DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

NUMERICAL SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS

by
Mustafa Kemal ALTINBAŞ

June, 2023
İZMİR

NUMERICAL SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS

A Thesis Submitted to the
Graduate School of Natural And Applied Sciences of Dokuz Eylül University
In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

by
Mustafa Kemal ALTINBAŞ

June, 2023
İZMİR

M.Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**NUMERICAL SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS**" completed by **MUSTAFA KEMAL ALTINBAŞ** under supervision of **ASSIST. PROF. DR. MELTEM ADIYAMAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

.....

Assist. Prof. Dr. Meltem Adıyaman

Supervisor

.....

Assist. Prof. Dr. Gülter Budakçı

Jury Member

.....

Assoc. Prof. Dr. Ahmet Yantır

Jury Member

Prof. Dr. Okan FISTIKOĞLU
Director
Graduate School of Natural and Applied Sciences

ACKNOWLEDGEMENTS

Words cannot express my gratitude to my supervisor Assist. Prof. Dr. Meltem ADIYAMAN for her invaluable patience, feedback, generously providing knowledge and expertise.

I am also grateful to Assist. Prof. Dr. Gülter BUDAKÇI for sharing her experiences, encouraging me to walk on the academic path with her encouraging words, and her moral support.

I would like to thank Dr. Cem ÇELİK and Dr. Volkan ÖĞER for their patience, technical supports and smiling faces too.

Finally, I would like to express my deepest appreciation to my family for supporting me in every moment of my life.

Mustafa Kemal ALTINBAŞ

NUMERICAL SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS

ABSTRACT

The aim of this thesis is to construct numerical solutions of the second-order integro-differential equations by applying residual method. The residual method is based on the construction of the approximate solution by using the Bézier curves and finding the unknown control points. First, the integral of the product of the Bernstein polynomials with monomials is generalized. After that, residual method is adapted to integro-differential equations. Error analysis is given for mentioned type of equations. Finally, the obtained method is applied to integro-differential equation to demonstrate the behaviour of the numerical solutions by graph and convergency of the method by tables. From graph and tables, it can be seen that the proposed method is efficient, applicable and has good accuracy.

Keywords: Integro-differential equations, Fredholm integro-differential equations, residual method, Bézier curves, Bernstein polynomials.

İNTEGRO-DİFERANSİYEL DENKLEMLERİN SAYISAL ÇÖZÜMLERİ

ÖZ

Bu tezin amacı, kalıntı yöntemi uygulayarak ikinci mertebeden integro-diferansiyel denklemlerin sayısal çözümlerini oluşturmaktır. Kalıntı yöntemi, Bézier eğrilerini kullanarak ve bilinmeyen kontrol noktalarını bularak yaklaşık çözümün oluşturulmasına dayanır. İlk olarak, Bernstein polinomlarının tek terimli polinomlarla çarpımının integrali genelleştirilir. Ardından, kalıntı yöntemi integro-diferansiyel denklemlere uyarlanır. Belirtilen tipteki denklemler için hata analizi verilir. Son olarak, sayısal çözümlerin grafikle davranışını ve yöntemin tablolarla yakınsamasını göstermek için elde edilen yöntem integro-diferansiyel denklemlere uygulanır. Grafik ve tablolardan, önerilen yöntemin verimli, uygulanabilir ve iyi bir doğruluğa sahip olduğu görülebilir.

Anahtar kelimeler: İntegro-diferansiyel denklemler, Fredholm integro-diferansiyel denklemler, kalıntı metodu, Bézier eğrileri, Bernstein polinomları.

CONTENTS

Page

M.Sc. THESIS EXAMINATION RESULT FORM ii
ACKNOWLEDGEMENTS iii
ABSTRACT iv
ÖZ v
LIST OF FIGURES vii
LIST OF TABLES viii

CHAPTER ONE – INTRODUCTION 1

CHAPTER TWO – ADAPTATION OF RESIDUAL METHOD TO INTEGRO-DIFFERENTIAL EQUATIONS 3

CHAPTER THREE – ERROR ANALYSIS 11

3.1 Algorithm of Method 13

CHAPTER FOUR – NUMERICAL EXAMPLES 15

CHAPTER FIVE – CONCLUSION 18

REFERENCES 19

APPENDICES 21

A.1 Properties of Bézier Curves 21
A.2 Residual Method 23

LIST OF FIGURES

Page

Figure 4.1 Graph of error functions of Example (4.2) for $n = 10, 13, 15$. 17

LIST OF TABLES

Page

Table 4.1 Exact values and the errors obtained by residual method for $n = 2$ of Example (4.1) 16

Table 4.2 Comparison of the exact values with the errors of the residual method for $n = 7, 10, 13$ and errors given in Darania & Ebadian (2007). 17

CHAPTER ONE
INTRODUCTION

In this thesis, we consider the initial value problem for the second-order integro-differential equation

$$y''(x) = F(x, y(x), y'(x)) + \int_0^1 x\, t^m\, y(t)dt \tag{1.1}$$

with initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{1.2}$$

where $F \in C^{n-1}([0,1] \times D_1 \times D_2)$, D_1 and D_2 are closed intervals in $\mathbb{R}$, α, β are finite constants, $m \in \mathbb{N}$ and n is the degree of approximate Bézier curves. This equation is also known as the second order Fredholm integro-differential equation.

The theory of integro-differential equations revealed when Volterra worked on elasticity. This equation includes the unknown function, its integral and its derivatives. Many researchers, including the famous scientists like Abel, Lotka, Fredholm, Malthus, Verhulst, interested in integro-differential equations after Volterra. There are many areas of usage of this modelling such as engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics. The most popular scientific studies in these fields using integro-differential equations are El-Gendi (1969), Medlock & Kot (2003), El-Shahed (2005), Arikoglu & Ozkol (2005) and Caffarelli & Silvestre (2009). Interest in the integro-differential equations has continued in recent years. Baleanu et al. (2020), Hajiseyedazizi et al. (2021) and Karthikeyan et al. (2022) are just some of the newest articles which are published to find existence of solution of the mentioned problems.

In order to find approximate solution of integro-differential equations, we apply residual method which is developed to find approximate solution of initial value problems. After initial value problems, with some changes, in chronological order, this method is applied to boundary value problems such as Troesch's problem (Oger, 2016), singular nonlinear equations of Lane-Emden type (Adiyaman & Oger, 2017),

nonlinear system of initial value problems (Adiyaman & Noyan, 2020) and non-linear eigenvalue problems such as Bratu problem (Adiyaman & Beler, 2022). The proposed method is based on the construction of the approximate solution by the linear combination of the Bernstein polynomials. The most important advantage of this method is finding unknown control points without need for any linearization. The unknown control points are found by solving linear system of equations.

In Chapter 2, residual method is adapted for the second-order integro-differential equations. In Chapter 3, error analysis is given. Two examples are illustrated to show the accuracy of the method and obtained results are compared with the another technique in Chapter 4. In the conclusion, we summarize our study and inform about our future studies.

CHAPTER TWO
ADAPTATION OF RESIDUAL METHOD TO INTEGRO-DIFFERENTIAL EQUATIONS

Consider the initial value problem for the second-order integro-differential equation

$$y''(x) = F(x, y(x), y'(x)) + \int_0^1 x\, t^m\, y(t)dt \tag{2.1}$$

with initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{2.2}$$

where $F \in C^{n-1}([0,1] \times D_1 \times D_2)$, D_1 and D_2 are closed intervals in $\mathbb{R}$, α, β are finite constants, $m \in \mathbb{N}$ and n is the degree of approximate Bézier curves. This equation is also known as the second order Fredholm integro-differential equation.

Residual method is constructed to approximate the solution of (2.1) and (2.2) using Bézier curves over $[0,1]$. For this, first, we define the nth degree Bézier curve as follows:

$$u(x) = \sum_{i=0}^{n} c_i B_i^n(x), \quad x \in [0,1], \tag{2.3}$$

where

$$B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}$$

is the Bernstein polynomial over the interval $[0,1]$ and c_i are unknown control points. Clearly, there are $n+1$ unknown control points for $u(x)$. Since the approximate solution must satisfy the initial conditions (2.2), we have

$$u(0) = \alpha, \quad u'(0) = \beta. \tag{2.4}$$

Then, from the end point property (A.3) and the end point derivative property (A.5) of the Bézier curves, we have first two control points of $u(x)$ for $a = 0$ and $b = 1$ such

that

$$\begin{aligned} c_0 &= \alpha, \\ c_1 &= \frac{\beta}{n} + \alpha. \end{aligned} \tag{2.5}$$

The other $n-1$ control points have to be found in order to obtain the approximate solution $u(x)$.

Now, we will use the same strategy given in Oger (2016); Adiyaman & Oger (2017). Putting (2.3) into integro-differential equation (2.1), we can define the residual function as

$$R(x) = u''(x) - F(x, u(x), u'(x)) - \int_0^1 x\, t^m\, u(t)dt, \tag{2.6}$$

where $F(x, u(x), u'(x))$ is a sufficiently differentiable function with respect to x.

In order to minimize the sufficiently differentiable residual function $R(x)$, we find unknown control points by forcing the first $n-1$ terms in Taylor's series expansion of $R(x)$ to be zero, that is,

$$\begin{aligned} R(0) &= 0, \\ R'(0) &= 0, \\ R''(0) &= 0, \\ &\vdots \\ R^{(n-2)}(0) &= 0. \end{aligned} \tag{2.7}$$

It follows from (2.6) that

$$u''(x) - F(x, u(x), u'(x)) - \int_0^1 x\, t^m\, u(t)dt = 0. \tag{2.8}$$

Taking derivatives of (2.8) with respect to x and combining with (2.7), we obtain

$$R^{(k)}(x) = u^{(k+2)}(x) - F^{(k)}(x, u(x), u'(x)) - \frac{d^k}{dx^k}\left(\int_0^1 x\, t^m\, u(t)dt\right) = 0, \tag{2.9}$$

where $F^{(k)}(x, y(x), z(x)) = \dfrac{\partial^k}{\partial x^k} F(x, y(x), z(x))$ for $k = 0, 1, \ldots, n-2$.

Now, we will give some lemmas to calculate the unknown control points by using the definition of Bernstein polynomials in $[a, b]$, $\quad a, b \in \mathbb{R}$.

Lemma 2.1. *For all $n, k \in \mathbb{N}$,*

$$\sum_{j=i+1}^{n+1} \sum_{s=j+k}^{n+1+k} \binom{s-1-j}{k-1} a_s = \sum_{s=i+k+1}^{n+k+1} \binom{s-1-i}{k} a_s,$$

where $i = 0, 1, \ldots, n$.

Proof. Let's start with changing the order of the summations

$$\begin{aligned} \sum_{j=i+1}^{n+1} \sum_{s=j+k}^{n+1+k} \binom{s-1-j}{k-1} a_s &= \sum_{s=i+1+k}^{n+1+k} \sum_{j=i+1}^{s-k} \binom{s-1-j}{k-1} a_s \\ &= \sum_{s=i+1+k}^{n+1+k} a_s \sum_{j=i+1}^{s-k} \binom{s-1-j}{k-1}. \end{aligned}$$

Shifting index j from $i+1$ to 0, we get

$$\sum_{j=i+1}^{n+1} \sum_{s=j+k}^{n+1+k} \binom{s-1-j}{k-1} a_s = \sum_{s=i+1+k}^{n+1+k} a_s \sum_{j=0}^{s-k-i-1} \binom{s-i-2-j}{k-1}.$$

Let $r = s-i-2-j$, then for $j = 0$ and $j = s-k-i-1$ we have $r = s-i-2$ and $r = k-1$, respectively, which implies

$$\begin{aligned} \sum_{j=i+1}^{n+1} \sum_{s=j+k}^{n+1+k} \binom{s-1-j}{k-1} a_s &= \sum_{s=i+1+k}^{n+1+k} a_s \sum_{r=k-1}^{s-i-2} \binom{r}{k-1} \\ &= \sum_{s=i+1+k}^{n+1+k} a_s \left(\sum_{r=0}^{s-i-2} \binom{r}{k-1} - \sum_{r=0}^{k-2} \binom{r}{k-1} \right) \\ &= \sum_{s=i+1+k}^{n+1+k} a_s \sum_{r=0}^{s-i-2} \binom{r}{k-1} \end{aligned}$$

Since for $c \geq 0$,

$$\sum_{k=0}^{n} \binom{k}{c} = \binom{n+1}{c+1}$$

which is the sum of the entries in column c of Pascal's triangle from row 0 down to row n (is given in Gross (2016)), we have

$$\sum_{s=i+1+k}^{n+1+k} a_s \sum_{r=0}^{s-i-2} \binom{r}{k-1} = \sum_{s=i+1+k}^{n+1+k} \binom{s-1-i}{k} a_s.$$

Hence the following identity is proved

$$\sum_{j=i+1}^{n+1} \sum_{s=j+k}^{n+1+k} \binom{s-1-j}{k-1} a_s = \sum_{s=i+1+k}^{n+1+k} \binom{s-1-i}{k} a_s.$$

□

Lemma 2.2. *For all non-negative integers m and n,*

$$\begin{aligned} \int t^m B_i^n(t;[a,b])dt = \sum_{k=1}^{m+1} \Bigg(& \frac{(-1)^{k-1}(b-a)^k m! n! t^{m-k+1}}{(m-k+1)!(n+k)!} \\ & \times \sum_{j=i+k}^{n+k} \binom{j-1-i}{k-1} B_j^{n+k}(t;[a,b]) \Bigg). \end{aligned} \tag{2.10}$$

for $i = 0, 1, \ldots, n$.

Proof. We will prove this lemma by principle of mathematical induction on m.

For $m = 0$, from Appendix (A.7), obviously, the given equation is hold for all $n \in \mathbb{N}$ and $i = 0, 1, \ldots, n$, that is,

$$\begin{aligned} \int B_i^n(t;[a,b])dt &= \frac{b-a}{n+1} \sum_{j=i+1}^{n+1} B_j^{n+1}(t;[a,b]) \\ &= \frac{(b-a)\, n!}{(n+1)!} \sum_{j=i+1}^{n+1} \binom{j-1-i}{0} B_j^{n+1}(t;[a,b]). \end{aligned}$$

Assume that (2.10) is true for $m-1$, $\forall n \in \mathbb{N}$, $i = 0, 1, \ldots, n$, i.e.,

$$\int t^{m-1} B_i^n(t;[a,b])dt = \sum_{k=1}^{m} \left(\frac{(-1)^{k-1}(b-a)^k (m-1)! n! t^{m-k}}{(m-k)!(n+k)!} \times \sum_{j=i+k}^{n+k} \binom{j-1-i}{k-1} B_j^{n+k}(t;[a,b]) \right).$$

Now, we will show that given statement is true for m, $\forall n \in \mathbb{N}$, $i = 0, 1, \ldots, n$. Then, using integration by parts, we get

$$\int t^m B_i^n(t;[a,b])dt = t^m \frac{b-a}{n+1} \sum_{j=i+1}^{n+1} B_j^{n+1}(t;[a,b]) - \int \frac{b-a}{n+1} \sum_{j=i+1}^{n+1} B_j^{n+1}(t;[a,b]) m t^{m-1} dt.$$

Rearranging the terms in the last equation gives

$$\int t^m B_i^n(t;[a,b])dt = t^m \frac{b-a}{n+1} \sum_{j=i+1}^{n+1} B_j^{n+1}(t;[a,b]) - \frac{m(b-a)}{n+1} \sum_{j=i+1}^{n+1} \int t^{m-1} B_j^{n+1}(t;[a,b])dt.$$

By induction hypothesis, we can write

$$\begin{aligned}
\int t^m B_i^n(t;[a,b])dt =& \frac{t^m(b-a)}{n+1} \sum_{j=i+1}^{n+1} B_j^{n+1}(t;[a,b]) \\
& - \frac{m(b-a)}{n+1} \sum_{j=i+1}^{n+1} \left(\sum_{k=1}^{m} \frac{(-1)^{k-1}(b-a)^k (m-1)!(n+1)! t^{m-k}}{(m-k)!(n+1+k)!} \right. \\
& \left. \times \sum_{s=j+k}^{n+1+k} \binom{s-1-j}{k-1} B_s^{n+1+k}(t;[a,b]) \right) \\
=& \sum_{j=i+1}^{n+1} \left(\frac{t^m(b-a)}{n+1} B_j^{n+1}(t;[a,b]) \right) \\
& - \frac{m(b-a)}{n+1} \sum_{k=1}^{m} \left(\frac{(-1)^{k-1}(b-a)^k (m-1)!(n+1)! t^{m-k}}{(m-k)!(n+1+k)!} \right. \\
& \left. \times \sum_{j=i+1}^{n+1} \sum_{s=j+k}^{n+1+k} \binom{s-1-j}{k-1} B_s^{n+1+k}(t;[a,b]) \right).
\end{aligned}$$

Using Lemma (2.1), we obtain

$$\int t^m B_i^n(t;[a,b])\,dt = \sum_{j=i+1}^{n+1}\left(\frac{t^m(b-a)}{n+1}B_j^{n+1}(t;[a,b])\right) - \frac{m(b-a)}{n+1}\sum_{k=1}^{m}\left(\frac{(-1)^{k-1}(b-a)^k(m-1)!(n+1)!t^{m-k}}{(m-k)!(n+1+k)!} \times \sum_{s=i+k+1}^{n+k+1}\binom{s-1-i}{k}B_s^{n+k+1}(t;[a,b])\right).$$

After some simplifications and shifting index k from 1 to 2, we have

$$\int t^m B_i^n(t;[a,b])\,dt = \sum_{j=i+1}^{n+1}\left(\frac{t^m(b-a)}{n+1}B_j^{n+1}(t;[a,b])\right) + \sum_{k=2}^{m+1}\left(\frac{(-1)^{k-1}(b-a)^k m!n!t^{m-k+1}}{(m-k+1)!(n+k)!} \times \sum_{s=i+k}^{n+k}\binom{s-1-i}{k-1}B_s^{n+k}(t;[a,b])\right).$$

Finally, we arrive the following equation which completes the proof

$$\int t^m B_i^n(t;[a,b])\,dt = \sum_{k=1}^{m+1}\left(\frac{(-1)^{k-1}(b-a)^k m!n!t^{m-k+1}}{(m-k+1)!(n+k)!} \times \sum_{s=i+k}^{n+k}\binom{s-1-i}{k-1}B_s^{n+k}(t;[a,b])\right).$$

□

Lemma (2.2) yields the following corollary.

Corollary 2.3. *For all $m,n\in\mathbb{N}$ and $i=0,1,\ldots,n$,*

$$\int_a^b t^m B_i^n(t;[a,b])dt = \frac{b^m(b-a)}{n+1}{}_2F_1\left(-m,n-i+1;n+2;1-\frac{a}{b}\right), \qquad (2.11)$$

where ${}_2F_1\left(-m,n-i+1;n+2;1-\frac{a}{b}\right) = \sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(n-i+1)_k}{(n+2)_k}\left(1-\frac{a}{b}\right)^k$ *is hypergeometric function and* $(a)_k = \frac{(a+k-1)!}{(a-1)!}$.

Now, the obtained results are substituted in (2.8) at $x = 0$ so that we have the following linear equation

$$\frac{n!}{(n-2)!}(c_2 - 2c_1 + c_0) - F(0, c_0, n(c_1 - c_0)) = 0. \tag{2.12}$$

Then, for $k = 1$, (2.9) gives

$$\frac{n!}{(n-3)!}(c_3 - 3c_2 + 3c_1 - c_0) - F'(0, c_0, n(c_1 - c_0)) - \sum_{i=0}^{n} c_i \sum_{p=1}^{m+1} \omega_{p,i} \; = 0, \tag{2.13}$$

where $\omega_{p,i} = \dfrac{(-1)^{p-1}\, m!\, n!}{(m-p+1)!(n+p)!}\binom{n+p-i-1}{p-1}$.
The others for $k = 2, 3, \ldots, n-2$ in (2.9) become

$$\frac{n!}{(n-(k+2))!}\Delta^{k+2}c_0 - F^{(k)}(0, c_0, n(c_1 - c_0)) = 0, \tag{2.14}$$

where $F^{(k)}(x, y(x), z(x)) = \dfrac{\partial^k}{\partial x^k}F(x, y(x), z(x))$ and

$$\Delta^{k+2}c_0 = \sum_{i=0}^{k+2}\binom{k+2}{i}(-1)^{k+2-i}c_i.$$

From equations (2.12), (2.13) and (2.14), we obtain the following system

$$\begin{bmatrix} \frac{n!}{(n-2)!} & 0 & \cdots & \cdots & \cdots & 0 \\ -\frac{n!\,3}{(n-3)!} - \sum_{p=1}^{m+1}\omega_{p,2} & \frac{n!}{(n-3)!} - \sum_{p=1}^{m+1}\omega_{p,3} & -\sum_{p=1}^{m+1}\omega_{p,4} & \cdots & \cdots & -\sum_{p=1}^{m+1}\omega_{p,n} \\ \frac{n!\,6}{(n-4)!} & -\frac{n!\,4}{(n-4)!} & \frac{n!}{(n-4)!} & 0 & \cdots & 0 \\ -\frac{n!\,10}{(n-5)!} & \frac{n!\,10}{(n-5)!} & -\frac{n!\,5}{(n-5)!} & -\frac{n!}{(n-5)!} & \ddots & \vdots \\ \vdots & & & & \ddots & 0 \\ (-1)^{n-2}\, n!\, \binom{n}{n-2} & (-1)^{n-3}\, n!\, \binom{n}{n-3} & \cdots & \cdots & \cdots & n! \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \\ c_4 \\ c_5 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} g_2 \\ g_3 \\ g_4 \\ g_5 \\ \vdots \\ g_n \end{bmatrix},$$

where

$$
\begin{aligned}
g_i =& c_0 \left(\frac{(-1)^{i-1}\, n!}{(n-i)!} + \sigma_i \sum_{k=1}^{m+1} \frac{(-1)^{k-1}\, m!\, n!}{(m-k+1)!\, (n+k)!} \binom{n+k-1}{k-1} \right) \\
&+ c_1 \left(\frac{(-1)^{i}\, n!}{(n-i)!} \binom{i}{1} + \sigma_i \sum_{k=1}^{m+1} \frac{(-1)^{k-1}\, m!\, n!}{(m-k+1)!\, (n+k)!} \binom{n+k-2}{k-1} \right) \\
&+ F^{(i-2)}\left(0, c_0, n(c_1 - c_0)\right), \quad \sigma_i = \begin{cases} 1, & i = 2 \\ 0, & i = 3, \ldots, n \end{cases},
\end{aligned}
\tag{2.15}
$$

$i = 2, 3, \ldots, n.$

Then, solving above system, we can determine unknown control points $c_2, c_3, \ldots, c_n$. So we have both the approximate solution $u(x)$ and minimal $R(x)$ in $[0, 1]$.

CHAPTER THREE
ERROR ANALYSIS

Lemma 3.1. *Let $y(x)$ be the exact solution and $u(x)$ be the approximation solution defined in (2.3) with conditions (2.4) of (2.1) with (2.2). Then, we get*

$$\left|y^{(k+2)}(0) - u^{(k+2)}(0)\right| = \begin{cases} \frac{1}{(m+1)!}\left|y(\xi) - u(\xi)\right|, & k = 1 \\ 0, & otherwise, \end{cases} \tag{3.1}$$

where $k = 0, 1, \ldots, n-2$ and $0 < \xi < x \leqslant 1$.

Proof. From (2.2) and (2.4), we have

$$u(0) = y(0) \quad \text{and} \quad u'(0) = y'(0). \tag{3.2}$$

Using (3.2), we rewrite (2.9) at $x = 0$ as

$$u^{(k+2)}(0) - F^{(k)}(0, y(0), y'(0)) - \frac{d^k}{dx^k}\left(\int_0^1 x\, t^m\, u(t)dt\right)\Big|_{x=0} = 0 \tag{3.3}$$

From (2.1), we get

$$y^{(k+2)}(0) - F^{(k)}(0, y(0), y'(0)) - \frac{d^k}{dx^k}\left(\int_0^1 x\, t^m\, y(t)dt\right)\Big|_{x=0} = 0. \tag{3.4}$$

Substituting (3.4) into (3.3) yields

$$u^{(k+2)}(0) - y^{(k+2)}(0) + \frac{d^k}{dx^k}\left(\int_0^1 x\, t^m\, y(t)dt\right)\Big|_{x=0} - \frac{d^k}{dx^k}\left(\int_0^1 x\, t^m\, u(t)dt\right)\Big|_{x=0} = 0$$

Calculating the above equation, we obtain

$$y^{(k+2)}(0) - u^{(k+2)}(0) = \begin{cases} \int_0^1 t^m\,(y(t) - u(t))dt, & k = 1 \\ 0, & otherwise \end{cases}$$

for $k = 0, 1, \ldots, n-2$.

From the Mean Value Theorem for Integrals, we have

$$\begin{aligned}\int_0^1 t^m \left(y(t)-u(t)\right)dt &= \left(y(\xi)-u(\xi)\right)\int_0^1 t^m dt \\ &= \frac{1}{(m+1)!}(y(\xi)-u(\xi)), \quad 0<\xi<x\leq 1,\end{aligned}$$

which gives

$$y^{(k+2)}(0)-u^{(k+2)}(0)=\begin{cases}\frac{1}{(m+1)!}\left(y(\xi)-u(\xi)\right), & k=1\\ 0, & otherwise,\end{cases}$$

as desired, clearly. □

Lemma 3.2. *Let $y(x)$ be the exact solution of (2.1) with (2.2) and $u(x)$ be the corresponding nth degree approximate function defined in (2.3) with conditions (2.4). Then, we have the following upper bound of the error*

$$|y(x)-u(x)|\leq \frac{C}{(n+1)!}, \tag{3.5}$$

where

$$C=\frac{1}{\left(6\ (m+1)!-1\right)}\ \max_{0\leq x\leq 1}\left|y^{n+1}(x)\right|. \tag{3.6}$$

Proof. First, consider the Taylor's series expansion of $y(\xi)-u(\xi)$ around the point $x=0$:

$$\begin{aligned}y(\xi)-u(\xi) =&(y(0)-u(0))+\xi(y'(0)-u'(0))+\frac{\xi^2}{2!}(y''(0)-u''(0))\\ &+\frac{\xi^3}{3!}(y'''(0)-u'''(0))+\cdots+\frac{\xi^{n+1}}{(n+1)!}(y^{(n+1)}(\tau)-0),\end{aligned}$$

where $0<\tau<\xi<1$. After taking absolute values of both sides and using Lemma(3.2), we have

$$|y(\xi)-u(\xi)|=\left|\frac{\xi^3}{3!\ (m+1)!}(y(\xi)-u(\xi))+\frac{\xi^{n+1}}{(n+1)!}y^{(n+1)}(\tau)\right|$$

which gives

$$\left(1-\frac{\xi^3}{3!\ (m+1)!}\right)|y(\xi)-u(\xi)| \leq \left|\frac{\xi^{n+1}}{(n+1)!}y^{(n+1)}(\tau)\right|$$

by triangle inequality. From (3.6), since $\xi \leq 1$, we obtain

$$|y(\xi)-u(\xi)| \leq \frac{3!\ (m+1)!}{(3!\ (m+1)!-1)\ (n+1)!}\ \max_{0\leq x\leq 1}\left|y^{n+1}(x)\right|. \tag{3.7}$$

Now using the Taylor's formula around the point $x=0$ for $y(x)-u(x)$, we get

$$\begin{aligned} y(x)-u(x) =&(y(0)-u(0))+x(y'(0)-u'(0))+\frac{x^2}{2!}(y''(0)-u''(0)) \\ &+\frac{x^3}{3!}(y'''(0)-u'''(0))+\ldots. \end{aligned}$$

After taking absolute value of both sides and using Lemma(3.2), we have

$$\begin{aligned} |y(x)-u(x)| =&\frac{|x|^3}{3!}\ |y'''(0)-u'''(0)| \\ =&\left|\frac{x^3}{3!\ (m+1)!}(y(\xi)-u(\xi))\right|. \end{aligned}$$

Since $x \leq 1$, we obtain

$$|y(x)-u(x)| \leq \frac{C}{(n+1)!},$$

where $C=\dfrac{1}{(6\ (m+1)!-1)}\ \max\limits_{0\leq x\leq 1}\left|y^{n+1}(x)\right|$ □

3.1 Algorithm of Method

The algorithm of the method can be written as

S 1 Set n and m.

S 2 Define the function

$$f_k(x,y,z) := F^{(k)}(x,y,z), \quad k=0,1,\ldots,n-2$$

and prepare

$$\omega_{p,i} = \frac{(-1)^{p-1}\, m!\, n!}{(m-p+1)!\, (n+p)!} \binom{n+p-i-1}{p-1}, \quad i = 0, 1, \ldots, n, \;\; p = 1, \ldots, m+1.$$

S 3 Set $c_0 = \alpha$, $c_1 = \frac{\beta}{n} + \alpha$.

S 4 Solve c_2, from

$$\frac{n!}{(n-2)!}\Delta^2 c_0 - f_0\left(0, c_0, n(c_1 - c_0)\right) = 0,$$

where Δ is difference operator.

S 5 Solve c_3, from

$$\frac{n!}{(n-3)!}\Delta^3 c_0 - f_1\left(0, c_0, n(c_1 - c_0)\right) - \sum_{i=0}^{n} c_i \sum_{p=1}^{m+1} \omega_{p,i} = 0.$$

S 6 Solve c_k from

$$\frac{n!}{(n-k)!}\Delta^k c_0 - f_{k-2}\left(0, c_0, n(c_1 - c_0)\right) = 0, \quad k = 4, 5, \ldots, n$$

S 7 Using computed control points c_i for $i = 0, 1, \ldots, n$, write the approximate solution as

$$u(x) = \sum_{i=0}^{n} c_i B_i^n(x),$$

where

$$B_i^n(x) = \binom{n}{i}(x)^i(1-x)^{n-i},$$

CHAPTER FOUR
NUMERICAL EXAMPLES

In this chapter, we illustrate our method using the following examples. Example 4.1 is the test example. Since our method is based on polynomial approximation, it is expected that when it is applied to a differential equation whose solution is a polynomial, the exact solution has to be obtained by using the proposed method if both polynomials have the same degree. This expectation has been proven in error analysis. In order to see this in the numerical results, the test problem Example 4.1 is considered. Comparison of the exact and numerical results are given in Table 4.1. Table 4.2 given in Example 4.2 demonstrates the exact results, errors of the obtained approximate solutions applying residual method for $n = 7, 10, 13$ and comparison of the errors obtained using the differential transformation method given in Darania & Ebadian (2007). The graph of error functions are presented in Figure 4.1.

Example 4.1. Consider the second-order integro differential equation

$$y''(x) = 2 - \frac{x}{4} + \int_0^1 x\, t\, y(t)dt \tag{4.1}$$

$$y(0) = 0, \quad y'(0) = 0,$$

with the exact solution $y(x) = x^2$, $x \in [0, 1]$. Table 4.1 shows exact values and errors of the proposed method at $x = 0, 0.1, \ldots, 1$, for $n = 2$.

Table 4.1 Exact values and the errors obtained by residual method for $n = 2$ of Example (4.1)

x	$Exact$	$n = 2$
0	0	0
0.1	0.01	0
0.2	0.04	0
0.3	0.09	0
0.4	0.16	0
0.5	0.25	0
0.6	0.36	0
0.7	0.49	0
0.8	0.64	0
0.9	0.81	0
1	1	0

Example 4.2. Consider the second-order linear Fredholm IDE

$$y''(x) = e^x - x + \int_0^1 x\ t\ y(t)dt \tag{4.2}$$

$$y(0) = 1, \quad y'(0) = 1,$$

with the exact solution $y(x) = e^x$, $x \in [0, 1]$. Table 4.2 gives the comparison of the exact solutions of Example (4.2) with the errors obtained by proposed for $n = 7, 10, 13$ and errors given in Darania & Ebadian (2007) at $x = 0, 0.1, \ldots, 1$. It can be easily seen that residual method gets better results than the results given in Darania & Ebadian (2007), and gets much closer ones to the exact solution as n gets larger. Figure 4.1 shows error functions of the method for $n = 7, 10, 13$.

Table 4.2 Comparison of the exact values with the errors of the residual method for $n = 7, 10, 13$ and errors given in Darania & Ebadian (2007).

x	$Exact$	$n = 7$	$n = 10$	$n = 13$	Darania & Ebadian (2007)
0	1	0	0	0	0
0.1	1.10517	4.75377 (-10)	3.59268 (-13)	1.11022 (-15)	0.00000 (-9)
0.2	1.22140	3.86594 (-09)	2.87770 (-12)	5.32907 (-15)	4.63105 (-4)
0.3	1.34986	1.45116 (-08)	9.76064 (-12)	7.32747 (-15)	1.42980 (-3)
0.4	1.49182	4.74144 (-08)	2.41140 (-11)	7.10543 (-15)	2.94384 (-3)
0.5	1.64872	1.61936 (-07)	5.77369 (-11)	2.59792 (-14)	5.05241 (-3)
0.6	1.82212	5.48732 (-07)	1.73366 (-10)	1.82212 (-13)	7.80640 (-3)
0.7	2.01375	1.71224 (-06)	6.49301 (-10)	6.25588 (-12)	1.12606 (-2)
0.8	2.22554	4.80604 (-06)	2.48899 (-09)	2.01363 (-11)	1.54740 (-2)
0.9	2.45960	1.21948 (-05)	8.75712 (-09)	5.46048 (-11)	2.05100 (-2)
1	2.71828	2.83353 (-05)	2.76724 (-08)	3.07966 (-10)	2.64366 (-2)

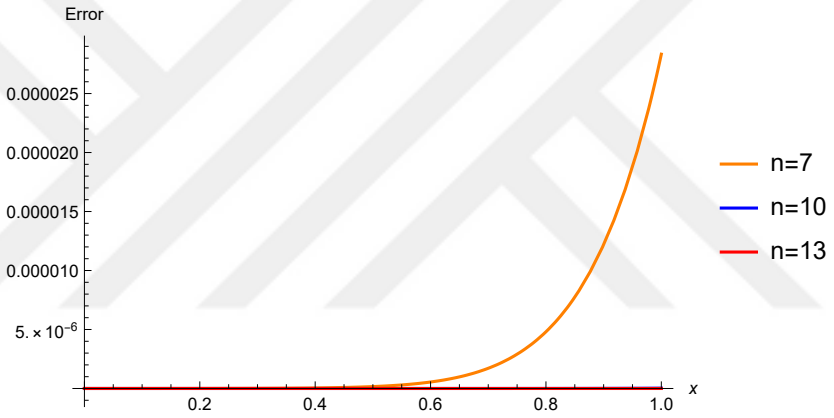


Figure 4.1 Graph of error functions of Example (4.2) for $n = 10, 13, 15$.

CHAPTER FIVE
CONCLUSION

In this work, we applied residual method to integro-differential equations. In order to do this we get formulas for the definite and indefinite integrals of the multiplication of monomials and Bernstein polynomials. Formulas are used to find the approximate solutions. Obtained numerical results are given in tables and figures. From the tables and figures, it can be seen that the accuracy of the approximation is good. We can conclude that this method is efficient and strong for this type of problems. We aim to improve our method to find approximate solutions of nth order initial value problems.

REFERENCES

Adiyaman, M., & Beler, A. (2022). Improved residual method for approximating bratu problem. *Computational Methods for Differential Equations*.

Adiyaman, M., & Noyan, B. (2020). Residual method for nonlinear system of initial value problems. *Computational Methods for Differential Equations*, *8*(4), 733–744.

Adiyaman, M., & Oger, V. (2017). A residual method using bezier curves for singular nonlinear equations of lane-emden type. *Kuwait Journal of Science*, *44*(4).

Arikoglu, A., & Ozkol, I. (2005). Solution of boundary value problems for integro-differential equations by using differential transform method. *Applied Mathematics and Computation*, *168*(2), 1145–1158.

Baleanu, D., Etemad, S., & Rezapour, S. (2020). On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators. *Alexandria Engineering Journal*, *59*(5), 3019–3027.

Caffarelli, L., & Silvestre, L. (2009). Regularity theory for fully nonlinear integro-differential equations. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, *62*(5), 597–638.

Darania, P., & Ebadian, A. (2007). A method for the numerical solution of the integro-differential equations. *Applied Mathematics and Computation*, *188*(1), 657–668.

Doha, E. H., Bhrawy, A. H., & Saker, M. (2011). Integrals of bernstein polynomials: an application for the solution of high even-order differential equations. *Applied Mathematics Letters*, *24*(4), 559–565.

El-Gendi, S. (1969). Chebyshev solution of differential, integral and integro-differential equations. *The Computer Journal*, *12*(3), 282–287.

El-Shahed, M. (2005). Application of he's homotopy perturbation method to volterra's integro-differential equation. *International Journal of Nonlinear Sciences and Numerical Simulation*, *6*(2), 163–168.

Gross, J. L. (2016). *Combinatorial methods with computer applications*. CRC Press.

Hajiseyedazizi, S. N., Samei, M. E., Alzabut, J., & Chu, Y.-m. (2021). On multi-step methods for singular fractional q-integro-differential equations. *Open Mathematics*, *19*(1), 1378–1405.

Joy, K. I. (2000). Bernstein polynomials. *On-Line Geometric Modeling Notes*, *13*(5).

Karthikeyan, K., Karthikeyan, P., Baskonus, H. M., Venkatachalam, K., & Chu, Y.-M. (2022). Almost sectorial operators on ψ-hilfer derivative fractional impulsive integro-differential equations. *Mathematical Methods in the Applied Sciences*, *45*(13), 8045–8059.

Medlock, J., & Kot, M. (2003). Spreading disease: integro-differential equations old and new. *Mathematical biosciences*, *184*(2), 201–222.

Oger, V. (2016). *Numerical solutions of differential equations using residual method* (Publication No. 10111172) [Doctoral dissertation, Dokuz Eylül University]. Dokuz Eylül University Graduate School of Natural and Apllied Sciences. http://panel.fbe.deu.edu.tr/ArchieveDetail.aspx?Yayin=&Type=Thesis&Ogrenci=2003900436.

APPENDICES

A.1 Properties of Bézier Curves

In this part, we will introduce some properties of Bézier curves.

Bézier curves are defined with Bernstein polynomials, that is, any nth degree Bézier curves can be expressed over $[a, b]$ as

$$u(x) = \sum_{i=0}^{n} c_i B_i^n(x; [a, b]), \tag{A.1}$$

where

$$B_i^n(x; [a, b]) = \binom{n}{i} \frac{1}{(b-a)^n} (x-a)^i (b-x)^{n-i}$$

are Bernstein polynomials in the interval $[a, b]$. Bézier curves, which are used in this thesis, is known as the classical Bézier curve which is

$$u(x) = \sum_{i=0}^{n} c_i B_i^n(x), \tag{A.2}$$

where

$$B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}.$$

Bézier curves usually does not interpolate to its all control points. However, it always interpolates to their the first and last control points. This property is called the end point interpolation which is

$$u(a) = c_0 \quad \text{and} \quad u(b) = c_n. \tag{A.3}$$

Now, we give one of the most important properties of Bernstein polynomials. The first derivative of Bernstein polynomials can be defined in terms of two Bernstein

polynomials having less degree. So, for all $n \in \mathbb{N}$ and $i = 0, 1, \ldots, n$,

$$\frac{d}{dt}\left(B_i^n(t;[a,b])\right) = \frac{n}{b-a}\left(B_{i-1}^{n-1}(t;[a,b]) - B_i^{n-1}(t;[a,b])\right) \tag{A.4}$$

which one can find the proof of above lemma in Joy (2000) for $t \in [0,1]$ and by the similar way, it can be stretched over $[a,b]$.

The control points play an important role in the calculation of many operations related to Bézier curves. One of them is the derivation of (A.1) at the end points, which can be expressed in terms of the control points, that is,

$$\left.\frac{d^k}{dx^k}u(x)\right|_{x=a} = \frac{1}{(b-a)^k}\frac{n!}{(n-k)!}\Delta^k c_0 \tag{A.5}$$

and

$$\left.\frac{d^k}{dx^k}u(x)\right|_{x=b} = \frac{1}{(b-a)^k}\frac{n!}{(n-k)!}\Delta^k c_{n-k} \tag{A.6}$$

where Δ is difference operator such that $\Delta c_j = c_{j+1} - c_j$. Above formulas help us to connect Bézier segments for continuity C^k.

The next properties show that indefinite integral of nth degree Bernstein polynomials can be expressed as a sum of $(n+1)$st degree Bernstein polynomials in two ways:

$$\int B_i^n(t;[a,b])dt = \frac{b-a}{n+1}\sum_{j=i+1}^{n+1} B_j^{n+1}(t;[a,b]) \tag{A.7}$$

and

$$\int B_i^n(t;[a,b])dt = -\frac{b-a}{n+1}\sum_{j=0}^{i} B_j^{n+1}(t;[a,b]).$$

Then, the definite integral is given for all $n \in \mathbb{N}$, as

$$\int_a^b B_i^n(t;[a,b]) = \frac{b-a}{n+1},$$

where $a, b \in \mathbb{R}$. These integral properties of Bernstein polynomials is stated in Doha et al. (2011).

A.2 Residual Method

In this section, we describe the residual method for second order non-linear initial value problems constructed in (Adiyaman & Oger (2017); Oger (2016)).

First, consider the second order non-linear initial value problems

$$y''(x) = F(x, y(x), y'(x)) \tag{A.8}$$

with initial conditions

$$y(a) = \alpha \quad y'(a) = \beta, \tag{A.9}$$

where $F \in C^{n-1}([a, b] \times D_1 \times D_2)$, D_1 and D_2 are closed intervals in $\mathbb{R}$, α, β are finite constants and n is the degree of approximate Bézier curve.

Then, let we divide the interval $[a, b]$ into equally spaced sub-intervals $[a_{i-1}, a_i]$, where

$$a_i = a + ih, \quad i = 1, 2, \ldots, N,$$

$h = \dfrac{b-a}{N}$, N is the number of intervals. So we can write the initial value problem (A.8) and (A.9) piecewisely as

$$y_i''(x) = F(x, y_i(x), y_i'(x)), \quad x \in S_i, \tag{A.10}$$

where $S_i = [a_{i-1}, a_i]$ for $i = 1, \ldots, N$ and

$$\begin{aligned} y_1(a_0) = \alpha, \quad y_1'(a_0) = \beta, \\ y_i(a_{i-1}) = y_{i-1}(a_{i-1}), \quad y_i'(a_{i-1}) = y_{i-1}'(a_{i-1}) \end{aligned} \tag{A.11}$$

for $i = 2, \ldots, N$.

The goal is to approximate to the exact solution $y_i(x)$ over the interval S_i using a

Bézier curve. Let we define the nth degree Bézier curve as

$$u_i(x) = \sum_{j=0}^{n} c_j^i B_j^n(x; S_i), \quad x \in S_i, \tag{A.12}$$

where

$$B_j^n(x; S_i) = \binom{n}{j} \frac{1}{h^n} (x - a_{i-1})^j (a_i - x)^{n-j}$$

is the Bernstein polynomials over the interval S_i and c_j^i are unknown control points. The number of unknown control points is $n + 1$.

Applying the initial conditions to the approximate solution, we obtain

$$u_1(a_0) = \alpha, \quad u_1'(a_0) = \beta. \tag{A.13}$$

Since the given differential equation is second order, we need continuously differentiable approximate solution. For this, $u_i(x)$ must satisfy

$$u_i(a_{i-1}) = u_{i-1}(a_{i-1}), \quad u_i'(a_{i-1}) = u_{i-1}'(a_{i-1}) \tag{A.14}$$

for $i = 2, \ldots, N$. Thus, we can write $u(x)$ piecewise as

$$u(x) = \begin{cases} u_1(x), & \text{if } x \in S_1 \\ u_2(x), & \text{if } x \in S_2 \\ \vdots & \\ u_N(x), & \text{if } x \in S_N, \end{cases} \tag{A.15}$$

which is continuously differentiable.

First, we calculate the first two control points from the initial conditions in each sub-interval. In order to do this, we use the end point property and the end point derivative

property of Bézier curves in (A.10) which gives

$$\begin{aligned} c_0^1 &= \alpha, \\ c_1^1 &= \beta\frac{h}{n} + \alpha \end{aligned} \tag{A.16}$$

for the first interval.

Using (A.5) and (A.6), in the other sub-interval first two control points are obtained as

$$\begin{aligned} c_0^i &= c_n^{i-1}, \\ c_1^i &= 2c_n^{i-1} - c_{n-1}^{i-1}, \end{aligned} \tag{A.17}$$

for $i = 2, \ldots, N$. Thus, C^1 continuity of Bézier curves are satisfied and there are $n-1$ unknown control points for each sub-interval S_i from on now.

Putting (A.12) into the given differential equation (A.10) we describe the piecewise residual function as

$$R(x) = \begin{cases} R_1(x), & \text{if } x \in S_1 \\ R_2(x), & \text{if } x \in S_2 \\ \vdots & \\ R_N(x), & \text{if } x \in S_N, \end{cases} \tag{A.18}$$

where

$$R_i(x) = u_i''(x) - F(x, u_i(x), u_i'(x)), \quad x \in S_i. \tag{A.19}$$

The purpose is to find unknown control points c_j^i such that they minimize the Taylor's series expansion of sufficiently differentiable residual function $R_i(x)$ at $x = a_{i-1}$. For this, the strategy is the same as in Chapter 2. So, to determine $n-1$ unknown control

points c_j^i, we consider

$$\begin{aligned} R_i(a_{i-1}) &= 0 \\ R_i'(a_{i-1}) &= 0 \\ R_i''(a_{i-1}) &= 0 \\ &\vdots \\ R_i^{(n-2)}(a_{i-1}) &= 0. \end{aligned} \tag{A.20}$$

Then, using equations (A.5), (A.16) and (A.17), for any $k = 1, \ldots, n-2$, equations (A.20)

$$R_i^{(k)}(a_{i-1}) = u_i^{(k+2)}(a_{i-1}) - F^{(k)}(a_{i-1}, u_i(a_{i-1}), u_i'(a_{i-1})) = 0,$$

become linear equations

$$\frac{n!}{(n-k-2)!\, h^{k+2}} \Delta^{k+2} c_0^i - F^{(k)}\left(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\right) = 0,$$

where

$$F^{(k)}(x, y(x), z(x)) = \frac{\partial^k}{\partial x^k} f(x, y(x), z(x))$$

and

$$\Delta^{k+2} c_0^i = \sum_{j=0}^{k+2} \binom{k+2}{j} (-1)^{k+2-j} c_j^i.$$

Aboce linear equations give the following lower triangular system

$$\begin{bmatrix} 1 & 0 & \ldots & \ldots & 0 \\ -3 & 1 & 0 & \ldots & 0 \\ 6 & -4 & 1 & 0 & \vdots \\ \vdots & & & \ddots & 0 \\ (-1)^{n-2}\binom{n}{n-2} & (-1)^{n-3}\binom{n}{n-3} & \ldots & (-1)\binom{n}{1} & 1 \end{bmatrix} \begin{bmatrix} c_2^i \\ c_3^i \\ c_4^i \\ \vdots \\ c_n^i \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{bmatrix}$$

where

$$g_j = (-1)^{j-1}\, c_0^i + (-1)^{j-2}\, j\, c_1^i - \frac{(n-j)!}{n!}\, h^{j+1}\, F^{(j-2)}\left(a_{i-1}, \alpha, \frac{n}{h}(c_1^i - c_0^i)\right).$$

The unknown control points c_j^i are determined using above lower triangular system in each sub-intervals for $j = 2, 3, \ldots, n$. Thus, we have the approximate solution $u_i(x)$ such that $R_i(x)$ will be minimum in S_i, for $i = 1, \ldots, N$.

Lemma A.2.1. *Residual function $R(x)$ is order of h^{n-1}.*

One can find the proof of this lemma in Oger (2016).