_1}{\partial x_3^2} + R_{2321}\frac{\partial^2 w_2}{\partial x_3 \partial x_1} + R_{2322}\frac{\partial^2 w_2}{\partial x_3 \partial x_2} + R_{2323}\frac{\partial^2 w_2}{\partial x_3^2}
\end{aligned}
$$

$$+ c'(x_3)\frac{1}{2}\Big(C_{2311}\frac{\partial u_1}{\partial x_1} + C_{2311}\frac{\partial u_1}{\partial x_1} + C_{2312}\frac{\partial u_1}{\partial x_2} + C_{2312}\frac{\partial u_2}{\partial x_1}$$
$$+ C_{2313}\frac{\partial u_1}{\partial x_3} + C_{2313}\frac{\partial u_3}{\partial x_1} + C_{2321}\frac{\partial u_2}{\partial x_1} + C_{2321}\frac{\partial u_1}{\partial x_2} + C_{2322}\frac{\partial u_2}{\partial x_2}$$
$$+ C_{2322}\frac{\partial u_2}{\partial x_2} + C_{2323}\frac{\partial u_2}{\partial x_3} + C_{2323}\frac{\partial u_3}{\partial x_2} + C_{2331}\frac{\partial u_3}{\partial x_1} + C_{2331}\frac{\partial u_1}{\partial x_3}$$
$$+ C_{2332}\frac{\partial u_3}{\partial x_2} + C_{2332}\frac{\partial u_2}{\partial x_3} + C_{2333}\frac{\partial u_3}{\partial x_3} + C_{2333}\frac{\partial u_3}{\partial x_3}\Big)$$
$$+ r'(x_3)\Big(R_{2311}\frac{\partial w_1}{\partial x_1} + R_{2312}\frac{\partial w_1}{\partial x_2} + R_{2313}\frac{\partial w_1}{\partial x_3} + R_{2321}\frac{\partial w_2}{\partial x_1}$$
$$+ R_{2322}\frac{\partial w_2}{\partial x_2} + R_{2323}\frac{\partial w_2}{\partial x_3}\Big) + f_2(x,t).$$

For i=3 from the equation (3.9) we have:

$$\begin{aligned}\rho\frac{\partial^2 u_3}{\partial t^2} &= \sum_{j=1}^{3}\frac{\partial \sigma_{3j}}{\partial x_j} + f_3(x,t)\\ &= \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3(x,t),\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}\sigma_{31} = {} & C_{3111}\varepsilon_{11} + C_{3112}\varepsilon_{12} + C_{3113}\varepsilon_{13} + C_{3121}\varepsilon_{21} + C_{3122}\varepsilon_{22} + C_{3123}\varepsilon_{23}\\ & + C_{3131}\varepsilon_{31} + C_{3132}\varepsilon_{32} + C_{3133}\varepsilon_{33} + R_{3111}w_{11} + R_{3112}w_{12}\\ & + R_{3113}w_{13} + R_{3121}w_{21} + R_{3122}w_{22} + R_{3123}w_{23},\end{aligned} \tag{3.21}$$

$$\begin{aligned}\sigma_{32} = {} & C_{3211}\varepsilon_{11} + C_{3212}\varepsilon_{12} + C_{3213}\varepsilon_{13} + C_{3221}\varepsilon_{21} + C_{3222}\varepsilon_{22} + C_{3223}\varepsilon_{23}\\ & + C_{3231}\varepsilon_{31} + C_{3232}\varepsilon_{32} + C_{3233}\varepsilon_{33} + R_{3211}w_{11} + R_{3212}w_{12}\\ & + R_{3213}w_{13} + R_{3221}w_{21} + R_{3222}w_{22} + R_{3223}w_{23}\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\sigma_{33} = {} & C_{3311}\varepsilon_{11} + C_{3312}\varepsilon_{12} + C_{3313}\varepsilon_{13} + C_{3321}\varepsilon_{21} + C_{3322}\varepsilon_{22} + C_{3323}\varepsilon_{23}\\ & + C_{3331}\varepsilon_{31} + C_{3332}\varepsilon_{32} + C_{3333}\varepsilon_{33} + R_{3311}w_{11} + R_{3312}w_{12}\\ & + R_{3313}w_{13} + R_{3321}w_{21} + R_{3322}w_{22} + R_{3323}w_{23}.\end{aligned} \tag{3.23}$$

Subsituting $\sigma_{3j}, j = 1, 2, 3$ into (3.20) we obtain:

$$\begin{aligned}
\rho(x_3)\frac{\partial^2 u_3}{\partial t^2} =& \frac{1}{2}\Big(C_{3111}\frac{\partial^2 u_1}{\partial x_1^2} + C_{3111}\frac{\partial^2 u_1}{\partial x_1^2} + C_{3112}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + C_{3112}\frac{\partial^2 u_2}{\partial x_1^2} \\
&+ C_{3113}\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + C_{3113}\frac{\partial^2 u_3}{\partial x_1^2} + C_{3121}\frac{\partial^2 u_2}{\partial x_1^2} + C_{3121}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} \\
&+ C_{3122}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + C_{3122}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + C_{3123}\frac{\partial^2 u_2}{\partial x_1 \partial x_3} + C_{3123}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} \\
&+ C_{3131}\frac{\partial^2 u_3}{\partial x_1^2} + C_{3131}\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + C_{3132}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} + C_{3132}\frac{\partial^2 u_2}{\partial x_1 \partial x_3} \\
&+ C_{3133}\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + C_{3133}\frac{\partial^2 u_3}{\partial x_1 \partial x_3}\Big) + R_{3111}\frac{\partial^2 w_1}{\partial x_1^2} + R_{3112}\frac{\partial^2 w_1}{\partial x_1 \partial x_2} \\
&+ R_{3113}\frac{\partial^2 w_1}{\partial x_1 \partial x_3} + R_{3121}\frac{\partial^2 w_2}{\partial x_1^2} + R_{3122}\frac{\partial^2 w_2}{\partial x_1 \partial x_2} + R_{3123}\frac{\partial^2 w_2}{\partial x_1 \partial x_3} \\
&+ \frac{1}{2}\Big(C_{3211}\frac{\partial^2 u_1}{\partial x_2 \partial x_1} + C_{3211}\frac{\partial^2 u_1}{\partial x_2 \partial x_1} + C_{3212}\frac{\partial^2 u_1}{\partial x_2^2} + C_{3212}\frac{\partial^2 u_2}{\partial x_2 \partial x_1} \\
&+ C_{3213}\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + C_{3213}\frac{\partial^2 u_3}{\partial x_2 \partial x_1} + C_{3221}\frac{\partial^2 u_2}{\partial x_2 \partial x_1} + C_{3221}\frac{\partial^2 u_1}{\partial x_2^2} \\
&+ C_{3222}\frac{\partial^2 u_2}{\partial x_2^2} + C_{3222}\frac{\partial^2 u_2}{\partial x_2^2} + C_{3223}\frac{\partial^2 u_2}{\partial x_2 \partial x_3} + C_{3223}\frac{\partial^2 u_3}{\partial x_2^2} \\
&+ C_{3231}\frac{\partial^2 u_3}{\partial x_2 \partial x_1} + C_{3231}\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + C_{3232}\frac{\partial^2 u_3}{\partial x_2^2} + C_{3232}\frac{\partial^2 u_2}{\partial x_2 \partial x_3} \\
&+ C_{3233}\frac{\partial^2 u_3}{\partial x_2 \partial x_3} + C_{3233}\frac{\partial^2 u_3}{\partial x_2 \partial x_3}\Big) + R_{3211}\frac{\partial^2 w_1}{\partial x_2 \partial x_1} + R_{3212}\frac{\partial^2 w_1}{\partial x_2^2} \\
&+ R_{3213}\frac{\partial^2 w_1}{\partial x_2 \partial x_3} + R_{3221}\frac{\partial^2 w_2}{\partial x_2 \partial x_1} + R_{3222}\frac{\partial^2 w_2}{\partial x_2^2} + R_{3223}\frac{\partial^2 w_2}{\partial x_2 \partial x_3} \\
&+ \frac{1}{2}\Big(C_{3311}\frac{\partial^2 u_1}{\partial x_3 \partial x_1} + C_{3311}\frac{\partial^2 u_1}{\partial x_3 \partial x_1} + C_{3312}\frac{\partial^2 u_1}{\partial x_3 \partial x_2} + C_{3312}\frac{\partial^2 u_2}{\partial x_3 \partial x_1} \\
&+ C_{3313}\frac{\partial^2 u_1}{\partial x_3^2} + C_{3313}\frac{\partial^2 u_3}{\partial x_3 \partial x_1} + C_{3321}\frac{\partial^2 u_2}{\partial x_3 \partial x_1} + C_{3321}\frac{\partial^2 u_1}{\partial x_3 \partial x_2} \\
&+ C_{3322}\frac{\partial^2 u_2}{\partial x_3 \partial x_2} + C_{3322}\frac{\partial^2 u_2}{\partial x_3 \partial x_2} + C_{3323}\frac{\partial^2 u_2}{\partial x_3^2} + C_{3323}\frac{\partial^2 u_3}{\partial x_3 \partial x_2} \\
&+ C_{3331}\frac{\partial^2 u_3}{\partial x_3 \partial x_1} + C_{3331}\frac{\partial^2 u_1}{\partial x_3^2} + C_{3332}\frac{\partial^2 u_3}{\partial x_3 \partial x_2} + C_{3332}\frac{\partial^2 u_2}{\partial x_3^2} \\
&+ C_{3333}\frac{\partial^2 u_3}{\partial x_3^2} + C_{3333}\frac{\partial^2 u_3}{\partial x_3^2}\Big) + R_{3311}\frac{\partial^2 w_1}{\partial x_3 \partial x_1} + R_{3312}\frac{\partial^2 w_1}{\partial x_3 \partial x_2} \\
&+ R_{3313}\frac{\partial^2 w_1}{\partial x_3^2} + R_{3321}\frac{\partial^2 w_2}{\partial x_3 \partial x_1} + R_{3322}\frac{\partial^2 w_2}{\partial x_3 \partial x_2} + R_{3323}\frac{\partial^2 w_2}{\partial x_3^2}
\end{aligned}$$

$$
\begin{aligned}
&+ c'(x_3)\frac{1}{2}\Big(C_{3311}\frac{\partial u_1}{\partial x_1} + C_{3311}\frac{\partial u_1}{\partial x_1} + C_{3312}\frac{\partial u_1}{\partial x_2} + C_{3312}\frac{\partial u_2}{\partial x_1} \\
&+ C_{3313}\frac{\partial u_1}{\partial x_3} + C_{3313}\frac{\partial u_3}{\partial x_1} + C_{3321}\frac{\partial u_2}{\partial x_1} + C_{3321}\frac{\partial u_1}{\partial x_2} + C_{3322}\frac{\partial u_2}{\partial x_2} \\
&+ C_{3322}\frac{\partial u_2}{\partial x_2} + C_{3323}\frac{\partial u_2}{\partial x_3} + C_{3323}\frac{\partial u_3}{\partial x_2} + C_{3331}\frac{\partial u_3}{\partial x_1} + C_{3331}\frac{\partial u_1}{\partial x_3} \\
&+ C_{3332}\frac{\partial u_3}{\partial x_2} + C_{3332}\frac{\partial u_2}{\partial x_3} + C_{3333}\frac{\partial u_3}{\partial x_3} + C_{3333}\frac{\partial u_3}{\partial x_3}\Big) \\
&+ r'(x_3)\Big(R_{3311}\frac{\partial w_1}{\partial x_1} + R_{3312}\frac{\partial w_1}{\partial x_2} + R_{3313}\frac{\partial w_1}{\partial x_3} + R_{3321}\frac{\partial w_2}{\partial x_1} \\
&+ R_{3322}\frac{\partial w_2}{\partial x_2} + R_{3323}\frac{\partial w_2}{\partial x_3}\Big) + f_3(x,t).
\end{aligned}
$$

For β=1, from the equation (3.10) we have:

$$
\rho\frac{\partial^2 w_1}{\partial t^2} = \frac{\partial H_{11}}{\partial x_1} + \frac{\partial H_{12}}{\partial x_2} + \frac{\partial H_{13}}{\partial x_3} + g_1(x,t), \tag{3.24}
$$

where

$$
\begin{aligned}
H_{11} = {} & R_{1111}\varepsilon_{11} + R_{1211}\varepsilon_{12} + R_{1311}\varepsilon_{13} + R_{2111}\varepsilon_{21} + R_{2211}\varepsilon_{22} + R_{2311}\varepsilon_{23} \\
&+ R_{3111}\varepsilon_{31} + R_{3211}\varepsilon_{32} + R_{3311}\varepsilon_{33} + K_{1111}w_{11} + K_{1112}w_{12} + K_{1113}w_{13} \\
&+ K_{1121}w_{21} + K_{1122}w_{22} + K_{1123}w_{23},
\end{aligned} \tag{3.25}
$$

$$
\begin{aligned}
H_{12} = {} & R_{1112}\varepsilon_{11} + R_{1212}\varepsilon_{12} + R_{1312}\varepsilon_{13} + R_{2112}\varepsilon_{21} + R_{2212}\varepsilon_{22} + R_{2312}\varepsilon_{23} \\
&+ R_{3112}\varepsilon_{31} + R_{3212}\varepsilon_{32} + R_{3312}\varepsilon_{33} + K_{1211}w_{11} + K_{1212}w_{12} + K_{1213}w_{13} \\
&+ K_{1221}w_{21} + K_{1222}w_{22} + K_{1223}w_{23}
\end{aligned} \tag{3.26}
$$

and

$$
\begin{aligned}
H_{13} = {} & R_{1113}\varepsilon_{11} + R_{1213}\varepsilon_{12} + R_{1313}\varepsilon_{13} + R_{2113}\varepsilon_{21} + R_{2213}\varepsilon_{22} + R_{2313}\varepsilon_{23} \\
&+ R_{3113}\varepsilon_{31} + R_{3213}\varepsilon_{32} + R_{3313}\varepsilon_{33} + K_{1311}w_{11} + K_{1312}w_{12} + K_{1313}w_{13} \\
&+ K_{1321}w_{21} + K_{1322}w_{22} + K_{1323}w_{23}.
\end{aligned} \tag{3.27}
$$

Substituting $H_{1j}, j = 1,2,3$ into the equation (3.24) we obtain

$$
\begin{aligned}
\rho\frac{\partial^2 w_1}{\partial t^2} \quad = \quad & \frac{1}{2}\Big(R_{1111}\frac{\partial^2 u_1}{\partial x_1^2} + R_{1111}\frac{\partial^2 u_1}{\partial x_1^2} + R_{1211}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + R_{1211}\frac{\partial^2 u_2}{\partial x_1^2} \\
& + R_{1311}\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + R_{1311}\frac{\partial^2 u_3}{\partial x_1^2} + R_{2111}\frac{\partial^2 u_2}{\partial x_1^2} + R_{2111}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} \\
& + R_{2211}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + R_{2211}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + R_{2311}\frac{\partial^2 u_2}{\partial x_1 \partial x_3} + R_{2311}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} \\
& + R_{3111}\frac{\partial^2 u_3}{\partial x_1^2} + R_{3111}\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + R_{3211}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} + R_{3211}\frac{\partial^2 u_2}{\partial x_1 \partial x_3} \\
& + R_{3311}\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + R_{3311}\frac{\partial^2 u_3}{\partial x_1 \partial x_3}\Big) + K_{1111}\frac{\partial^2 w_1}{\partial x_1^2} + K_{1112}\frac{\partial^2 w_1}{\partial x_1 \partial x_2} \\
& + K_{1113}\frac{\partial^2 w_1}{\partial x_1 \partial x_3} + K_{1121}\frac{\partial^2 w_2}{\partial x_1^2} + K_{1122}\frac{\partial^2 w_2}{\partial x_1 \partial x_2} + K_{1123}\frac{\partial^2 w_2}{\partial x_1 \partial x_3} \\
& + \frac{1}{2}\Big(R_{1112}\frac{\partial^2 u_1}{\partial x_2 \partial x_1} + R_{1112}\frac{\partial^2 u_1}{\partial x_2 \partial x_1} + R_{1212}\frac{\partial^2 u_1}{\partial x_2^2} + R_{1212}\frac{\partial^2 u_2}{\partial x_2 \partial x_1} \\
& + R_{1312}\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + R_{1312}\frac{\partial^2 u_3}{\partial x_2 \partial x_1} + R_{2112}\frac{\partial^2 u_2}{\partial x_2 \partial x_1} + R_{2112}\frac{\partial^2 u_1}{\partial x_2^2} \\
& + R_{2212}\frac{\partial^2 u_2}{\partial x_2^2} + R_{2212}\frac{\partial^2 u_2}{\partial x_2^2} + R_{2312}\frac{\partial^2 u_2}{\partial x_2 \partial x_3} + R_{2312}\frac{\partial^2 u_3}{\partial x_2^2} \\
& + R_{3112}\frac{\partial^2 u_3}{\partial x_2 \partial x_1} + R_{3112}\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + R_{3212}\frac{\partial^2 u_3}{\partial x_2^2} + R_{3212}\frac{\partial^2 u_2}{\partial x_2 \partial x_3} \\
& + R_{3312}\frac{\partial^2 u_3}{\partial x_2 \partial x_3} + R_{3312}\frac{\partial^2 u_3}{\partial x_2 \partial x_3}\Big) + K_{1211}\frac{\partial^2 w_1}{\partial x_2 \partial x_1} + K_{1212}\frac{\partial^2 w_1}{\partial x_2^2} \\
& + K_{1213}\frac{\partial^2 w_1}{\partial x_2 \partial x_3} + K_{1221}\frac{\partial^2 w_2}{\partial x_2 \partial x_1} + K_{1222}\frac{\partial^2 w_2}{\partial x_2^2} + K_{1223}\frac{\partial^2 w_2}{\partial x_2 \partial x_3} \\
& + \frac{1}{2}\Big(R_{1113}\frac{\partial^2 u_1}{\partial x_3 \partial x_1} + R_{1113}\frac{\partial^2 u_1}{\partial x_3 \partial x_1} + R_{1213}\frac{\partial^2 u_1}{\partial x_3 \partial x_2} + R_{1213}\frac{\partial^2 u_2}{\partial x_3 \partial x_1} \\
& + R_{1313}\frac{\partial^2 u_1}{\partial x_3^2} + R_{1313}\frac{\partial^2 u_3}{\partial x_3 \partial x_1} + R_{2113}\frac{\partial^2 u_2}{\partial x_3 \partial x_1} + R_{2113}\frac{\partial^2 u_1}{\partial x_3 \partial x_2} \\
& + R_{2213}\frac{\partial^2 u_2}{\partial x_3 \partial x_2} + R_{2213}\frac{\partial^2 u_2}{\partial x_3 \partial x_2} + R_{2313}\frac{\partial^2 u_2}{\partial x_3^2} + R_{2313}\frac{\partial^2 u_3}{\partial x_3 \partial x_2} \\
& + R_{3113}\frac{\partial^2 u_3}{\partial x_3 \partial x_1} + R_{3113}\frac{\partial^2 u_1}{\partial x_3^2} + R_{3213}\frac{\partial^2 u_3}{\partial x_3 \partial x_2} + R_{3213}\frac{\partial^2 u_2}{\partial x_3^2} \\
& + R_{3313}\frac{\partial^2 u_3}{\partial x_3^2} + R_{3313}\frac{\partial^2 u_3}{\partial x_3^2}\Big) + K_{1311}\frac{\partial^2 w_1}{\partial x_3 \partial x_1} + K_{1312}\frac{\partial^2 w_1}{\partial x_3 \partial x_2} \\
& + K_{1313}\frac{\partial^2 w_1}{\partial x_3^2} + K_{1321}\frac{\partial^2 w_2}{\partial x_3 \partial x_1} + K_{1322}\frac{\partial^2 w_2}{\partial x_3 \partial x_2} + K_{1323}\frac{\partial^2 w_2}{\partial x_3^2}
\end{aligned}
$$

$$+\frac{1}{2}r'(x_3)\Big(R_{1113}\frac{\partial u_1}{\partial x_1}+R_{1113}\frac{\partial u_1}{\partial x_1}+R_{1213}\frac{\partial u_1}{\partial x_2}+R_{1213}\frac{\partial u_2}{\partial x_1}$$
$$+R_{1313}\frac{\partial u_1}{\partial x_3}+R_{1313}\frac{\partial u_3}{\partial x_1}+R_{2113}\frac{\partial u_2}{\partial x_1}+R_{2113}\frac{\partial u_1}{\partial x_2}+R_{2213}\frac{\partial u_2}{\partial x_2}$$
$$+R_{2213}\frac{\partial u_2}{\partial x_2}+R_{2313}\frac{\partial u_2}{\partial x_3}+R_{2313}\frac{\partial u_3}{\partial x_2}+R_{3113}\frac{\partial u_3}{\partial x_1}+R_{3113}\frac{\partial u_1}{\partial x_3}$$
$$+R_{3213}\frac{\partial u_3}{\partial x_2}+R_{3213}\frac{\partial u_2}{\partial x_3}+R_{3313}\frac{\partial u_3}{\partial x_3}+R_{3313}\frac{\partial u_3}{\partial x_3}\Big)$$
$$+k'(x_3)\Big(K_{1311}\frac{\partial w_1}{\partial x_1}+K_{1312}\frac{\partial w_1}{\partial x_2}+K_{1313}\frac{\partial w_1}{\partial x_3}+K_{1321}\frac{\partial w_2}{\partial x_1}$$
$$+K_{1322}\frac{\partial w_2}{\partial x_2}+K_{1323}\frac{\partial w_2}{\partial x_3}\Big)+g_1(x,t).$$

For $\beta=2$, from (3.10) we have:

$$\rho\frac{\partial^2 w_2}{\partial t^2}=\frac{\partial H_{21}}{\partial x_1}+\frac{\partial H_{22}}{\partial x_2}+\frac{\partial H_{23}}{\partial x_3}+g_2(x,t), \tag{3.28}$$

where

$$\begin{aligned}H_{21}&=R_{1121}\varepsilon_{11}+R_{1221}\varepsilon_{12}+R_{1321}\varepsilon_{13}+R_{2121}\varepsilon_{21}+R_{2221}\varepsilon_{22}+R_{2321}\varepsilon_{23}\\&+R_{3121}\varepsilon_{31}+R_{3221}\varepsilon_{32}+R_{3321}\varepsilon_{33}+K_{2111}w_{11}+K_{2112}w_{12}+K_{2113}w_{13}\\&+K_{2121}w_{21}+K_{2122}w_{22}+K_{2123}w_{23}.\end{aligned} \tag{3.29}$$

$$\begin{aligned}H_{22}&=R_{1122}\varepsilon_{11}+R_{1222}\varepsilon_{12}+R_{1322}\varepsilon_{13}+R_{2122}\varepsilon_{21}+R_{2222}\varepsilon_{22}+R_{2322}\varepsilon_{23}\\&+R_{3122}\varepsilon_{31}+R_{3222}\varepsilon_{32}+R_{3322}\varepsilon_{33}+K_{2211}w_{11}+K_{2212}w_{12}+K_{2213}w_{13}\\&+K_{2221}w_{21}+K_{2222}w_{22}+K_{2223}w_{23}.\end{aligned} \tag{3.30}$$

$$\begin{aligned}H_{23}&=R_{1123}\varepsilon_{11}+R_{1223}\varepsilon_{12}+R_{1323}\varepsilon_{13}+R_{2123}\varepsilon_{21}+R_{2223}\varepsilon_{22}+R_{2323}\varepsilon_{23}\\&+R_{3123}\varepsilon_{31}+R_{3223}\varepsilon_{32}+R_{3323}\varepsilon_{33}+K_{2311}w_{11}+K_{2312}w_{12}+K_{2313}w_{13}\\&+K_{2321}w_{21}+K_{2322}w_{22}+K_{2323}w_{23}.\end{aligned} \tag{3.31}$$

Substituting $H_{2j},\ j=1,2,3$ into (3.28) we obtain:

$$
\begin{aligned}
\rho\frac{\partial^2 w_2}{\partial t^2} =& \frac{1}{2}\Big(R_{1121}\frac{\partial^2 u_1}{\partial x_1^2} + R_{1121}\frac{\partial^2 u_1}{\partial x_1^2} + R_{1221}\frac{\partial^2 u_1}{\partial x_1\partial x_2} + R_{1221}\frac{\partial^2 u_2}{\partial x_1^2} \\
&+ R_{1321}\frac{\partial^2 u_1}{\partial x_1\partial x_3} + R_{1321}\frac{\partial^2 u_3}{\partial x_1^2} + R_{2121}\frac{\partial^2 u_2}{\partial x_1^2} + R_{2121}\frac{\partial^2 u_1}{\partial x_1\partial x_2} \\
&+ R_{2221}\frac{\partial^2 u_2}{\partial x_1\partial x_2} + R_{2221}\frac{\partial^2 u_2}{\partial x_1\partial x_2} + R_{2321}\frac{\partial^2 u_2}{\partial x_1\partial x_3} + R_{2321}\frac{\partial^2 u_3}{\partial x_1\partial x_2} \\
&+ R_{3121}\frac{\partial^2 u_3}{\partial x_1^2} + R_{3121}\frac{\partial^2 u_1}{\partial x_1\partial x_3} + R_{3221}\frac{\partial^2 u_3}{\partial x_1\partial x_2} + R_{3221}\frac{\partial^2 u_2}{\partial x_1\partial x_3} \\
&+ R_{3321}\frac{\partial^2 u_3}{\partial x_1\partial x_3} + R_{3321}\frac{\partial^2 u_3}{\partial x_1\partial x_3}\Big) + K_{2111}\frac{\partial^2 w_1}{\partial x_1^2} + K_{2112}\frac{\partial^2 w_1}{\partial x_1\partial x_2} \\
&+ K_{2113}\frac{\partial^2 w_1}{\partial x_1\partial x_3} + K_{2121}\frac{\partial^2 w_2}{\partial x_1^2} + K_{2122}\frac{\partial^2 w_2}{\partial x_1\partial x_2} + K_{2123}\frac{\partial^2 w_2}{\partial x_1\partial x_3} \\
&+ \frac{1}{2}\Big(R_{1122}\frac{\partial^2 u_1}{\partial x_2\partial x_1} + R_{1122}\frac{\partial^2 u_1}{\partial x_2\partial x_1} + R_{1222}\frac{\partial^2 u_1}{\partial x_2^2} + R_{1222}\frac{\partial^2 u_2}{\partial x_2\partial x_1} \\
&+ R_{1322}\frac{\partial^2 u_1}{\partial x_2\partial x_3} + R_{1322}\frac{\partial^2 u_3}{\partial x_2\partial x_1} + R_{2122}\frac{\partial^2 u_2}{\partial x_2\partial x_1} + R_{2122}\frac{\partial^2 u_1}{\partial x_2^2} \\
&+ R_{2222}\frac{\partial^2 u_2}{\partial x_2^2} + R_{2222}\frac{\partial^2 u_2}{\partial x_2^2} + R_{2322}\frac{\partial^2 u_2}{\partial x_2\partial x_3} + R_{2322}\frac{\partial^2 u_3}{\partial x_2^2} \\
&+ R_{3122}\frac{\partial^2 u_3}{\partial x_2\partial x_1} + R_{3122}\frac{\partial^2 u_1}{\partial x_2\partial x_3} + R_{3222}\frac{\partial^2 u_3}{\partial x_2^2} + R_{3222}\frac{\partial^2 u_2}{\partial x_2\partial x_3} \\
&+ R_{3322}\frac{\partial^2 u_3}{\partial x_2\partial x_3} + R_{3322}\frac{\partial^2 u_3}{\partial x_2\partial x_3}\Big) + K_{2211}\frac{\partial^2 w_1}{\partial x_2\partial x_1} + K_{2212}\frac{\partial^2 w_1}{\partial x_2^2} \\
&+ K_{2213}\frac{\partial^2 w_1}{\partial x_2\partial x_3} + K_{2221}\frac{\partial^2 w_2}{\partial x_2\partial x_1} + K_{2222}\frac{\partial^2 w_2}{\partial x_2^2} + K_{2223}\frac{\partial^2 w_2}{\partial x_2\partial x_3} \\
&+ \frac{1}{2}\Big(R_{1123}\frac{\partial^2 u_1}{\partial x_3\partial x_1} + R_{1123}\frac{\partial^2 u_1}{\partial x_3\partial x_1} + R_{1223}\frac{\partial^2 u_1}{\partial x_3\partial x_2} + R_{1223}\frac{\partial^2 u_2}{\partial x_3\partial x_1} \\
&+ R_{1323}\frac{\partial^2 u_1}{\partial x_3^2} + R_{1323}\frac{\partial^2 u_3}{\partial x_3\partial x_1} + R_{2123}\frac{\partial^2 u_2}{\partial x_3\partial x_1} + R_{2123}\frac{\partial^2 u_1}{\partial x_3\partial x_2} \\
&+ R_{2223}\frac{\partial^2 u_2}{\partial x_3\partial x_2} + R_{2223}\frac{\partial^2 u_2}{\partial x_3\partial x_2} + R_{2323}\frac{\partial^2 u_2}{\partial x_3^2} + R_{2323}\frac{\partial^2 u_3}{\partial x_3\partial x_2} \\
&+ R_{3123}\frac{\partial^2 u_3}{\partial x_3\partial x_1} + R_{3123}\frac{\partial^2 u_1}{\partial x_3^2} + R_{3223}\frac{\partial^2 u_3}{\partial x_3\partial x_2} + R_{3223}\frac{\partial^2 u_2}{\partial x_3^2} \\
&+ R_{3323}\frac{\partial^2 u_3}{\partial x_3^2} + R_{3323}\frac{\partial^2 u_3}{\partial x_3^2}\Big) + K_{2311}\frac{\partial^2 w_1}{\partial x_3\partial x_1} + K_{2312}\frac{\partial^2 w_1}{\partial x_3\partial x_2} \\
&+ K_{2313}\frac{\partial^2 w_1}{\partial x_3^2} + K_{2321}\frac{\partial^2 w_2}{\partial x_3\partial x_1} + K_{2322}\frac{\partial^2 w_2}{\partial x_3\partial x_2} + K_{2323}\frac{\partial^2 w_2}{\partial x_3^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ r'(x_3)\frac{1}{2}(R_{1123}\frac{\partial u_1}{\partial x_1} + R_{1123}\frac{\partial u_1}{\partial x_1} + R_{1223}\frac{\partial u_1}{\partial x_2} + R_{1223}\frac{\partial u_2}{\partial x_1}\\
&+ R_{1323}\frac{\partial u_1}{\partial x_3} + R_{1323}\frac{\partial u_3}{\partial x_1} + R_{2123}\frac{\partial u_2}{\partial x_1} + R_{2123}\frac{\partial u_1}{\partial x_2} + R_{2223}\frac{\partial u_2}{\partial x_2}\\
&+ R_{2223}\frac{\partial u_2}{\partial x_2} + R_{2323}\frac{\partial u_2}{\partial x_3} + R_{2323}\frac{\partial u_3}{\partial x_2} + R_{3123}\frac{\partial u_3}{\partial x_1} + R_{3123}\frac{\partial u_1}{\partial x_3}\\
&+ R_{3223}\frac{\partial u_3}{\partial x_2} + R_{3223}\frac{\partial u_2}{\partial x_3} + R_{3323}\frac{\partial u_3}{\partial x_3} + R_{3323}\frac{\partial u_3}{\partial x_3})\\
&+ k'(x_3)\Big(K_{2311}\frac{\partial w_1}{\partial x_1} + K_{2312}\frac{\partial w_1}{\partial x_2} + K_{2313}\frac{\partial w_1}{\partial x_3} + K_{2321}\frac{\partial w_2}{\partial x_1}\\
&+ K_{2322}\frac{\partial w_2}{\partial x_2} + K_{2323}\frac{\partial w_2}{\partial x_3}\Big) + g_2(x,t).
\end{aligned}
$$

3.2 Vector Form of the Initial Value Problem (3.9)-(3.11)

Using equations (3.3), (3.4) and symmetry properties (3.5)-(3.7), the equations (3.9) - (3.11) can be written in the matrix form:

$$\rho\frac{\partial^2 \boldsymbol{U}}{\partial t^2} = \sum_{j,l=1}^{3}\boldsymbol{A}_{jl}\frac{\partial^2 \boldsymbol{U}}{\partial x_j \partial x_l} + \sum_{j,l=1}^{3}\boldsymbol{B}_{jl}\frac{\partial^2 \boldsymbol{W}}{\partial x_j \partial x_l} + \sum_{j,l=1}^{3}\boldsymbol{T}_{jl}\frac{\partial \boldsymbol{U}}{\partial x_l} + \sum_{j,l=1}^{3}\boldsymbol{Y}_{jl}\frac{\partial \boldsymbol{W}}{\partial x_l} + \boldsymbol{f}(x,t), \tag{3.32}$$

$$\rho\frac{\partial^2 \boldsymbol{W}}{\partial t^2} = \sum_{j,l=1}^{3}\boldsymbol{M}_{jl}\frac{\partial^2 \boldsymbol{U}}{\partial x_j \partial x_l} + \sum_{j,l=1}^{3}\boldsymbol{N}_{jl}\frac{\partial^2 \boldsymbol{W}}{\partial x_j \partial x_l} + \sum_{j,l=1}^{3}\boldsymbol{R}_{jl}\frac{\partial \boldsymbol{U}}{\partial x_l} + \sum_{j,l=1}^{3}\boldsymbol{Z}_{jl}\frac{\partial \boldsymbol{W}}{\partial x_l} + \boldsymbol{g}(x,t), \tag{3.33}$$

$$\boldsymbol{U}(x,t)|_{t=0} = 0, \quad \boldsymbol{W}(x,t)|_{t=0} = 0, \tag{3.34}$$

where $\boldsymbol{U} = (u_1(x,t), u_2(x,t), u_3(x,t))^T$ and $\boldsymbol{W} = (w_1(x,t), w_2(x,t))^T$ are vectors of phonon and phason displacements, respectively. $\boldsymbol{f} = (f_1, f_2, f_3)^T$ and $\boldsymbol{g} = (g_1, g_2)^T$ are vectors of body forces. The entries of the matrices $\boldsymbol{A}_{jl} = [a_{ik}]_{3\times3}$, $\boldsymbol{B}_{jl} = [b_{i\alpha}]_{3\times2}$, $\boldsymbol{M}_{jl} = [m_{\beta k}]_{2\times3}$, $\boldsymbol{N}_{jl} = [n_{\beta k}]_{2\times2}$, $\boldsymbol{T}_{jl} = [t_{ik}]_{3\times3}$, $\boldsymbol{Y}_{il} = [y_{i\alpha}]_{3\times2}$, $\boldsymbol{R}_{jl} = [r_{\beta k}]_{2\times3}$ and $\boldsymbol{Z}_{jl} = [z_{\beta\alpha}]_{2\times2}$ are defined as follows:

$$a_{ik} = A_{jl}^{ik} = \frac{1}{2}(C_{ijkl} + C_{ilkj}); \quad b_{i\alpha} = B_{jl}^{i\alpha} = \frac{1}{2}(R_{ij\alpha l} + R_{il\alpha j}), \tag{3.35}$$

$$t_{ik} = T_{jl}^{ik} = c'(x_3)C_{ijkl}; \quad y_{i\alpha} = Y_{jl}^{i\alpha} = r'(x_3)R_{ij\alpha l}, \tag{3.36}$$

$$m_{\beta k} = M_{jl}^{\beta k} = \frac{1}{2}(R_{kl\beta j} + R_{kj\beta l}); \quad n_{\beta k} = N_{jl}^{\beta\alpha} = \frac{1}{2}(K_{\beta j\alpha l} + K_{\beta l\alpha j}), \tag{3.37}$$

$$r_{\beta k} = R_{jl}^{\beta k} = r'(x_3)R_{kj\beta l}; \quad z_{\beta\alpha} = Z_{jl}^{\beta\alpha} = k'(x_3)K_{\beta j\alpha l}. \tag{3.38}$$

3.3 Matrix Form of the Initial Value Problem (3.9)-(3.11)

If phonon and phason displacement are defined as components of a vector such as $\boldsymbol{V} = [u_1, u_2, u_3, w_1, w_2]^T$ and $\boldsymbol{F} = [f_1, f_2, f_3, g_1, g_2]^T$ then equations (3.32) - (3.34) can be written in the form:

$$\rho \frac{\partial^2 \boldsymbol{V}(x,t)}{\partial t^2} = \sum_{j,l=1}^{3} P_{jl}^1 \frac{\partial^2 \boldsymbol{V}(x,t)}{\partial x_j \partial x_l} + \sum_{j,l=1}^{3} P_{jl}^2 \frac{\partial \boldsymbol{V}(x,t)}{\partial x_l} + \boldsymbol{F}(x,t), \tag{3.39}$$

$$\boldsymbol{V}(x,t)|_{t=0} = 0, \quad \frac{\partial \boldsymbol{V}(x,t)}{\partial t}|_{t=0} = 0. \tag{3.40}$$

where $\boldsymbol{P}_{jl}^1$ and $\boldsymbol{P}_{jl}^2$ are block matrices of the size (5×5) defined by

$$\boldsymbol{P}_{jl}^1 = \begin{bmatrix} A_{jl} & B_{jl} \\ M_{jl} & N_{jl} \end{bmatrix}; \quad \boldsymbol{P}_{jl}^2 = \begin{bmatrix} T_{jl} & Y_{jl} \\ R_{jl} & Z_{jl} \end{bmatrix};$$

Remark. The submatrices $A_{jl}, B_{jl}, M_{jl}, N_{jl}, T_{jl}, Y_{jl}, R_{jl}, Z_{jl}$ are given (3.35) - (3.38) using the symmetry properties (3.5) - (3.7) and equalities (3.34)-(3.37) the following identities are obtained.

$$\begin{aligned} a_{ik} &= A_{jl}^{ik} = A_{jl}^{ki} = a_{ki}, \\ b_{i\alpha} &= B_{jl}^{i\alpha} = M_{jl}^{\alpha i} = m_{\alpha i}, \\ m_{\beta k} &= M_{jl}^{\beta k} = B_{jl}^{k\beta} = b_{k\beta}, \\ n_{\beta\alpha} &= N_{jl}^{\beta\alpha} = N_{jl}^{\alpha\beta} = n_{\alpha\beta}. \end{aligned} \tag{3.41}$$

It follows that the square matrix $\boldsymbol{A}_{jl} = [a_{ik}]_{3\times3}$ and $\boldsymbol{N}_{jl} = [n_{\beta\alpha}]_{2\times2}$ are symmetric and also $\boldsymbol{M}_{jl} = [m_{\beta k}]_{3\times2}$ is a transpose of $\boldsymbol{B}_{jl} = [b_{i\alpha}]_{2\times3}$. Thus, it is clear that $\boldsymbol{P}_{jl}^1$ is a symmetric matrix.

3.4 Reduction to Integral Equations of the Initial Value Problem (3.9)-(3.11)

Using the following change of variable $\boldsymbol{V}=\wedge\tilde{\boldsymbol{V}}$ (3.39), (3.40) we obtain;

$$\rho\frac{\partial^2(\wedge\tilde{\boldsymbol{V}})}{\partial t^2}=\boldsymbol{P}_{33}^1\frac{\partial^2(\wedge\tilde{\boldsymbol{V}})}{\partial x_3^2}+\sum_{\substack{j,l=1\\ j,l\neq 3}}^{3}\boldsymbol{P}_{jl}^1\frac{\partial^2(\wedge\tilde{\boldsymbol{V}})}{\partial x_j\partial x_l}+\sum_{j,l=1}^{3}\boldsymbol{P}_{jl}^2\frac{\partial(\wedge\tilde{\boldsymbol{V}})}{\partial x_l}+\boldsymbol{F}, \tag{3.42}$$

$$\tilde{\boldsymbol{V}}(x,t)|_{t=0}=0,\quad \frac{\partial\tilde{\boldsymbol{V}}(x,t)}{\partial t}|_{t=0}=0. \tag{3.43}$$

Equation (3.42) may be written

$$\rho\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial t^2}=\boldsymbol{P}_{33}^1\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial x_3^2}+2\boldsymbol{P}_{33}^1\wedge'\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_3}+\boldsymbol{P}_{33}^1\wedge''\tilde{\boldsymbol{V}}+\sum_{j,l=1}^{2}\boldsymbol{P}_{jl}^1\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial x_j\partial x_l}$$
$$+2\sum_{j=1}^{2}\boldsymbol{P}_{j3}^1\frac{\partial(\wedge'\tilde{\boldsymbol{V}}+\wedge\tilde{\boldsymbol{V}}')}{\partial x_j}+\sum_{\substack{j=1\\ l\neq 3}}^{3}\boldsymbol{P}_{jl}^2\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_l}+\sum_{j=1}^{3}\boldsymbol{P}_{j3}^2(\wedge'\tilde{\boldsymbol{V}}+\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_3})+\boldsymbol{F}. \tag{3.44}$$

It follows from the above equation;

$$\rho\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial t^2}=\boldsymbol{P}_{33}^1\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial x_3^2}+2\boldsymbol{P}_{33}^1\wedge'\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_3}+\boldsymbol{P}_{33}^1\wedge''\tilde{\boldsymbol{V}}+\boldsymbol{P}_{11}^1\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial x_1^2}$$
$$+\boldsymbol{P}_{12}^1\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial x_1\partial x_2}+\boldsymbol{P}_{13}^1\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial x_1\partial x_3}+\boldsymbol{P}_{21}^1\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial x_2\partial x_1}+\boldsymbol{P}_{22}^1\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial x_2^2}$$
$$+\boldsymbol{P}_{23}^1\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial x_2\partial x_3}+\boldsymbol{P}_{31}^1\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial x_3\partial x_1}+\boldsymbol{P}_{32}^1\wedge\frac{\partial^2\tilde{\boldsymbol{V}}}{\partial x_3\partial x_2}+2\boldsymbol{P}_{13}^1\frac{\partial(\wedge'\tilde{\boldsymbol{V}}+\wedge\tilde{\boldsymbol{V}}')}{\partial x_1}$$
$$+2\boldsymbol{P}_{23}^1\frac{\partial(\wedge'\tilde{\boldsymbol{V}}+\wedge\tilde{\boldsymbol{V}}')}{\partial x_2}+\boldsymbol{P}_{11}^2\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_1}+\boldsymbol{P}_{12}^2\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_2}+\boldsymbol{P}_{13}^2\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_3}+\boldsymbol{P}_{21}^2\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_1}$$
$$+\boldsymbol{P}_{22}^2\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_2}+\boldsymbol{P}_{23}^2\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_3}+\boldsymbol{P}_{31}^2\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_1}+\boldsymbol{P}_{32}^2\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_2}+\boldsymbol{P}_{13}^2(\wedge'\tilde{\boldsymbol{V}}+\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_3})$$
$$+\boldsymbol{P}_{23}^2(\wedge'\tilde{\boldsymbol{V}}+\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_3})+\boldsymbol{P}_{33}^2(\wedge'\tilde{\boldsymbol{V}}+\wedge\frac{\partial\tilde{\boldsymbol{V}}}{\partial x_3})+\boldsymbol{F}. \tag{3.45}$$

Multiplying both sides of equation (3.45) by $\wedge^T$, we get

$$
\begin{aligned}
\rho \wedge^T \wedge \frac{\partial^2 \tilde{\boldsymbol{V}}}{\partial t^2} &= \wedge^T \boldsymbol{P}_{33}^1 \wedge \frac{\partial^2 \tilde{\boldsymbol{V}}}{\partial x_3^2} + 2 \wedge^T \boldsymbol{P}_{33}^1 \wedge' \frac{\partial \tilde{\boldsymbol{V}}}{\partial x_3} + \wedge^T \boldsymbol{P}_{33}^1 \wedge'' \tilde{\boldsymbol{V}} \\
&+ \sum_{\substack{j,l=1 \\ j,l \neq 3}}^{2} \wedge^T \boldsymbol{P}_{jl}^1 \wedge \frac{\partial^2 \tilde{\boldsymbol{V}}}{\partial x_j \partial x_l} + 2 \sum_{j=1}^{2} \wedge^T \boldsymbol{P}_{j3}^1 \frac{\partial}{\partial x_j} (\wedge' \tilde{\boldsymbol{V}} + \wedge \tilde{\boldsymbol{V}}') \\
&+ \sum_{\substack{j=1 \\ l \neq 3}}^{3} \wedge^T \boldsymbol{P}_{jl}^2 \wedge \frac{\partial}{\partial x_l} \tilde{\boldsymbol{V}} + \sum_{j=1}^{3} \wedge^T \boldsymbol{P}_{j3}^2 (\wedge' \tilde{\boldsymbol{V}} + \wedge \frac{\partial \tilde{\boldsymbol{V}}}{\partial x_3}) + \wedge^T \boldsymbol{F}.
\end{aligned}
\tag{3.46}
$$

Since P_{33}^1 is symmetric and positive definite, by the theorem (see Appendix 3) we have

$$
\wedge^T \boldsymbol{P}_{33}^1 \wedge = \boldsymbol{D}, \tag{3.47}
$$

where $\boldsymbol{D}$ is diagonal matrix with entries $d_j, j = 1, \ldots, 5$.

Let

$$
\wedge^T \boldsymbol{P}_{jl}^1 \wedge = \tilde{\boldsymbol{P}}_{jl}^1, \quad \wedge^T \boldsymbol{P}_{jl}^2 \wedge = \tilde{\boldsymbol{P}}_{jl}^2, \quad \wedge^T \boldsymbol{F} = \tilde{\boldsymbol{F}}. \tag{3.48}
$$

Then, equation (3.46) may be written as

$$
\begin{aligned}
\rho \frac{\partial^2 \tilde{\boldsymbol{V}}}{\partial t^2} &= \boldsymbol{D} \frac{\partial^2 \tilde{\boldsymbol{V}}}{\partial x_3^2} + 2 \wedge^T \boldsymbol{P}_{33}^1 \wedge' \frac{\partial \tilde{\boldsymbol{V}}}{\partial x_3} + \wedge^T \boldsymbol{P}_{33}^1 \wedge'' \tilde{\boldsymbol{V}} + \sum_{\substack{j,l=1 \\ j,l \neq 3}}^{2} \tilde{\boldsymbol{P}}_{jl}^1 \frac{\partial^2 \tilde{\boldsymbol{V}}}{\partial x_j \partial x_l} \\
&+ 2 \sum_{j=1}^{2} \wedge^T \boldsymbol{P}_{j3}^1 \frac{\partial}{\partial x_j} (\wedge' \tilde{\boldsymbol{V}} + \wedge \tilde{\boldsymbol{V}}') + \sum_{\substack{j=1 \\ l \neq 3}}^{3} \tilde{\boldsymbol{P}}_{jl}^2 \frac{\partial}{\partial x_l} \tilde{\boldsymbol{V}} \\
&+ \sum_{j=1}^{3} \wedge^T \boldsymbol{P}_{j3}^2 (\wedge' \tilde{\boldsymbol{V}} + \wedge \frac{\partial \tilde{\boldsymbol{V}}}{\partial x_3}) + \tilde{\boldsymbol{F}}.
\end{aligned}
\tag{3.49}
$$

Applying the Fourier transform with respect to x_1 and x_2 and using the properties of the Fourier transform (see Appendix 1), equations (3.49) can be written in terms of the Fourier image $\hat{\mathbf{V}}(v, x_3, t)$ as follows:

$$\rho\frac{\partial^2\hat{\boldsymbol{V}}}{\partial t^2} =\hat{\boldsymbol{D}}\frac{\partial^2\hat{\boldsymbol{V}}}{\partial x_3^2} + 2\wedge^T\boldsymbol{P}_{33}^1\wedge'\frac{\partial\hat{\boldsymbol{V}}}{\partial x_3} + \wedge^T\boldsymbol{P}_{33}^1\wedge''\hat{\boldsymbol{V}} - \sum_{j,l=1}^{2}\tilde{\boldsymbol{P}}_{jl}^1 v_j v_l\hat{\boldsymbol{V}}$$
$$- 2i\sum_{l=1}^{2}(\tilde{\boldsymbol{P}}_{l3}^1 + \tilde{\boldsymbol{P}}_{3l}^1)v_l\frac{\partial\hat{\boldsymbol{V}}}{\partial x_3} - i\sum_{j=1}^{2}\tilde{\boldsymbol{P}}_{jl}^2 v_l\hat{\boldsymbol{V}} + \sum_{j=1}^{3}\tilde{\boldsymbol{P}}_{j3}^2\frac{\partial\hat{\boldsymbol{V}}}{\partial x_3} \tag{3.50}$$
$$+ \wedge^T\tilde{\boldsymbol{P}}_{13}^2\wedge'\hat{\boldsymbol{V}} + \wedge^T\tilde{\boldsymbol{P}}_{23}^2\wedge'\hat{\boldsymbol{V}} + \wedge^T\tilde{\boldsymbol{P}}_{33}^2\wedge'\hat{\boldsymbol{V}} + \hat{\boldsymbol{F}}.$$

Let us denote the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ as

$$\boldsymbol{A} = 2\wedge^T P_{33}^1\wedge' + \sum_{j,l=1}^{3}(-2i\tilde{\boldsymbol{P}}_{l3}^1 + \tilde{\boldsymbol{P}}_{3l}^1)v_l + \tilde{\boldsymbol{P}}_{j3}^2. \tag{3.51}$$

$$\boldsymbol{B} = \wedge^T P_{33}^1\wedge'' - \sum_{j,l=1}^{2}(\tilde{\boldsymbol{P}}_{jl}^1 v_j v_l + i\tilde{\boldsymbol{P}}_{jl}^2 v_l + \wedge^T\sum_{j=1}^{3}\tilde{\boldsymbol{P}}_{j3}^2\wedge'), \tag{3.52}$$

Thus, for each component $\hat{V}_j(v, x_3, t)$ of $\hat{\boldsymbol{V}}(v, x_3, t)$ satisfies

$$\rho\frac{\partial^2\hat{V}_j}{\partial t^2} = d_j\frac{\partial^2\hat{V}_n}{\partial x_3^2} + \sum_{n=1}^{5}A_{jn}\frac{\partial\,\hat{V}_n}{\partial x_3} + \sum_{n=1}^{5}B_{jn}\hat{V}_n + \hat{F}_j, \quad j = 1,\ldots,5, \tag{3.53}$$

$$\hat{V}_j(v, x_3, t)|_{t=0} = 0, \quad \frac{\partial\hat{V}_j(v, x_3, t)}{\partial t}|_{t=0} = 0, \tag{3.54}$$

where A_{jn} and B_{jn} are components of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ given in (3.51) and (3.52), respectively.

Let us consider the following transform

$$y_j = \tau_j(x_3); \quad \tau_j(x_3) = \int_0^{x_3}c_j(\xi)d\xi, \quad c_j^2(\xi) = \frac{1}{d_j(\xi)},$$
$$\phi_j(v, y_j, t) = \hat{V}_j(v, x_3, t)\big|_{x_3=\tau_j^{-1}(y_j)} \quad j = 1,\ldots,5. \tag{3.55}$$

Using the above transformation, the equations (3.53) and (3.54) becomes:

$$\frac{\partial^2\phi_j}{\partial t^2} - \frac{\partial^2\phi_j}{\partial y_j^2} = -K_j(y_j)\frac{\partial\phi_j}{\partial y_j} + \frac{1}{\rho(x_3)}\Big(\sum_{\substack{n=1\\n\neq j}}^{5} A_{jn}\frac{\partial\phi_n}{\partial x_3} + \sum_{n=1}^{5} B_{jn}\phi_n + \hat{F}_j\Big)\Big|_{x_3=\tau_j^{-1}(y_j)}, \tag{3.56}$$

$$\phi_j|_{t=0} = 0, \quad \frac{\partial\phi_j}{\partial t}|_{t=0} = 0, \tag{3.57}$$

where

$$K_j(y_j) = \frac{1}{2}\sqrt{\frac{d_j}{\rho}}\frac{d}{d_{y_j}}\Big(ln(\frac{\rho}{d_j})\Big) + \frac{A_{jj}}{\sqrt{\rho\, d_j}}\Big|_{x_3=\tau_j^{-1}(y_j)} \tag{3.58}$$

We will seek a solution of the problem (3.56), (3.57) in the following form

$$\phi_j(v, y_j, t) = S_j(y_j)\tilde{\phi}_j(v, y_j, t), \tag{3.59}$$

where the function $S_j(y_j)$ defined by

$$S_j(y) = \exp(\frac{1}{2}\int_0^{y_j} K_j(\xi)d\xi), \tag{3.60}$$

Then (3.56) and (3.57) becomes

$$\begin{aligned}\frac{\partial^2\tilde{\phi}_j}{\partial t^2} - \frac{\partial^2\tilde{\phi}_j}{\partial y_j^2} =&(\frac{1}{2}K_j{}'(y_j) - \frac{1}{4}K_j^2(y_j))\tilde{\phi}_j + \frac{1}{S_j(x_3)\rho(x_3)}\Big(\sum_{\substack{n=1\\n\neq j}}^{5} A_{jn}\frac{\partial\tilde{\phi}_j}{\partial x_3} \\ &+ \sum_{n=1}^{5} B_{jn}\tilde{\phi}_j + \hat{F}_j\Big)\Big|_{x_3=\tau_j^{-1}(y_j)}\end{aligned} \tag{3.61}$$

$$\tilde{\phi}_j|_{t=0} = 0, \quad \frac{\partial\tilde{\phi}_j}{\partial t}|_{t=0} = 0 \tag{3.62}$$

Let

$$q_j(y_j) = \frac{1}{2}K_j^{'}(y_j) - \frac{1}{4}K_j^2(y_j), \ \tilde{c}_j(y_j) = c_j(x_3)|_{x_3=\tau_j^{-1}(y_j)},$$

$$M_j(y_j) = S_j(y_j)\rho(\tau_j^{-1}(y_j)), \quad L_j^1(y_j) = \frac{A_{jj}(\tau_j^{-1}(y_j))}{M_j(y_j)}$$

$$L_j^3(y_j) = \frac{\tilde{c}_j(y_j)}{M_j(y_j)}\Big(\sum_{j,l=1}^{2}(-2i\tilde{P}_{l3}^1 + \tilde{P}_{3l}^1)v_l + \tilde{P}_{j3}^2\Big),$$

$$M_j(y_j)L_{jn}^2(y_j) = \Big(\wedge^T P_{33}^1 \frac{\partial\wedge}{\partial y_j}\Big)\Big|_{x_3=\tau_j^{-1}(y_j)},$$

$$M_j(y_j)L_j^4(y_j) = \sum_{j,l=1}^{2}(\tilde{P}_{jl}^1 v_j v_l + i\tilde{P}_{jl}^2 v_l), \quad \frac{F_j(v, \tau_j^{-1}(y_j), t)}{M_j(y_j)} = \tilde{F}_j(v, y_j, t),$$

$$M_j(y_j)L_j^5(y_j) = \tilde{c}(x_3) \wedge^T \Big(\tilde{P}_{j3}^2 \frac{\partial\wedge}{\partial y_j} + P_{33}^1(\frac{\partial^2\wedge}{\partial y_j^2}\tilde{c}_j + \frac{\partial\wedge}{\partial y_j}\tilde{c}_j')\Big)\Big|_{x_3=\tau_j^{-1}(y_j)}.$$

Then equations (3.61) and (3.62) may be written as

$$\begin{aligned}\frac{\partial^2\tilde{\phi}_j}{\partial t^2} - \frac{\partial^2\tilde{\phi}_j}{\partial y_j^2} =& \Big(q_j(y_j) + L_j^1(y_j)\Big)\tilde{\phi}_j + \sum_{\substack{n=1\\n\neq j}}^{5} 2\tilde{c}_j^2(y_j)L_{jn}^2\frac{\partial\tilde{\phi}_n}{y_j}\\ &+ \sum_{\substack{n=1\\n\neq j}}^{5} L_j^2(y_j)\frac{\partial\tilde{\phi}_n}{\partial y_j} + \sum_{\substack{n=1\\n\neq j}}^{5} L_j^4(y_j)\tilde{\phi}_n + \sum_{\substack{n=1\\n\neq j}}^{5} L_j^5(y_j)\tilde{\phi}_n\\ &+ \tilde{F}_j(v,\ y_j\ t).\end{aligned} \tag{3.63}$$

$$\tilde{\phi}_j(v,\ y_j\ t)\Big|_{t=0} = 0, \quad \frac{\partial\tilde{\phi}_j(v,\ y_j\ t)}{\partial t}\Big|_{t=0} = 0. \tag{3.64}$$

Using d'Alembert's formula (see Appendix 2) the solution of (3.63) and (3.64) can be written in the form:

$$
\begin{aligned}
\tilde{\phi}_j(v,y_j,t) &= \frac{1}{2}\int_0^t \int_{y_j-(t-\tau)}^{y_j+(t-\tau)} (q_j(\xi)+L_j^1(\xi))\tilde{\phi}_j(v,\xi,\tau)d\xi d\tau \\
&+ \frac{1}{2}\int_0^t \int_{y_j-(t-\tau)}^{y_j+(t-\tau)} \sum_{\substack{n=1\\ j\neq n}}^{5} \big(\tilde{L}_j^4(\xi)+L_j^5(\xi)\big)\tilde{\phi}_n(v,\tau^{-1}(\xi),n)d\xi dn \\
&- \frac{1}{2}\int_0^t \int_{y_j-(t-\tau)}^{y_j+(t-\tau)} \sum_{\substack{n=1\\ j\neq n}}^{5} \frac{\partial}{\partial \xi}\Big(2\tilde{c}_j^2(\xi)L_{jn}^2(\xi) \\
&+ L_j^3(\xi)\Big)\tilde{\phi}_n(v,\xi,n)d\xi dn + \frac{1}{2}\int_0^t \int_{y_j-(t-\tau)}^{y_j+(t-\tau)} \tilde{F}_j(v,\xi,n)d\xi dn \\
&+ \frac{1}{2}\int_0^t \sum_{\substack{n=1\\ j\neq n}}^{5} \big(2c_j^2(\xi)L_{jn}^2(\xi)+L_j^3(\xi)\big)\tilde{\phi}_n(v,\tau_j^{-1}(\xi),n)\Big|_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)} d\tau
\end{aligned}
\tag{3.65}
$$

Using the equalities given in (3.55) and (3.59) the above equation may be written as

$$
\begin{aligned}
\hat{V}_j(v,x_3,t) &= \frac{S_j(\tau_j(x_3))}{2}\int_0^t \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \Big\{\big(q_j(\xi)+L_j^1(\xi)\big)\frac{\hat{V}_j}{S_j(\xi)}(v,\xi,\tau) \\
&+ \sum_{\substack{n=1\\ j\neq n}}^{5} \big(\tilde{L}_j^4(\xi)+L_j^5(\xi)\big)\hat{V}_n(v,\tau_j^{-1}(\xi),n) - \sum_{\substack{n=1\\ j\neq n}}^{5} \frac{\partial}{\partial \xi}(2\tilde{c}_j^2(\xi)L_{jn}^2(\xi) \\
&+ L_j^3(\xi))\hat{V}_n(v,\xi,n) + \tilde{F}_j(v,\xi,n)\Big\}d_\xi dn + \frac{S_j(\tau_j(x_3)}{2}\int_0^t \big(2\tilde{c}_j^2(\xi)L_{jn}^2(\xi) \\
&+ L_j^3(\xi)\big)\hat{V}_n(v,\tau_j^{-1}(\xi),n)\Big|_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)} dn
\end{aligned}
\tag{3.66}
$$

As a result of the above fact that we obtain the following result.

Lemma 3.4.1 Fourier image solution of the initial value problem (3.39), (3.40) is equivalent to the following operator integral equation

$$
\hat{\boldsymbol{V}}(v,x_3,t) = \boldsymbol{G}(v,x_3,t) + \int_0^t (\boldsymbol{K}\hat{\boldsymbol{V}})(v,x_3,t,\tau)d\tau \tag{3.67}
$$

where

$$G(v, x_3, t) = \frac{S_j(\tau_j(x_3))}{2} \int_0^t \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \hat{F}_j(v, \xi, \tau) d\xi d\tau, \tag{3.68}$$

$$\begin{aligned}(K_j \hat{\boldsymbol{V}})(v, x_3, t, \tau) =& \frac{S_j(\tau_j(x_3))}{2} \Big(\int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \big\{ (q_j(\xi) + L_j^1(\xi)) \frac{\hat{V}_j(v, \xi, \tau)}{S_j(\xi)} \\ &+ \sum_{\substack{n=1 \\ j \neq n}}^{5} (L_j^4(\xi) + L_j^5(\xi)) \hat{V}_n(v, \tau_j^{-1}(\xi), n) \\ &- \sum_{\substack{n=1 \\ j \neq n}}^{5} (2\tilde{c}_j^2(\xi) L_{jn}^2 + L_j^3(\xi)) \hat{V}_n(v, \xi, n) \big\} d\xi \\ &+ \sum_{\substack{n=1 \\ j \neq n}}^{5} (2\tilde{c}_j^2(\xi) L_{jn}^2 xi + L_j^3(\xi)) \hat{V}_n(v, \tau_j^{-1}(\xi), n) \Big|_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)} \Big). \end{aligned} \tag{3.69}$$

CHAPTER FOUR
CONCLUSION

The basic equations of the elasticity theory of quasicrystals were considered. An initial value problem for two dimensional quasicrystals was studied. Reduction to the vector integral equation of the studied problem is the main result of the thesis.

The studied problem of the thesis and obtained results were presented in the following workshop and symposium:

- 6^{th} Workshop of Association for Turkish Women in Maths, 26-28 April, 2019, Selcuk University, Konya, TURKEY.
- Izmir Mathematics Days, 26-27 June, 2018, Yasar University, Izmir, TURKEY.

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APPENDICES

Appendix 1. The Fourier Transform

Let $x = (x_1, x_2, x_3) \in R^3$, $t \in R$ and $\mathcal{F}_{x_1x_2}$ be the operator of the Fourier transform with respect to variables x_1, x_2. This operator defined by the formula (see for example Myint-U & Debnath (2007), Vladimirov (1976), Reed & Simon (1980), Sevimlican (2007))

$$\mathcal{F}_{x_1x_2}[U_j](\nu, x_3, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_j(x,t)e^{i(\nu_1x_1+\nu_2x_2)}dx_1dx_2,\ j = 1, 2, 3,$$

$i^2 = -1$, $\nu = (\nu_1, \nu_2) \in R^2$ is the parameter of Fourier transform.

If $\mathbf{U}(x, t) = (U_1(x, t), U_2(x, t), U_3(x, t))$ is the vector function then we define its Fourier transform $\mathcal{F}_{x_1x_2}[\mathbf{U}](\nu, x_3, t) = \widehat{\mathbf{U}}(\nu, x_3, t)$ as follows

$$\mathcal{F}_{x_1x_2}[\mathbf{U}](\nu, x_3, t) = \big(\mathcal{F}_{x_1x_2}[U_1](\nu, x_3, t), \mathcal{F}_{x_1x_2}[U_2](\nu, x_3, t)\mathcal{F}_{x_1x_2}[U_3](\nu, x_3, t)\big).$$

The operator $\mathcal{F}_{x_1x_2}$ have the following property for $j = 1, 2$,

$$\mathcal{F}_{x_1x_2}[\frac{\partial^n \mathbf{U}}{\partial x_j^n}](\nu, x_3, t) = (-i\nu_j)^n\widehat{\mathbf{U}}(\nu, x_3, t), \quad n = 0, 1, \ldots.$$

Appendix 2. d'Alembert's Formula

Consider the following Cauchy problem for one-dimensional homogenous wave equation Asmar (2005)

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \qquad t > 0, \tag{A.1}$$

$$u(x,0)=\phi(x), \qquad \frac{\partial u}{\partial t}=\psi(x), \qquad -\infty<x<\infty. \tag{A.2}$$

The general solution of (A.1), (A.2) is given by

$$u(x,t)=\frac{1}{2}[\phi(x-ct)+\phi(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)d\xi. \tag{A.3}$$

(A.3) is called d'Alembert's solution (formula). It is named after the mathematician Jean Le Rond d'Alembert who derived it in 1747.

By Duhammel's principle and d'Alembert's formula, the solution of the following nonhomogeneous wave equation (Asmar, 2005)

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}+f(x,t), \quad -\infty<x<\infty, \quad t>0, \tag{A.4}$$

subject to the initial conditions given in (A.2) is given by

$$u(x,t)=\frac{1}{2}[\phi(x-ct)+\phi(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)d\xi+\frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t+\tau)}f(\xi,\tau)d\xi d\tau. \tag{A.5}$$

Appendix 3. Symmetric Matrix, Positive Definiteness and Diagonanalization

The following basic definitions and theorems are written from the linear algebra book (Hoffman & Kunze, 1971).

Definiton. A symmetric matrix is a square matrix that is equal to its transpose.

Theorem. A square matrix **P** is an orthogonal matrix if $\boldsymbol{P}^{-1}=\boldsymbol{P}^T$.

Theorem. If A is a real symmetric matrix, then there exists an orthogonal matrix **P** such that $\boldsymbol{P}^{-1}A\boldsymbol{P}=\boldsymbol{D}$ where **D** is a diagonal matrix.

Definiton. A real symmetric $n\times n$ matrix A is positive definite if $x^T\boldsymbol{A}x>0$ for all $x\in\mathbb{R}^n\backslash\{\mathbf{0}\}$.