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GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

LIE GROUP APPLICATIONS IN SPACE KINEMATICS

by
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İZMİR

LIE GROUP APPLICATIONS IN SPACE KINEMATICS

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Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**LIE GROUP APPLICATIONS IN SPACE KINEMATICS**" completed by **DERYA BAYRIL AYKUT** under the supervision of **ASSOC. PROF. DR. İLHAN KARAKILIÇ** and **ASSOC. PROF. DR. JONATHAN MARK SELIG** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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LIE GROUP APPLICATIONS IN SPACE KINEMATICS

ABSTRACT

This thesis deals with the kinematics of the rigid body motions. The product of two group elements of Special Euclidean Group in 3-dimension is investigated. This problem corresponds to the Baker-Campell-Hausdorff formula and geometrically the screw triangle. The main focus of this work is to find the Baker-Campell-Hausdorff formula for space kinematics and investigate the screw triangle. Also, the same problem is studied for the group of rotations in 3-dimension and the group of planar displacements. From the geometric point of view, they correspond to the spherical triangle and the pole triangle, respectively. Then the Baker-Campell-Hausdorff formula is obtained for 2-systems of screws.

In the next study, the relations between the exponential map and Cayley maps are derived. Then the product of two group elements of Special Euclidean Group in 3-dimension is investigated by using the 4×4 and the 6×6 Cayley maps. The corresponding twists for the product of two group elements are written in terms of Lie brackets as in the Baker-Campell-Hausdorff formula.

Plane symmetric motions are studied in the rest of the thesis. The velocity twists, axodes, acceleration centres and inflection points of these motions, which are generated by reflections in the osculating, normal and rectifying planes of a space curve, are obtained. Finally, line-symmetric motions generated by reflections in the tangent, normal and binormal lines of a space curve are determined and their velocity twists and fixed axodes are obtained.

Keywords: Kinematics, screw triangle, Cayley maps, plane symmetric motion.

UZAY KİNEMATİĞİNDE LIE GRUP UYGULAMALARI

ÖZ

Bu tezde katı cisim hareketlerinin kinematiği incelenmiştir. İlk çalışma uzay kinematiğinde iki farklı grup elemanının çarpımını geometrik ve cebirsel olarak incelemektedir. Bu problem cebirsel olarak Baker-Campell-Hausdorff formülüne, geometrik olarak da screw üçgenine karşılık gelmektedir. İlk çalışmanın amacı 3-boyutta katı cisim hareketleri grubu için Baker-Campell-Hausdorff formülünü elde etmek ve screw üçgenini incelemektir. Aynı problem küresel kinematikte ve düzlem kinematiğinde de çalışılmıştır. Bu problemler geometrik olarak, küresel üçgen ve pole üçgenine karşılık gelmektedir. Çalışmanın son bölümünde ise Baker-Campell-Hausdorff formülü Gibson-Hunt sınıflandırmasındaki ikili sistemler kullanılarak bazı özel gruplar için elde edilmiştir.

İkinci çalışmada Cayley dönüşümleri ve üstel dönüşüm arasındaki ilişkiler elde edilmiştir. Daha sonra 4×4 ve 6×6 Cayley dönüşümleri kullanılarak uzayda katı cisim hareketleri grubunun elemanları çarpımı Baker-Campell-Hausdorff formülüne benzer bir yapıyla Lie parantezleri kullanılarak ifade edilmiştir.

Son çalışmada düzlemsel simetrik hareketler incelenmektedir. Bir uzay eğrisinin oskülatör (dokunum) düzlemleri, rectifiye (doğrultma) düzlemleri ve normal düzlemleri üzerinde yansımalarla elde edilen düzlem simetrik katı cisim hareketinin hız eksenleri, axodları, ivlemeleri, bükülme noktaları, hız ve ivmenin birbirine göre dik olduğu durumlar (Bresse hiperbolü) ayrıntılı olarak çalışılmıştır. Son olarak, bir uzay eğrisinin teğet, normal ve binormal doğruları kullanılarak elde edilen doğru simetrik katı cisim hareketleri tanımlanmıştır ve bu tür hareketlerin hız eksenleri ve axodları elde edilmiştir.

Anahtar kelimeler: Kinematik, screw üçgeni, Cayley dönüşümleri, düzlemsel simetrik hareketler.

CONTENTS

Page

Ph.D. THESIS EXAMINATION RESULT FORM ii
ACKNOWLEDGEMENTS iii
ABSTRACT iv
ÖZ v
LIST OF FIGURES ix

CHAPTER ONE – INTRODUCTION 1

1.1 Quaternions and Study Quadric 4
1.2 The Group of Rigid Body Motions and Its Lie Algebra 8
1.2.1 Representations of $SE(3)$ 9
1.2.1.1 4×4 Homogeneous Representation of $SE(3)$ 9
1.2.1.2 6×6 Adjoint Representation of $SE(3)$ 10
1.2.2 Exponential and Cayley Maps for $SE(3)$ 11
1.2.2.1 Exponential Map 11
1.2.2.2 Cayley Maps 12
1.3 Rigid Body Motion as a Screw Motion 14

CHAPTER TWO – BAKER-CAMPELL-HAUSDORFF THEOREM AND SCREW TRIANGLE 19

2.1 The Pole Triangle 20
2.2 The Spherical Triangle 22
2.3 The Baker-Campbell-Hausdorff Formula 25
2.3.1 Iterated Commutators 25
2.4 The Baker-Campbell-Hausdorff Formula for $SE(3)$ 28
2.5 Screw Systems 32
2.5.1 The $I\ A$ System 33
2.5.2 The IB System 33

2.5.3 The IB^0 System 35
2.5.4 The IIA System 35
2.5.5 The IIB Systems 36
2.5.5.1 The completion group SE(2) 36
2.5.5.2 $H_p \ltimes \mathbb{R}^2$, 3-dimensional nonalgebraic group 38
2.5.6 The IIC System 39

CHAPTER THREE – BAKER-CAMPEL-HAUSDORFF THEOREM LIKE FORMULA FOR CAYLEY MAPS 40

3.1 The Exponential and Cayley Maps 40
3.1.1 The exponential map 40
3.1.2 The 4×4 Cayley map 41
3.1.3 The inverse 4×4 Cayley map 42
3.1.4 The 6×6 Cayley map 42
3.1.5 The inverse 6×6 Cayley map 43
3.1.6 Relations between the exponential and Cayley maps 44
3.2 Formulas for Products of Group Elements 47
3.2.1 Baker-Campell-Hausdorff Like Formula for $SO(3)$ Using 4×4 Cayley Map 47
3.2.2 Baker-Campell-Hausdorff Like Formula for $SE(3)$ Using 4×4 Cayley Map 48
3.2.3 Baker-Campell-Hausdorff Like Formula for SE(3) Using 6×6 Cayley Map 50

CHAPTER FOUR – ON PLANE-SYMMETRIC RIGID-BODY MOTIONS... 53

4.1 General Plane Symmetric Motions 53
4.2 Reflections in the Osculating Planes of a Space Curve 61
4.3 Reflections in the Normal Planes of a Curve 63
4.4 Reflections in the Rectifying Planes of a Curve 65
4.5 Example 67

CHAPTER FIVE – ACCELERATION, INFLECTION POINTS AND BRESSE HYPERBOLOID OF PLANE SYMMETRIC MOTIONS 73

5.1 Acceleration 73
5.2 Inflection Points 76
5.3 Bresse Hyperboloid 79

CHAPTER SIX – MOTIONS BASED ON CURVES WITH MATES AND LINE SYMMETRIC MOTIONS 86

6.1 Motions Based on Curves with Mates 86
6.2 Line Symmetric Motions 88

CHAPTER SEVEN – CONCLUSION 95

REFERENCES 97

LIST OF FIGURES

Page

Figure 2.1 The Screw Triangle formed by the lines l_1, l_2, l_3 and their common normals n_1, n_2 and n_{12}. 19

Figure 2.2 The Pole Triangle formed by three pole points p_1, p_2 and p_3 of the planar displacement..... 22

Figure 2.3 A Spherical Triangle 23

Figure 4.1 Reflection in a plane 53

Figure 4.2 The fixed axodes of the osculating, normal and rectifying planes motions (left to right) associated to a twisted cubic curve. 67

CHAPTER ONE
INTRODUCTION

Kinematics is a branch of mechanics that describes motion without considering the forces and torques that cause it. The position, velocity, acceleration and high order derivatives of a moving point (body) with respect to the time or other variables are the concern of kinematics. Kinematics is also known as the geometry of motion. In this thesis we are interested in kinematics of rigid body motions. Rigid body motions are important in robotics, since the parts of the robots are rigid in general. In n-dimensional space rigid body displacements form a group called the Special Euclidean Group, denoted by $SE(n)$. $SE(n)$ is a Lie group. Being a Lie group has properties such as; the tangent space to the identity linearizes the Lie group. That is, one can work on the Lie algebra to the group near identity instead of working on the group.

There are many maps between a Lie group and its corresponding Lie algebra. The exponential and Cayley maps are in common use. Each map has different advantages. Although Cayley maps depend on a representation, the exponential map is independent from the choice of representation. Cayley maps are rational but the exponential map is not. We will use both the exponential and Cayley maps for investigating the Special Euclidean Group $SE(3)$ and its Lie algebra.

In the theory of Lie groups the Baker-Campbell-Hausdorff formula is an important link between the multiplication of elements in the group and the Lie algebra to the group. Usually only the first few terms of the formula are given in textbooks, but particular formulas for particular groups are not given. The more important point for these texts is that the formula exists and can be written entirely as sums of Lie brackets of different orders as this has important theoretical consequences. The Baker-Campell-Hausdorff theorem was investigated by many famous mathematicians. It was studied by Schur (1889), Campbell (1896), Poincaré (1900), Baker (1901), Hausdorff (1906) and Dynkin (1947). The history of the Baker-Campell-Hausdorff theorem with the contributions of the scientists to the

theorem are given in detail by Achilles & Bonfiglioli (2012).

The almost final version of the theorem were given by Dynkin (1947) such that,

$$e^X e^Y = e^Z$$

where

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots$$

and X, Y are the elements of the Lie algebra to the Lie the group. This is the common version of the theorem in use. With the contributions of Dynkin to the theorem, the formula is called as Baker-Campell-Hausdorff-Dykin formula.

The Baker-Campbell-Hausdorff formula is very general, but it is possible to write a finite formula for small groups. In the literature, there is a formula for the group of rotations, $SO(3)$. For the group of rigid body motions in 3-dimension, Condurache & Ciureanu (2019) used the principle of transference to write the BCH formula in terms of dual vector products. The aim of chapter 2 is to give a finite formula for $SE(3)$ in terms of iterated commutators by using dual quaternions. In kinematics the same problem is called the screw triangle. The axes of two spatial displacement and the axis of their product with their three common normals form the screw triangle. A subsidiary aim is to clarify the relationship between the screw triangle and the Baker-Campbell-Hausdorff formula. Also similar arguments for $SO(3)$ and $SE(2)$ will be given, but now the triangle is a spherical triangle for $SO(3)$, and planar triangle for $SE(2)$. Finally, the Baker-Campbell-Hausdorff formula for $SE(3)$ is investigated for particular clases of 2-system of screws.

In the Chapter 3, the relations between exponential and Cayley maps are derived. Then the Baker-Campbell-Hausdorff formula is extended to the Cayley maps. The product of group elements is studied for the 4×4 Cayley map and the 6×6 Cayley map using quaternions. Then the twist corresponding to the product of group elements is written in terms of commutators using quaternions.

The next chapters concern plane symmetric rigid-body motions. In (Bottema & Roth, 1990, ch. IX §8) a short section is devoted to the subject of plane symmetric rigid-body motions. The theoretical kinematics literature contains very little else on this subject. By contrast, in the theory of mechanisms and robotics plane symmetric mechanism are well known. Both single loop mechanisms and parallel robots with this type of symmetry are well represented in the literature. In this thesis we will look in some detail at the properties of these rigid-body motions and their relationships with the geometry of curves in space.

We will begin to the Chapter 4 with the definition of plane symmetric motions. Then the axodes of these motions are investigated. The next sections deal with plane symmetric motions generated by reflections in the osculating, normal and rectifying planes of a space curve. Then the motion based on the osculating, normal and rectifying planes of a twisted cubic curve are studied as an example. The following chapter deals with the acceleration properties, inflection points and Bresse hyperboloid of these plane symmetric motions. The next chapter deals with particular types of curves that have partner curves. In these cases relations between the motions determined by the curve and the motion determined by its partner curve are studied. Finally, line-symmetric motions generated by reflections in the tangent, normal and binormal lines of a space curve are determined and their velocity twists and fixed axodes are found.

Now, some basic concepts in the study will be given briefly. The terms rigid body motion in three-dimensional space, screw motion, spatial displacement, or proper rigid transformation in three-dimension all have the same meaning for us. The 4×4 standard matrix representation, 6×6 adjoint representation, and dual quaternion representation will be used to investigate rigid body motions.

1.1 Quaternions and Study Quadric

Quaternions were introduced by Hamilton (1844). The general form of a quaternion consists of a scalar part a_0 and a vectorial part $\vec{a} = (a_1, a_2, a_3)$. They have the form

$$q = a_0 + a_1 i + a_2 j + a_3 k$$

where a_i (i=0,1,2,3) are just real numbers and i, j, k are basic (unit) quaternions which satisfies

$$i^2 = k^2 = j^2 = -1$$

and

$$ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j.$$

If the scalar part of the quaternion a_0 is zero and at least one of the coefficients a_1, a_2, a_3 is nonzero then the quaternion is called *vector quaternion*, *right quaternion*, or *pure quaternion*. A quaternion that has no vectorial part is called a *scalar quaternion* (just a real number). The addition of two quaternions is componentwise and multiplying a quaternion by a scalar is like the multiplication of four-dimensional vectors by a scalar. Product of two quaternions $p = b_0 + \vec{b}$ and $q = a_0 + \vec{a}$ defined as

$$pq = (b_0 + \vec{b})(a_0 + \vec{a}) = (b_0 a_0 - \vec{b} \cdot \vec{a}) + (b_0 \vec{a} + a_0 \vec{b} + \vec{b} \times \vec{a})$$

where the $\cdot$ and $\times$ denote the usual vectorial dot and cross products respectively. This formula is also known as the *Gibbs relation*. Note that multiplication is not commutative. Quaternions form an associative algebra and are denoted by $\mathbb{H}$. The conjugate of a quaternion $q = a_0 + \vec{a}$ is $q^* = a_0 - \vec{a}$, and the norm of the quaternion q is the square root of

$$qq^* = a_0^2 + a_1^2 + a_2^2 + a_3^2 = |q|^2.$$

Quaternions with a norm equal to 1 are called *unit quaternions*. Dividing a nonzero

quaternion q by its norm gives a unit quaternion,

$$\tilde{q} = \frac{q}{|q|}.$$

Any quaternion can be written as the product of its norm and a unit quaternion, $q = |q|\tilde{q}$. This form is called *polar decompositon*.

The inverse of a quaternion q is written by,

$$q^{-1} = \frac{q^*}{|q|^2}.$$

Rotations in three dimension can be represented by quaternions, especially by unit quaternions. Any rotation r of $\mathbb{R}^3$ can be written as a unit quaternion as follows,

$$r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\vec{\hat{\omega}} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\hat{\omega}_x i + \hat{\omega}_y j + \hat{\omega}_z k)$$

where θ is the angle of rotation and $\vec{\hat{\omega}}$ is the unit pure quaternion directed along the rotation axis.

Unit quaternions are in bijection with the points of the real 3-sphere. Each rotation corresponds to a pair of points on the unit sphere S^3. The $3-$sphere double covers the rotations. The group manifold of the group of rotations $SO(3)$ is 3-dimensional real projective space. Like the representation of rotations by quaternions, rigid body motions can be represented by dual quaternions.

Dual quaternions were obtained by Clifford (1871) as a generalization of Hamilton's quaternions. He called these numbers *biquaternions*. Dual quaternions consist of a pair of quaternions p, q and the dual unit ε which satisfies $\varepsilon^2 = 0$. They have the form,

$$h = p + \varepsilon q.$$

Here, the dual unit ε commutes with the basic quaternions i, j, k. If the quaternions p and q have no real part then the dual quaternion h is called *pure dual quaternion*. The

algebra of dual quaternions is the tensor product of quaternions $\mathbb{H}$ and the ring of dual numbers $\mathbb{D}$. Dual quaternions can be added together and multiplied by scalar like eight-dimensional vectors. Product of dual quaternions $h_1 = p_1 + \varepsilon q_1$ and $h_2 = p_2 + \varepsilon q_2$ is given by the formula,

$$h_1 h_2 = p_1 p_2 + \varepsilon(p_1 q_2 + q_1 p_2),$$

and their cross product, also known as dual vector product, is written by

$$h_1 \times h_2 = p_1 \times q_1 + \varepsilon(p_1 \times q_2 + p_2 \times q_1).$$

There are three types of dual quaternion conjugation. Quaternion conjugation, dual number conjugation, and the combination of both conjugations. The quaternion conjugate of $h = p + \varepsilon q$ is given by

$$h^* = p^* + \varepsilon q^*,$$

the dual number conjugate is as follows

$$h^* = p - \varepsilon q,$$

and the third one their combination

$$h^* = p^* - \varepsilon q^*.$$

We will use the first one as a conjugation. A dual quaternion h is called as a *unit dual quaternion* if $hh^* = 1$. Note that hh^* does not need to be a real number, it is a dual number in general. The product of h and h^* is given by

$$\begin{aligned} hh^* &= (p + \varepsilon q)(p^* + \varepsilon q^*) \\ &= pp^* + \varepsilon(pq^* + qp^*) \\ &= |p|^2 + 2\varepsilon(b_0 a_0 + \vec{b} \cdot \vec{a}). \end{aligned} \tag{1.1}$$

A dual quaternion h is a unit quaternion if the quaternion p is a unit quaternion and $b_0a_0 + \vec{b} \cdot \vec{a}$ is equal to zero. Unit dual quaternions are important because they represent rigid body motions in three-dimensional space.

Each rigid body motion can be represented by a unit dual quaternion g as follows,

$$g = r + \frac{1}{2}\varepsilon tr$$

where r is a unit quaternion and t is a pure quaternion representing the rotation and the translation respectively.

Now, one can ask "What is the group manifold when rigid body motions are represented by dual quaternions?"

Suppose g is an element of the group of rigid body motions

$$g = (a_0 + a_1 i + a_2 j + a_3 k) + \varepsilon(c_0 + c_1 i + c_2 j + c_3 k),$$

then the following two relations are hold,

$$\begin{aligned} a_0^2 + a_1^2 + a_2^2 + a_3^2 &= 1 \\ a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 &= 0, \end{aligned}$$

since g is a unit dual quaternion. This set of points is a six-dimensional affine algebraic variety of $\mathbb{R}^8$. It is called the image space of spatial displacement. After Study (1891), thinking of these variables as homogeneous coordinates of $\mathbb{P}^7$, each rigid displacement can be seen as a point on the six-dimensional quadric in $\mathbb{P}^7$. This quadric is called the Study quadric. Except for the points $a_0 = a_1 = a_2 = a_3 = 0$, all points on the quadric represent distinct rigid body motions. Note that, g and $-g$, which represent the same rigid body motion, are identified by considering the coordinates to be homogeneous. For more and clear information see Selig (2005).

1.2 The Group of Rigid Body Motions and Its Lie Algebra

The group of proper rigid body motions is also the group of direct affine isometries that preserves the Euclidean metric. That is, rigid transformations preserve the distance between every pair of points on the rigid body. Any rigid transformation is the combination of a rotation followed by a translation, given by

$$\vec{x}' = R\vec{x} + \vec{t} \tag{1.2}$$

where R is the orthogonal (rotation) $n \times n$ matrix and $\vec{t}$ is the n-dimensional constant translation vector. The determinant of the rotation matrix R is equal to 1. The group of orthogonal matrices with determinant 1 is called *special orthogonal group* (or *group of rotations*), denoted by $SO(n)$. Orthogonal matrices with determinant -1 correspond to reflections. Reflections are rigid transformations but they can not be performed with a physical machine. The qualifier $proper$ is used to exculude reflections from the group of rigid body motions. The group of all rigid body motions is denoted by $E(n)$ and the group of proper rigid body motions denoted by $SE(n)$. Both $E(n)$ and $SE(n)$ are Lie groups. That is, multiplication and inverse operators of the group are continuous and differentiable.

The Lie algebra of a Lie group is the tangent space of the manifold at the identitiy. The Lie algebra of the rigid body motion group is denoted by $se(n)$. The term "infinitesimal group element" is used for the Lie algebra elements. The algebra structure is given by the Lie bracket,

$$[X, Y] = XY - YX.$$

This operator is bilinear and satisfies the Jacobi identity.

It is much easier to work with the matrix representations of $SE(3)$ rather than with the transformation (1.2). In this thesis, we are interested in space kinematics, so our main focus is $SE(3)$. Now, the most useful homogeneous 4×4 representation and

6×6 adjoint representation of $SE(3)$ will be given.

1.2.1 Representations of $SE(3)$

1.2.1.1 4×4 *Homogeneous Representation of* $SE(3)$

Rigid transformations can be represented by matrices. When $\mathbb{R}^n$ is embedded in $\mathbb{R}^{n+1}$, every rigid motion in $\mathbb{R}^n$ has a convenient $(n+1) \times (n+1)$ matrix representation such as,

$$G = \begin{pmatrix} R & \vec{t} \\ 0 & 1 \end{pmatrix} \tag{1.3}$$

where R is the $n \times n$ rotation matrix and $\vec{t}$ is the translation vector in $\mathbb{R}^n$. This representation is called the *homogeneous* (or *standard*) *representation* of $SE(n)$. For a spatial displacement, R is a 3×3 rotation matrix and $\vec{t}$ is a 3×1 translation matrix. A typical Lie algebra element of the group $SE(n)$ has the form,

$$S = \begin{pmatrix} \Omega & \vec{v} \\ 0 & 0 \end{pmatrix} \tag{1.4}$$

where Ω is a skew-symmetric matrix and $\vec{v}$ is a vector in $\mathbb{R}^n$. The set of skew-symmetric matrices corresponds to Lie algebra of rotations, $so(n)$. For rigid body motions in three dimension, Ω is a 3×3 skew-symmetric matrix and $\vec{v} = (v_1, v_2, v_3)^T \in \mathbb{R}^3$.

For any 3×3 skew-symmetric matrix Ω,

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \tag{1.5}$$

there exist a vector $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ which satisfies,

$$\Omega\vec{y} = \vec{\omega} \times \vec{y} \quad \forall \vec{y} \in \mathbb{R}^3.$$

Here $\times$ denotes cross product in $\mathbb{R}^3$. Using this property, Lie algebra elements of $SE(3)$ in equation (1.4) can be written as a 6 dimensional vector such that,

$$s = \begin{pmatrix} \vec{\omega} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{1.6}$$

Since the dimension of a manifold is equal to the dimension of its tangent space, the Lie group and the corresponding Lie algebra have the same dimension. The dimension of $se(3)$ is 6, 3 comes from the set of 3×3 skew symmetric matrices and 3 comes from the vectors $\vec{v} \in \mathbb{R}^3$. The dimension of $se(n)$ is equal to $n(n+1)/2$.

1.2.1.2 6×6 *Adjoint Representation of* $SE(3)$

The group of rigid displacements has different representations. One of the important ones is the 6×6 adjoint representation. This is the action of $SE(3)$ to its own Lie algebra $se(3)$. It is given by,

$$\begin{pmatrix} R & 0 \\ TR & R \end{pmatrix} \tag{1.7}$$

where R is the 3×3 rotation matrix and T is the 3×3 skew-symmetric matrix corresponds to the translation vector $\vec{t} = (t_1, t_2, t_3)^T$ such that

$$T\vec{y} = \vec{t} \times \vec{y} \quad \forall \vec{y} \in \mathbb{R}^3.$$

A typical Lie algebra element of $SE(3)$ is represented by,

$$Z = \begin{pmatrix} \Omega & 0 \\ V & \Omega \end{pmatrix} \tag{1.8}$$

where Ω and V are 3×3 skew-symmetric matrices.

1.2.2 Exponential and Cayley Maps for $SE(3)$

1.2.2.1 Exponential Map

The exponential map is an important mapping from the Lie algebra to the Lie group. In the geometric point of view, it is the map from tangent space to the manifold. In general, the exponential map is not surjective but it is surjective for the group of rotations $SO(n)$ and the group of rigid body motions $SE(n)$ near identity elements of the groups. For any $n \times n$ matrix A, the exponential map defined as,

$$e^A = I_n + \sum_{r \geq 1} \frac{A^r}{r!}$$

where I_n denotes $n \times n$ identity matrix.

Let Ω be the element of $so(3)$ in the form (1.5). The exponential of a Lie algebra element gives the group element (rotation matrix) R,

$$e^{\Omega} = R$$

such that

$$e^{\Omega} = I_3 + \frac{1}{\theta} \sin\theta\Omega + \frac{1}{\theta^2}(1 - \cos\theta)\Omega^2$$

where $\theta^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$.

For $se(3)$, we will give the exponential of Lie algebra elements both in the 4×4 standard matrix representation and the 6×6 adjoint matrix representation.

$$e^S = I_4 + S + \frac{1}{\theta^2}(1 - \cos\theta)S^2 + \frac{1}{\theta^3}(\theta - \sin\theta)S^3$$

and

$$e^Z = I_6 + \frac{1}{2\theta}(3\sin\theta - \theta\cos\theta)Z + \frac{1}{2\theta^2}(4 - 4\cos\theta - \theta\sin\theta)Z^2 + \frac{1}{2\theta^3}(\sin\theta - \theta\cos\theta)Z^3 + \frac{1}{2\theta^4}(2 - 2\cos\theta - \theta\sin\theta)Z^4$$

where S and Z denotes the Lie algebra elements of $SE(3)$ in the standard and adjoint representation respectively, $\theta^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$. For more information see Selig (2005).

In this thesis, dual quaternion representation of $SE(3)$ will be used. In Selig (2010b) a formula was found for the exponential map using dual quaternion representation,

$$\begin{aligned} e^s &= \frac{1}{2}\big(2\cos|\vec{\omega}| + |\vec{\omega}|\sin|\vec{\omega}|\big) - \frac{1}{2|\vec{\omega}|}\big(|\vec{\omega}|\cos|\vec{\omega}| - 3\sin|\vec{\omega}|\big)s + \\ &\quad \frac{1}{2|\vec{\omega}|}\big(\sin|\vec{\omega}|\big)s^2 - \frac{1}{2|\vec{\omega}|^3}\big(|\vec{\omega}|\cos|\vec{\omega}| - \sin|\vec{\omega}|\big)s^3 \end{aligned} \quad (1.9)$$

where $s = \vec{\omega} + \varepsilon\vec{v}$ is an element of the Lie algebra $se(3)$ written as a dual quaternion.

1.2.2.2 Cayley Maps

A Cayley map is a rational map from Lie algebra to the Lie group. Like the exponential map, it linearizes the group near the identity element. Cayley maps depend on the representation. In general, the Cayley map is not defined for all groups, see Lemire et al. (2006).

There are different Cayley maps for the group of rotations $SO(3)$, however, the main Cayley map is defined as follows and the others are rarely considered, see Tsiotras et al. (1997),

$$\begin{aligned} \text{Cay}(\Omega) &= (I_3 + \Omega)(I_3 - \Omega)^{-1} \\ &= I_3 + \frac{2}{1+\theta^2}\Omega + \frac{2}{1+\theta^2}\Omega^2 \\ &= R \end{aligned}$$

where $\Omega \in so(3)$ is the 3×3 skew symmetric matrix and $\theta^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$.

Cayley maps for $SE(3)$ were studied by Selig (2007b) using the standard and adjoint matrix representations. Now, the resulting equations will be given here. Let S and Z denotes the Lie algebra elements of $SE(3)$ in the 4×4 standard and 6×6 adjoint matrix representation respectively, then

$$\text{Cay}(S) = I_4 + 2S + \frac{2}{1+\theta^2}S^2 + \frac{2}{1+\theta^2}S^3$$

and

$$\text{Cay}(Z) = I_6 + \frac{2(1+2\theta^2)}{(1+\theta^2)^2}Z + \frac{2(1+2\theta^2)}{(1+\theta^2)^2}Z^2 + \frac{2}{(1+\theta^2)^2}Z^3 + \frac{2}{(1+\theta^2)^2}Z^4$$

where $\theta^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$.

Also in Selig (2010b), a formula was given for the Cayley map associated with the standard representation and adjoint representation of $SE(3)$ using dual quaternions. In order to avoid confusion between the matrix representations of $SE(3)$ and dual quaternion representations, the notations "Cay_4" and "Cay_6" will be used for standard 4×4 representation and 6×6 adjoint representation in terms of dual quaternions respectively. Writing the twist z as a dual quaternion $z = \vec{w} + \varepsilon\vec{u}$ then,

$$\text{Cay}_4(z) = \frac{1}{2\sqrt{1+|\vec{w}|^2}}\big((2+|\vec{w}|^2) + 2z + z^2\big)$$

and

$$\text{Cay}_6(z) = \frac{1}{2(1+|\vec{w}|^2)^{3/2}}\big((2+3|\vec{w}|^2) + (2+3|\vec{w}|^2)z + z^2 + z^3\big).$$

The formulations for Cayley maps, inverse Cayley map and exponential map will be given more detailed in chapter 3. Also, The relations between Cayley maps and exponential map as dual quaternions will be investigated in chapter 3. For more detailed information see Selig (2007b), Selig (2010b).

1.3 Rigid Body Motion as a Screw Motion

To understand rigid body motions we need to know the location of all points of the rigid body. Two coordinate systems are used to identify the location of the body . One is the reference frame, which is also known as *the fixed frame* or *global frame*, and the other one is *the moving frame*. The moving frame is a coordinate system attached to the moving body. The location of the moving frame with respect to the fixed frame is sufficient to know the position of all points of the rigid body. Initially, the fixed frame and the moving frame are coincident.

In plane kinematics, except for pure translations, for each planar displacement there is a single point whose coordinates are the same both fixed frame and moving frame before and after the transformation. That is, it is an invariant of the displacement. This point is called the *pole point* of the displacement. Except for pure translation, a planar displacement can be seen as a pure rotation about this pole point (centre of rotation). When the rigid body moves, there is an instantaneous centre of rotation (also known as instantaneous velocity centre, velocity pole, or the pole) at any instant. These pole points form a path called the *centrode* of the motion. There are two types of centrode, the fixed centrode, and the moving centrode. The pole points trace out a path in the fixed frame (plane) is called the *fixed centrode* and the path in the moving frame is called the *moving centrode*. At any instant, the moving centrode and the fixed centrode are coincident at the pole point. The moving centrode rolls on the fixed centrode without sliding.

In space kinematics, if one point is fixed after the transformation, it is called *spherical motion*, or *rotation*. When the origins of both moving and fixed frames are located at this fixed point, the displacement can be seen as a pure rotation. This fixed point is the center of the sphere.

Chasles (1830) or Mozzi-Chasles theorem says that each Euclidean displacement in three-dimensional space is a screw motion. That is, except for pure translations, any rigid body motion can be decomposed into a rotation about a line, actually screw axis,

and a translation along this line (translation in the same direction with this line). Like the pole of a planar displacement, the screw axis is an invariant of spatial displacement. A screw motion is determined by a screw axis and a pitch. The *pitch* is a real number defined as the ratio of translation to rotation. When the rotation angle of the finite screw motion is very small, it is called *infinitesimal screw motion*. Infinitesimal rigid body motions (*twists*) are the Lie algebra elements of the group of rigid body motions. In robotics, mechanical joints correspond to twists. A screw (or twist) can be determined by a pair of three-component vector as in equation (1.6),

$$s = \begin{pmatrix} \vec{\omega} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \vec{\omega} \\ \vec{c} \times \vec{\omega} + p\vec{\omega} \end{pmatrix} \tag{1.10}$$

where $\vec{\omega}$ is a vector pointing along the direction of the motion (or infinitesimal screw motion) and the vector $\vec{v} \in \mathbb{R}^3$,

$$\vec{v} = \vec{c} \times \vec{\omega} + p\vec{\omega}.$$

Here $\vec{c}$ is the position vector of any point on the axis of motion, the moment of the axis is $\vec{c} \times \vec{\omega}$ and p is the pitch. Lines are called zero pitch screws.

For pure rotation, the pitch $p = 0$ hence the twist reduces to

$$s = \begin{pmatrix} \vec{\omega} \\ \vec{c} \times \vec{\omega} \end{pmatrix}$$

and corresponds to revolute joint.

For pure translational motion, the twist has infinite pitch, $\vec{\omega} = \vec{0}$ and this corresponds to prismatic joint.

Also, a twist in (1.10) can be represented by dual quaternions such as,

$$s = \omega + \varepsilon v$$

where ω and v pure quaternions.

To better understand any rigid body motion, it is necessary to know about its velocities. Thinking of any rigid body displacement as a curve on the manifold $SE(3)$ with the parameter time $\tilde{t}$. Taking the derivative with respect to time at the identity $(\tilde{t} = 0)$ will give the velocities. Therefore velocities are the Lie algebra elements of $SE(3)$. This Lie algebra element is called *velocity twist* (or *twist velocity*). Velocity twists consist of the angular velocity and a linear velocity. There are many linear velocities associated to a rigid body motion. Each point in the body has a linear velocity. The first three components of the velocity twist give the angular velocity vector of the motion and the linear velocity vector $\vec{v}$ in the twist is the velocity of the point in the body which is instantaneously coincident with the origin of the world (global) coordinate system. The velocity twist of a motion G can be computed as follows,

$$S_d = \dot{G}G^{-1} = \begin{pmatrix} \Omega & \vec{v} \\ 0 & 0 \end{pmatrix} \tag{1.11}$$

and as a six component vector,

$$s_d = \begin{pmatrix} \vec{\omega} \\ \vec{v} \end{pmatrix}$$

where $\Omega = \dot{R}R^{-1}$ is 3×3 skew-symmetric matrix and $\vec{v}$ is 3×1 vector. The vector $\vec{\omega}$ obtained from Ω using the property in (1.5) gives the angular velocity vector and the vector $\vec{v}$ is the linear velocity vector of the motion. To avoid confusion between the velocity twist in the fixed frame and the velocity twist in the moving frame, the notations S_d and s_d are used for velocity twist of a motion in the fixed frame and the notations S_b and s_b are used for the velocity twist in the moving frame. The velocity twist in the moving frame is given by

$$S_b = G^{-1}\dot{G}.$$

Velocity twist is also known as the *instantaneous screw*. The axes of the instantaneous screws form a ruled surface as the rigid body moves this is called the *axode* of the motion. The ruled surfaces generated by s_d and s_b are called the *fixed axode* and the *moving axode* respectively. For more information see Bottema & Roth (1990), Tsai (1999), Selig (2005).

In chapter 4, we will investigate the axodes of plane symmetric motions. Axodes are ruled surfaces so now we will give a little bit information about ruled surfaces.

A ruled surface is determined by a straight line l moving along the space curve β. The generating line l is called *the generator (or ruling)* and β is *the base curve (or directrix)*. Let $\vec{\delta}$ be the unit vector along the generator, the position of a general point on the ruled surface is given by,

$$\vec{X}(u,v) = \vec{\beta}(u) + v\vec{\delta}(u) \tag{1.12}$$

where $u, v \in \mathbb{R}$.

In general, ruled surfaces are not developable surfaces, for example, the hyperboloid is a ruled surface but not a developable. Although all developable surfaces in $\mathbb{R}^3$ are ruled surfaces. They are generalized cones, generalized cylinders or the tangent developable of a space curve. Developable surfaces are special ruled surfaces which have the same tangent plane at all points along the generator. The generator of a tangent developable surfaces is always tangent to a curve called the *cuspidal edge (or edge of regression)* of the surface. The planes tangent to the surface are the osculating planes of the cuspidal edge. Since any developable surface is isometric to a plane, they have zero Gaussian curvature. Because Gaussian curvature is preserved under the isometries and planes have zero Gaussian curvature. Also, zero Gaussian curvature is a necessary and sufficient condition for a surface to be a developable. Developable surfaces are also known as flat surfaces.

For ruled surfaces this necessary and sufficient condition turns out to be equivalent to the vanishing of the distribution parameter. *The distribution parameter* of a ruled

surface defined in equation (1.12) is given by,

$$\delta_d = \frac{\det\left(\dot{\beta},\, \delta,\, \dot{\delta}\right)}{|\dot{\delta}|^2}.$$

The origin of the zero distribution parameter of developable ruled surfaces comes from the Gaussian curvature of the surface, for more information about developable surfaces and ruled surfaces see Eisenhart (1964), Forsyth (1912).

CHAPTER TWO
BAKER-CAMPELL-HAUSDORFF THEOREM AND SCREW TRIANGLE

The spatial triangle was named by Yang (1963) as a geometrical configuration formed by three lines and their common normals. Then using the screw axes instead of the lines, the screw triangle was developed as a tool in kinematics by Roth (1967) to study three positions of a rigid body in space. The screw triangle is analogus to the pole triange. The pole triangle shows the geometrical relationship between the poles of two planar displacements and the pole of their composition. The pole triangle and spherical triangle are special cases of the screw triangle. The axes of two rotational displacements with the axis of their product form the spherical triangle.

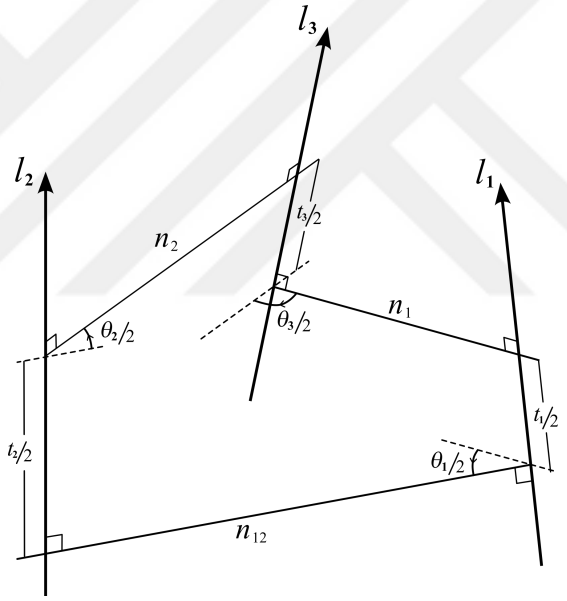


Figure 2.1 The Screw Triangle formed by the lines l_1, l_2, l_3 and their common normals n_1, n_2 and n_{12}.

Classically the product of two successive rigid-body displacements can be computed from a geometrical figure known as the screw triangle. The justification for this construction is not often given in modern accounts. It results from the fact that any rigid-body displacement can be factored as a product of two reflections in lines. A reflection in a line is simply a rotation by π about a line in space and is often also called a half-turn. Given a finite screw displacement g the axes of the two half-turns

producing g will be perpendicular to the axis of g. The angle between the axes of the half-turns will be half the rotation angle of g and the perpendicular distance between the axes will be half the translation along the axis of g. So if the displacement g is a pure rotation, the axes of the half-turns and the axis of g will be coincident. While if g is a pure translation the axes of the half-turns will be parallel. No other restriction is placed on the axes of the half-turns which generate g, so they can be arbitrarily translated along and rotated about the axis of g so long as the angle and distance between the lines is maintained.

So, to find the product of two finite screw displacements suppose that the axes of the displacements are located on the lines ℓ_1 and ℓ_2. Now decompose the two displacements into half-turns as described above, but choose the axis for the second half-turn of the first displacement and the first half-turn of the second displacement to coincide. This can be done if the axis of the two half-turns is the common perpendicular to the lines ℓ_1 and ℓ_2. With this choice the other two half-turns are fixed. Now the product of the two displacements is the product of the four half-turns but two of these are coincident and hence cancel, half-turns are self-inverse. So the product of the two displacements can be found from the two remaining half-turns the first half-turn of the first displacement and the second half-turn from the second displacement. In particular the axis of the product is the common perpendicular to the axes of these two half-turns. The figure consisting of all these lines is called the screw triangle, see Figure 2.1. The angles and lengths in the screw triangle correspond to half the rotation angles and translations of the displacements and given two sets of lengths and angles the third can be found using trigonometry. This classical approach is explained more full in texts such as Bottema & Roth (1990) and McCarthy & Soh (2010). Note also that this figure generalises the pole triangle from planar kinematics and the spherical triangle of spherical kinematics.

2.1 The Pole Triangle

The pole of two successive planar displacements with the pole of their product form the pole triangle. Almost all planar displacements are rotations about some point in

the plane, the exceptions are pure translations which could be thought of as rotations about points at infinity. Moreover, a rotation about a point with rotation angle θ can be thought of as a pair of reflections in line meeting at the centre of rotation and separated by half the rotation angle, $\theta/2$. The exact orientation of the lines is not important, so if we have two rotations, one followed by another, we can chose the second line of the first rotation to be the same as the first line of the second rotation, that is the line determined by the two rotation centres. In this way we get a reflection in this line followed by another reflection in the line, which of course gives the identity. The four reflections thus reduce to two reflections: in the first line for the first point and then the second line for the second point. The new rotation centre will then be the point where these two lines meet. This reduces the problem of computing the rotation centre and rotation angle of the combination of the two rotations to a problem in plane trigonometry. Rotation centers (poles) with the rotation agles form the pole triangle, see Figure 2.2, McCarthy & Soh (2010). Without lost of generality we will assume the pole triangle as a acute triangle. In particular it is easy to see that a rotation through angle θ_1 followed by a rotation by angle θ_2 will always combine to give a rotation with angle $(2\pi - \theta_1 - \theta_2)$. (Because all the angles in a triangle add to π radians). Hence the rotation angle θ_3,

$$\theta_3 = \theta_1 + \theta_2.$$

Now we will obtain the coordinates of the pole point p_3 geometrically. Rotating the vector $\vec{p_1} - \vec{p_2}$ by the angle $\theta_2/2$ and applying the sine rules for the triangle gives the vector $\vec{p_3} - \vec{p_2}$, such as

$$\frac{1}{\sin\theta_1/2}(\vec{p_3} - \vec{p_2}) = \frac{1}{\sin\theta_3/2} e^{i\frac{\theta_2}{2}}(\vec{p_1} - \vec{p_2}).$$

Note that $\sin(\pi - \theta_3/2) = \sin\theta_3/2$ and the rotation angle $\theta_3/2 = (\theta_1 + \theta_2)/2$. We can write the sinus functions as,

$$\sin\frac{\theta_1}{2} = \frac{1}{2i}(e^{i\frac{\theta_1}{2}} - e^{-i\frac{\theta_1}{2}}) \quad \text{and} \quad \sin\frac{\theta_3}{2} = \frac{1}{2i}(e^{i\frac{\theta_3}{2}} - e^{-i\frac{\theta_3}{2}}).$$

Rearranging the terms gives the resulting formula,

$$(1 - e^{i\frac{\theta_3}{2}})\vec{p_3} = (1 - e^{i\frac{\theta_2}{2}})\vec{p_2} + e^{i\frac{\theta_2}{2}}(1 - e^{i\frac{\theta_1}{2}})\vec{p_1}. \tag{2.1}$$

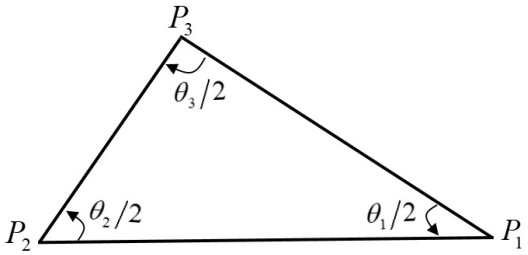


Figure 2.2 The Pole Triangle formed by three pole points p_1, p_2 and p_3 of the planar displacement

2.2 The Spherical Triangle

The spherical triangle shows the geometrical relationship between the axes of two planar displacements and the axis of their composition. The spherical triangle is a triangle on a sphere whose sides are great circles of the sphere. To study spherical triangles we will use spherical trigonometry.

Let $\hat{\omega}_1$, $\hat{\omega}_2$ and $\hat{\omega}_3$ be unit vectors forming the verticles of the triangle. For simplicity, suppose it is acute triangle, that is its sides A, B, C less than $\pi/2$. The angles between the sides of the triangle $\theta_1/2, \theta_2/2$ and $\theta_3/2$, see Figure 2.3

The scalar products of three vectors are given by,

$$\hat{\omega}_1 \cdot \hat{\omega}_2 = \cos A, \qquad \hat{\omega}_1 \cdot \hat{\omega}_3 = \cos B, \qquad \hat{\omega}_2 \cdot \hat{\omega}_3 = \cos C,$$

and the lenghts of the cross products are

$$|\hat{\omega}_1 \times \hat{\omega}_2| = \sin A, \qquad |\hat{\omega}_1 \times \hat{\omega}_3| = \sin B, \qquad |\hat{\omega}_2 \times \hat{\omega}_3| = \sin C.$$

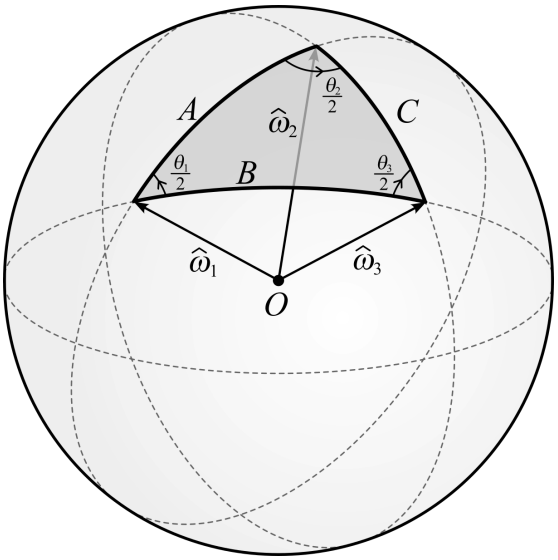


Figure 2.3 A Spherical Triangle

Also we will use the spherical analogues of Cosine and Sine rule of planar trigonometry. The spherical analoque of the Sine rule is given by,

$$\frac{\sin\theta_1/2}{\sin C} = \frac{\sin\theta_2/2}{\sin B} = \frac{\sin\theta_3/2}{\sin A} = \frac{1}{k}$$

where the constant k is less than 1 since the triangle is acute. The spherical analogue of the Cosine rule is

$$\begin{aligned}
\cos A &= \cos B\cos C + \sin B\sin C\cos\theta_3/2,\\
\cos B &= \cos A\cos C + \sin A\sin C\cos\theta_2/2,\\
\cos C &= \cos A\cos B + \sin A\sin B\cos\theta_1/2.
\end{aligned}$$

Using the polar triangle of this spherical triangle, see Todhunter (1863), we can find the cosine rules for the angles. The polar triangle is obtained by replacing the sides and the angles such that,

$$A = \pi - \theta_1/2, \quad B = \pi - \theta_2/2, \quad C = \pi - \theta_3/2$$

and

$$\theta_1/2 = \pi - A, \quad \theta_2/2 = \pi - B, \quad \theta_3/2 = \pi - C.$$

Applying the Cosine rule for the polar triange, we get

$$\cos\theta_1/2 = -\cos\theta_2/2\cos\theta_3/2 + \sin\theta_2/2\sin\theta_3/2\cos C,$$
$$\cos\theta_2/2 = -\cos\theta_1/2\cos\theta_3/2 + \sin\theta_1/2\sin\theta_3/2\cos B,$$
$$\cos\theta_3/2 = -\cos\theta_1/2\cos\theta_2/2 + \sin\theta_1/2\sin\theta_2/2\cos A.$$

Hence,

$$\cos(\pi - \frac{\theta_3}{2}) = -\cos\frac{\theta_3}{2} = \cos\frac{\theta_1}{2}\cos\frac{\theta}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\hat{\omega}_1\cdot\hat{\omega}_2. \tag{2.2}$$

To find $\hat{\omega}_3$, we can use the independent vectors $\hat{\omega}_1, \hat{\omega}_2$ and their vector product $\hat{\omega}_1 \times \hat{\omega}_2$. Assume that

$$\hat{\omega}_3 = \lambda_1\hat{\omega}_1 + \lambda_2\hat{\omega}_2 + \lambda_3\hat{\omega}_1\times\hat{\omega}_2,$$

then

$$\hat{\omega}_1\cdot\hat{\omega}_3 = \lambda_1 + \lambda_2\hat{\omega}_1\cdot\hat{\omega}_2 = \lambda_1 + \lambda_2\cos A,$$
$$\hat{\omega}_2\cdot\hat{\omega}_3 = \lambda_1\hat{\omega}_1\cdot\hat{\omega}_2 + \lambda_2 = \lambda_1\cos A + \lambda_2$$

and

$$\hat{\omega}_3\cdot(\hat{\omega}_1\times\hat{\omega}_2) = \lambda_3|\hat{\omega}_1\times\hat{\omega}_2|^2 = \lambda_3\sin^2 A.$$

Multiplying the second equation with $\cos A$ and rearranging the terms gives,

$$\lambda_1 = \frac{\cos B - \cos A\cos C}{\sin^2 A}.$$

Now using cosine and sine rules for spherical trigonometry we get,

$$\lambda_1\sin\frac{\theta_3}{2} = \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}.$$

Using similar computations,

$$\lambda_2\sin\frac{\theta_3}{2} = \sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} \quad \text{and} \quad \lambda_3\sin\frac{\theta_3}{2} = \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}.$$

Hence the formula for the vertex $\hat{\omega}_3$ of the spherical triangle is,

$$\hat{\omega}_3 \sin\frac{\theta_3}{2} = \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\hat{\omega}_1 + \sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\hat{\omega}_2 + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}(\hat{\omega}_1 \times \hat{\omega}_2). \tag{2.3}$$

This is really a classical result, see Selig (2005), McCarthy & Soh (2010). Equations (2.2) and (2.3) will be computed again as a Baker-Campell-Hausdorff formula for $SO(3)$, see (2.5.4).

2.3 The Baker-Campbell-Hausdorff Formula

Before launching into main body of this section we will need some results on the iterated commutators, these results will be collected here.

2.3.1 Iterated Commutators

First we note that in the Lie algebra $se(3)$, of the group of rigid body displacements the commutator correspond to the dual vector product, with $z_1 = \vec{w}_1 + \varepsilon\vec{u}_1$ and similar for z_2 we have,

$$[z_1,\, z_2] = \frac{1}{2}(z_1 z_2 - z_2 z_1) = \vec{w}_1 \times \vec{w}_2 + \varepsilon(\vec{w}_1 \times \vec{u}_2 + \vec{u}_1 \times \vec{w}_2).$$

This operation is usually called the dual vector product and denoted,

$$z_1 \times z_2 = \vec{w}_1 \times \vec{w}_2 + \varepsilon(\vec{w}_1 \times \vec{u}_2 + \vec{u}_1 \times \vec{w}_2). \tag{2.4}$$

In a free Lie algebra we could use a Philip Hall basis, for example, as a basis for the whole Lie algebra. The Lie algebra $se(3)$ is not free, it is finite dimensional, in fact it has dimension 6. So we need at least 6 Lie algebra elements to span the whole algebra. So, given a pair of Lie algebra elements z_1 and z_2 we can take as basis elements z_1 and

z_2 themselves, their commutator $z_1 \times z_2$ and several iterated commutator.

We investigate these iterated commutators in the following. The computations are facilitated by the screw analogue of the formula for the vector triple product,

$$z_1 \times (z_2 \times z_3) = \big((\vec{w}_1 \cdot \vec{w}_3) + \varepsilon(\vec{w}_1 \cdot \vec{u}_3 + \vec{u}_1 \cdot \vec{w}_3)\big) z_2 - \big((\vec{w}_1 \cdot \vec{w}_2) + \varepsilon(\vec{w}_1 \cdot \vec{u}_2 + \vec{u}_1 \cdot \vec{w}_3)\big) z_3.$$

This is just a simple consequence of the formula for the vector triple product.

The degree 3 commutators are then,

$$z_1 \times (z_1 \times z_2) = \big((\vec{w}_1 \cdot \vec{w}_2) + \varepsilon(\vec{w}_1 \cdot \vec{u}_2 + \vec{u}_1 \cdot \vec{w}_2)\big) z_1 - \big(|\vec{w}_1|^2 + 2\varepsilon(\vec{w}_1 \cdot \vec{u}_1)\big) z_2 \quad (2.5)$$

and

$$z_2 \times (z_1 \times z_2) = \big(|\vec{w}_2|^2 + 2\varepsilon(\vec{w}_2 \cdot \vec{u}_2)\big) z_1 - \big((\vec{w}_1 \cdot \vec{w}_2) + \varepsilon(\vec{w}_1 \cdot \vec{u}_2 + \vec{u}_1 \cdot \vec{w}_2)\big) z_2 \quad (2.6)$$

We will also need the following 4-fold iterated products,

$$z_1 \times (z_1 \times (z_1 \times z_2)) = -\big(|\vec{w}_1|^2 + 2\varepsilon(\vec{w}_1 \cdot \vec{u}_1)\big) z_1 \times z_2 \quad (2.7)$$

and

$$z_2 \times (z_2 \times (z_1 \times z_2)) = -\big(|\vec{w}_2|^2 + 2\varepsilon(\vec{w}_2 \cdot \vec{u}_2)\big) z_1 \times z_2. \quad (2.8)$$

Using equations between (2.4) and (2.8), we can find relations between the iterated commutators and three purely dual vectors εz_1, εz_2 and $\varepsilon(z_1 \times z_2)$,

$$-(\vec{w}_1 \cdot \vec{w}_2) z_1 + |\vec{w}_1|^2 z_2 + z_1 \times (z_1 \times z_2) = (\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1)\varepsilon z_1 - 2(\vec{w}_1 \cdot \vec{u}_1)\varepsilon z_2, \quad (2.9)$$

$$-|\vec{w}_2|^2 z_1 + (\vec{w}_1 \cdot \vec{w}_2) z_2 + z_2 \times (z_1 \times z_2) = 2(\vec{w}_2 \cdot \vec{u}_2)\varepsilon z_1 - (\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1)\varepsilon z_2, \quad (2.10)$$

$$|\vec{w}_1|^2(z_2 \times (z_2 \times (z_1 \times z_2)) - |\vec{w}_2|^2(z_1 \times (z_1 \times (z_1 \times z_2))$$
$$= 2(|\vec{w}_2|^2(\vec{w}_1 \cdot \vec{u}_1) - |\vec{w}_1|^2(\vec{w}_2 \cdot \vec{u}_2))\varepsilon(z_1 \times z_2) \quad (2.11)$$

and

$$|\vec{w}_1|^2 z_1 \times (z_1 \times z_2) + z_1 \times (z_1 \times (z_1 \times (z_1 \times z_2)))$$
$$= -2(\vec{w}_1 \cdot \vec{w}_2)(\vec{w}_1 \cdot \vec{u}_1)\varepsilon z_1 + 2|\vec{w}_1|^2(\vec{w}_1 \cdot \vec{u}_1)\varepsilon z_2, \quad (2.12)$$

$$|\vec{w}_2|^2 z_2 \times (z_1 \times z_2) + z_2 \times (z_2 \times (z_2 \times (z_1 \times z_2)))$$
$$= -2|\vec{w}_2|^2(\vec{w}_2 \cdot \vec{u}_2)\varepsilon z_1 + 2(\vec{w}_1 \cdot \vec{w}_2)(\vec{w}_2 \cdot \vec{u}_2)\varepsilon z_2. \quad (2.13)$$

These can be rearranged to give the following Lemma.

Lemma 2.3.1. Pure dual quaternions εz_1, εz_2 and $\varepsilon z_1 \times z_2$ are given by the following formulas:

$$\varepsilon z_1 = \frac{1}{4(\vec{w}_1 \cdot \vec{u}_1)(\vec{w}_2 \cdot \vec{u}_2) - (\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1)^2}$$
$$\times \Big(((\vec{w}_1 \cdot \vec{w}_2)(\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1) - 2|\vec{w}_2|^2(\vec{w}_1 \cdot \vec{u}_1))z_1$$
$$+ (2(\vec{w}_1 \cdot \vec{w}_2)(\vec{w}_1 \cdot \vec{u}_1) - |\vec{w}_1|^2(\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1))z_2$$
$$- (\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1)z_1 \times (z_1 \times z_2) + 2(\vec{w}_1 \cdot \vec{u}_1)z_2 \times (z_1 \times z_2)\Big), \quad (2.14)$$

$$\varepsilon z_2 = \frac{1}{4(\vec{w}_1 \cdot \vec{u}_1)(\vec{w}_2 \cdot \vec{u}_2) - (\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1)^2}$$
$$\times \Big((2(\vec{w}_1 \cdot \vec{w}_2)(\vec{w}_2 \cdot \vec{u}_2) - |\vec{w}_2|^2(\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1))z_1$$
$$+ ((\vec{w}_1 \cdot \vec{w}_2)(\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1) - 2|\vec{w}_1|^2(\vec{w}_2 \cdot \vec{u}_2))z_2$$
$$- 2(\vec{w}_2 \cdot \vec{u}_2)z_1 \times (z_1 \times z_2) + (\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1)z_2 \times (z_1 \times z_2)\Big) \quad (2.15)$$

and

$$\varepsilon(z_1 \times z_2) = \frac{|\vec{w}_1|^2(z_2 \times (z_2 \times (z_1 \times z_2)) - |\vec{w}_2|^2(z_1 \times (z_1 \times (z_1 \times z_2))}{2(|\vec{w}_2|^2(\vec{w}_1 \cdot \vec{u}_1) - |\vec{w}_1|^2(\vec{w}_2 \cdot \vec{u}_2))} \quad (2.16)$$

where $z_1 = \vec{w}_1 + \varepsilon\vec{u}_1$ and $z_2 = \vec{w}_2 + \varepsilon\vec{u}_2$.

2.4 The Baker-Campbell-Hausdorff Formula for $SE(3)$

Now, the Baker-Campbell-Hausdorff formula for Special Euclidean Group $SE(3)$ will be investigated. Let s_1 and s_2 are given twists, we seek a third twist s_3 such that,

$$e^{s_1}e^{s_2} = e^{s_3}.$$

To solve this we revisit the exponential map discussed in section 3.1.1 above. Consider an arbitrary twist $s = \vec{\omega} + \varepsilon\vec{v}$. Writing the modulus of the vector $\vec{\omega}$ as $|\vec{\omega}| = \theta/2$ the twist can be written as,

$$s = \frac{\theta}{2}\hat{s} = \frac{\theta}{2}(\hat{\omega} + \varepsilon\vec{\nu}).$$

Using the equation (3.3) above the exponential map can be written in the form,

$$\begin{aligned} e^{\frac{\theta}{2}\hat{s}} = &(\cos(\theta/2) + \sin(\theta/2)\hat{\omega}) + \\ &\varepsilon(-p(\theta/2)\sin(\theta/2) + \sin(\theta/2)\vec{\nu} + p((\theta/2)\cos(\theta/2) - \sin(\theta/2))\hat{\omega}), \end{aligned}$$

where the pitch $p = (\vec{\omega}\cdot\vec{v})/|\vec{\omega}|^2$ and the vector $\vec{\nu} = \vec{v}/|\vec{\omega}|$. This can be written simply in terms of a pair of dual numbers as,

$$e^{\frac{\theta}{2}\hat{s}} = (\alpha + \varepsilon\beta) + (\gamma + \varepsilon\delta)\hat{s},$$

where

$$\begin{aligned} \alpha &= \cos(\theta/2), \qquad \beta = -p(\theta/2)\sin(\theta/2), \\ \gamma &= \sin(\theta/2), \qquad \delta = p((\theta/2)\cos(\theta/2) - \sin(\theta/2)). \end{aligned}$$

In terms of these variable multiplying two exponentials is given by,

$$\begin{aligned} e^{\frac{\theta_1}{2}\hat{s}_1}e^{\frac{\theta_2}{2}\hat{s}_2} = &\big(\alpha_1\alpha_2 + \varepsilon(\alpha_1\beta_2 + \beta_1\alpha_2)\big) \\ &+ \big(\alpha_2\gamma_1 + \varepsilon(\alpha_2\delta_1 + \beta_2\gamma_1)\big)\hat{s}_1 \\ &+ \big(\alpha_1\gamma_2 + \varepsilon(\alpha_1\delta_2 + \beta_1\gamma_2)\big)\hat{s}_2 \\ &+ \big(\gamma_1\gamma_2 + \varepsilon(\gamma_1\delta_2 + \delta_1\gamma_2)\big)\hat{s}_1\hat{s}_2, \end{aligned} \tag{2.17}$$

where the subscripts 1 and 2 refer to the corresponding twists.

Now the product $\hat{s}_1\hat{s}_2$ can be expanded using the Gibbs relation,

$$\begin{aligned} \hat{s}_1\hat{s}_2 &= (\hat{\omega}_1 + \varepsilon\vec{\nu}_1)(\hat{\omega}_2 + \varepsilon\vec{\nu}_2) \\ &= (-\hat{\omega}_1\cdot\hat{\omega}_2 + \hat{\omega}_1\times\hat{\omega}_2) \\ &\quad + \varepsilon(-\hat{\omega}_2\cdot\vec{\nu}_1 - \vec{\nu}_2\cdot\hat{\omega}_1 + \hat{\omega}_1\times\vec{\nu}_2 + \vec{\nu}_1\times\hat{\omega}_2) \\ &= -\hat{s}_1\cdot\hat{s}_2 + \hat{s}_1\times\hat{s}_2. \end{aligned}$$

The operators $\cdot$ and $\times$ in the last line above are the dual versions of the scalar and vector products. Notice that this is really just the dual version of the Gibbs relation. This can be used to expand equation (2.17),

$$\begin{aligned} e^{\frac{\theta_1}{2}\hat{s}_1}e^{\frac{\theta_2}{2}\hat{s}_2} = &(\alpha_1\alpha_2 - \gamma_1\gamma_2\hat{\omega}_1\cdot\hat{\omega}_2) \\ &+ \varepsilon(\alpha_1\beta_2 + \beta_1\alpha_2 - (\gamma_1\delta_2 + \delta_1\gamma_2)\hat{\omega}_1\cdot\hat{\omega}_2 - \gamma_1\gamma_2(\hat{\omega}_1\cdot\vec{\nu}_2 + \vec{\nu}_1\cdot\hat{\omega}_2)) \\ &+ \alpha_2\gamma_1\hat{s}_1 + \alpha_1\gamma_2\hat{s}_2 + \gamma_1\gamma_2\hat{s}_1\times\hat{s}_2 \\ &+ (\alpha_2\delta_1 + \beta_2\gamma_1)\varepsilon\hat{s}_1 + (\alpha_1\delta_2 + \beta_1\gamma_2)\varepsilon\hat{s}_2 + (\gamma_1\delta_2 + \gamma_2\delta_1)\varepsilon\hat{s}_1\times\hat{s}_2. \end{aligned} \tag{2.18}$$

This result can now be compared with the exponential of the third twist to give first

$$\alpha_3 = \alpha_1\alpha_2 - \gamma_1\gamma_2\hat{\omega}_1\cdot\hat{\omega}_2.$$

That is,

$$\cos(\theta_3/2) = \cos(\theta_1/2)\cos(\theta_2/2) - \sin(\theta_1/2)\sin(\theta_2/2)\hat{\omega}_1\cdot\hat{\omega}_2. \tag{2.19}$$

Then we have,

$$\beta_3 = \alpha_1\beta_2 + \beta_1\alpha_2 - (\gamma_1\delta_2 + \delta_1\gamma_2)\hat{\omega}_1 \cdot \hat{\omega}_2 - \gamma_1\gamma_2(\hat{\omega}_1 \cdot \vec{\nu}_2 + \vec{\nu}_1 \cdot \hat{\omega}_2).$$

This can be written,

$$\begin{aligned} p_3(\theta_3/2)\sin(\theta_3/2) &= p_1(\theta_1/2)\sin(\theta_1/2)\cos(\theta_2/2) + p_2(\theta_2/2)\cos(\theta_1/2)\sin(\theta_2/2) \\ &+ \big(p_1(\theta_1/2)\cos(\theta_1/2)\sin(\theta_2/2) + p_2(\theta_2/2)\sin(\theta_1/2)\cos(\theta_2/2) \\ &- (p_1 + p_2)\sin(\theta_1/2)\sin(\theta_2/2)\big)\hat{\omega}_1 \cdot \hat{\omega}_2 + \sin(\theta_1/2)\sin(\theta_2/2)(\hat{\omega}_1 \cdot \vec{\nu}_2 + \vec{\nu}_1 \cdot \hat{\omega}_2). \end{aligned} \tag{2.20}$$

Taking the inverse cosine of equations (2.19) and doubling will give the rotation angle θ_3 of the composite displacement as a unique angle between 0 and 2π radians. Equation (2.20) then gives a unique value for the pitch p_3 of the composite displacement.

Comparing the dual vector part of equation (2.18) to the exponential gives

$$\begin{aligned} (\gamma_3 + \varepsilon\delta_3)\hat{s}_3 &= \alpha_2\gamma_1\hat{s}_1 + \alpha_1\gamma_2\hat{s}_2 + \gamma_1\gamma_2\hat{s}_1 \times \hat{s}_2 \\ &+ (\alpha_2\delta_1 + \beta_2\gamma_1)\varepsilon\hat{s}_1 + (\alpha_1\delta_2 + \beta_1\gamma_2)\varepsilon\hat{s}_2 + (\gamma_1\delta_2 + \gamma_2\delta_1)\varepsilon\hat{s}_1 \times \hat{s}_2. \end{aligned} \tag{2.21}$$

Notice that since now θ_3 and p_3 are known so are γ_3 and δ_3. To find $\hat{s}_3$ write,

$$\gamma_3\hat{s}_3 = \alpha_2\gamma_1\hat{s}_1 + \alpha_1\gamma_2\hat{s}_2 + \gamma_1\gamma_2\hat{s}_1 \times \hat{s}_2 + \eta_1\varepsilon\hat{s}_1 + \eta_2\varepsilon\hat{s}_2 + \eta_{12}\varepsilon\hat{s}_1 \times \hat{s}_2,$$

where the values of η_1, η_2 and η_{12} are to be determined. This can be used to substitute for $\hat{s}_3$ in the left-hand side of equation (2.21) after a cancelling we get an equation involving the three twist $\varepsilon\hat{s}_1$, $\varepsilon\hat{s}_2$ and $\varepsilon\hat{s}_1 \times \hat{s}_2$ only. In general, these twists will be linearly independent so we can set their coefficients to zero and then get,

$$\begin{aligned} \eta_1 &= \frac{1}{\gamma_3}(\alpha_2\delta_1\gamma_3 - \alpha_2\gamma_1\delta_3 + \beta_2\gamma_1\gamma_3), \\ \eta_2 &= \frac{1}{\gamma_3}(\alpha_1\delta_2\gamma_3 - \alpha_1\gamma_2\delta_3 + \beta_1\gamma_2\gamma_3), \\ \eta_{12} &= \frac{1}{\gamma_3}(\gamma_1\delta_2\gamma_3 - \gamma_1\gamma_2\delta_3 + \delta_1\gamma_2\gamma_3). \end{aligned}$$

The final step is to substitute for $\varepsilon\hat{s}_1,\ \varepsilon\hat{s}_2$ and $\varepsilon\hat{s}_1 \times \hat{s}_2$ using equations, (2.14), (2.15) and (2.16). The result is the equation,

$$\begin{aligned}\sin(\theta_3/2)\hat{s}_3 = \chi_1\hat{s}_1 + \chi_2\hat{s}_2 + \chi_{12}\hat{s}_1 \times \hat{s}_2 \\ + \chi_{112}\,\hat{s}_1 \times (\hat{s}_1 \times \hat{s}_2) + \chi_{212}\,\hat{s}_2 \times (\hat{s}_1 \times \hat{s}_2) \\ + \chi_{1112}\,\hat{s}_1 \times (\hat{s}_1 \times (\hat{s}_1 \times \hat{s}_2)) + \chi_{2212}\,\hat{s}_2 \times (\hat{s}_2 \times (\hat{s}_1 \times \hat{s}_2))\end{aligned} \tag{2.22}$$

where the coefficients of the iterated commutators are,

$$\begin{aligned}\chi_1 &= \sin(\theta_1/2)\cos(\theta_2/2) - (\hat{\omega}_1 \cdot \hat{\omega}_2)\chi_{112} - \chi_{212}, \\ \chi_2 &= \cos(\theta_1/2)\sin(\theta_2/2) + \chi_{112} + (\hat{\omega}_1 \cdot \hat{\omega}_2)\chi_{212}, \\ \chi_{12} &= \sin(\theta_1/2)\sin(\theta_2/2)\end{aligned}$$

and

$$\begin{aligned}\chi_{112} &= -\frac{\eta_1(\hat{\omega}_2 \cdot \vec{\nu}_1 + \hat{\omega}_1 \cdot \vec{\nu}_2) + 2\eta_2(\hat{\omega}_2 \cdot \vec{\nu}_2)}{4(\hat{\omega}_1 \cdot \vec{\nu}_1)(\hat{\omega}_2 \cdot \vec{\nu}_2) - (\hat{\omega}_2 \cdot \vec{\nu}_1 + \hat{\omega}_1 \cdot \vec{\nu}_2)^2}, \\ \chi_{212} &= \frac{2\eta_1(\hat{\omega}_1 \cdot \vec{\nu}_1) + \eta_2(\hat{\omega}_2 \cdot \vec{\nu}_1 + \hat{\omega}_1 \cdot \vec{\nu}_2)}{4(\hat{\omega}_1 \cdot \vec{\nu}_1)(\hat{\omega}_2 \cdot \vec{\nu}_2) - (\hat{\omega}_2 \cdot \vec{\nu}_1 + \hat{\omega}_1 \cdot \vec{\nu}_2)^2}, \\ \chi_{1112} &= -\chi_{2212} = \frac{-\eta_{12}}{2\big((\hat{\omega}_1 \cdot \vec{\nu}_1) - (\hat{\omega}_2 \cdot \vec{\nu}_2)\big)}.\end{aligned} \tag{2.23}$$

Note that $\hat{s}_3$ can be written using 6 independent twists $\hat{s}_1$, $\hat{s}_2$, $\hat{s}_1 \times \hat{s}_2$, and infinite pitch twists $-(\hat{\omega}_1 \cdot \hat{\omega}_2)\hat{s}_1 + \hat{s}_2 + \hat{s}_1 \times (\hat{s}_1 \times \hat{s}_2)$, $-\hat{s}_1 + (\hat{\omega}_1 \cdot \hat{\omega}_2)\hat{s}_2 + \hat{s}_2 \times (\hat{s}_1 \times \hat{s}_2)$ and $(\hat{s}_1 \times (\hat{s}_1 \times (\hat{s}_1 \times \hat{s}_2)) - \hat{s}_2 \times (\hat{s}_2 \times (\hat{s}_1 \times \hat{s}_2)))$.

Now we have the following Theorem.

Theorem 2.4.1 (Baker-Campbell-Hausdorff Formula for $SE(3)$). For two Lie algebra elements of $SE(3)$ $s_1 = \theta_1/2\hat{s}_1$ and $s_2 = \theta_2/2\hat{s}_2$, the product of exponentials can be written as follows,

$$e^{\frac{\theta_1}{2}\hat{s}_1} e^{\frac{\theta_2}{2}\hat{s}_2} = e^{\frac{\theta_3}{2}\hat{s}_3}$$

where

$$\theta_3/2 = \arccos(\cos(\theta_1/2)\cos(\theta_2/2) - \sin(\theta_1/2)\sin(\theta_2/2)\hat{\omega}_1 \cdot \hat{\omega}_2)$$

and

$$\begin{aligned}\sin(\theta_3/2)\hat{s}_3 = \chi_1\hat{s}_1 + \chi_2\hat{s}_2 + \chi_{12}\hat{s}_1 \times \hat{s}_2 \\ +\chi_{112}\,\hat{s}_1 \times (\hat{s}_1 \times \hat{s}_2) + \chi_{212}\,\hat{s}_2 \times (\hat{s}_1 \times \hat{s}_2) \\ + \chi_{1112}\,\hat{s}_1 \times (\hat{s}_1 \times (\hat{s}_1 \times \hat{s}_2)) + \chi_{2212}\,\hat{s}_2 \times (\hat{s}_2 \times (\hat{s}_1 \times \hat{s}_2))\end{aligned}$$

with the coefficients of the iterated commutators are given in equation (2.23), and the twists $\hat{s}_1 = \theta_1/2(\hat{\omega}_1 + \varepsilon\vec{\nu}_1)$ and $\hat{s}_2 = \theta_2/2(\hat{\omega}_2 + \varepsilon\vec{\nu}_2)$.

The Baker-Campbell-Hausdorff formula for $SE(3)$ has been written explicitly in terms of screw products using dual quaternions. A Closed form of the formula was studied by Condurache & Ciureanu (2019) using the principle of transference in terms of dual vector products.

2.5 Screw Systems

In the previous section the general Baker-Campbell-Hausdorff formula for $SE(3)$ was derived, but there are several special cases to consider. The most obvious of these are where the two twists s_1 and s_2 are pure rotations and have parallel or intersecting axes. In these cases the twists and their commutators close to form a subalgebra of the Lie algebrta to the group of rigid-body displacements. In the case where the axes are parallel the subalgebra formed is $se(2)$, the Lie algebra to the group of rigid-body displacements in the plane. When the axes meet the subalgebra is $so(3)$ the Lie algebra for the group of rotations about a fixed point.

More generally, the closure subalgebra depends on the type of the screw system, see (Selig, 2005, §8.1), so the special cases can be conveniently identified with the Gibson-Hunt type of the 2-system formed by the twists s_1 and s_2.

2.5.1 The $I\,A$ System

According to the classification given by Gibson & Hunt (1990), this is the general system. The closure subalgebra is all of $se(3)$ and hence the general formula given above applies.

2.5.2 The IB System

This is Hunt's 4th special system. Completion group is the Schönflies group $SE(2) \times \mathbb{R}$. Screws are parallel.

Using the canonical basis the generators of this system are given by,

$$\tilde{s}_1 = \begin{pmatrix} \vec{i} \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{s}_2 = \begin{pmatrix} 0 \\ \vec{i} + p\vec{j} \end{pmatrix}.$$

So,

$$\tilde{s}_1 \times \tilde{s}_2 = \begin{pmatrix} 0 \\ p\vec{k} \end{pmatrix} \quad \text{and} \quad \tilde{s}_1 \times (\tilde{s}_1 \times \tilde{s}_2) = \begin{pmatrix} 0 \\ -p\vec{j} \end{pmatrix}.$$

There are four non-zero basis element of the subalgebra. All other commutators are zero. Hence, any element of this subalgebra is written by,

$$s = \lambda_1 \tilde{s}_1 + \lambda_2 \tilde{s}_2 + \lambda_3 (\tilde{s}_1 \times \tilde{s}_2) + \lambda_4 (\tilde{s}_1 \times (\tilde{s}_1 \times \tilde{s}_2))$$

where λ_n, $n = 1 \cdots 4$ are elements of real numbers.

Suppose $s_1 = \theta_1/2(\vec{i} + \varepsilon\vec{\nu}_1) = \theta_1/2\hat{s}_1$ and $s_2 = \theta_2/2(\vec{i} + \varepsilon\vec{\nu}_2) = \theta_2/2\hat{s}_2$ are two Lie algebra elements of the Schönflies group. Then the product of two exponentials is given by,

$$e^{\frac{\theta_1}{2}\hat{s}_1}e^{\frac{\theta_2}{2}\hat{s}_2} = (\alpha_3 + \varepsilon\beta_3) + (\gamma_3 + \varepsilon\delta_3)\hat{s}_3$$

where

$$\begin{aligned} \alpha_3 = &= \cos(\theta_1/2)\cos(\theta_2/2) - \sin(\theta_1/2)\sin(\theta_2/2)(\vec{i}\cdot\vec{i}) \\ &= \cos(\tfrac{\theta_1+\theta_2}{2}) \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} \beta_3 &= \alpha_1\beta_2 + \beta_1\alpha_2 - (\gamma_1\delta_2 + \delta_1\gamma_2)(\vec{i}\cdot i) - \gamma_1\gamma_2(\vec{i}\cdot\vec{\nu}_2 + \vec{\nu}_1\cdot i) \\ &= -(p_1\tfrac{\theta_1}{2} + p_2\tfrac{\theta_2}{2})\sin(\tfrac{\theta_1+\theta_2}{2}). \end{aligned} \tag{2.25}$$

Note that $p_1 = (\vec{\nu}_1\cdot\vec{i})$ and $p_2 = (\vec{\nu}_2\cdot\vec{i})$. Taking the inverse cosine of α_3 and doubling will give the rotation angle θ_3 of the composite displacement. Rearranging the terms of β_3 gives a unique value for the pitch p_3 of the composite displacement. They are found such as,

$$\theta_3 = \theta_1 + \theta_2 \quad \text{and} \quad p_3 = (p_1\theta_1 + p_2\theta_2)/(\theta_1 + \theta_2).$$

Notice that since now θ_3 and p_3 are known so are γ_3 and δ_3. To find $\hat{s}_3$ write,

$$\gamma_3\hat{s}_3 = \cos(\theta_2/2)\sin(\theta_1/2)\hat{s}_1 + \cos(\theta_1/2)\sin(\theta_2/2)\hat{s}_2 + \sin(\theta_1/2)\sin(\theta_2/2)\hat{s}_1\times\hat{s}_2 + \eta_1\varepsilon\hat{s}_1,$$

twists are linearly independent, so we can find the coefficient η_1 as below,

$$\eta_1 = -p_1\cos(\theta_2/2)\sin(\theta_1/2) - p_2\cos(\theta_1/2)\sin(\theta_2/2) + \frac{p_1\theta_1 + p_2\theta_2}{\theta_1+\theta_2}\sin((\theta_1+\theta_2)/2).$$

After substituting for $\varepsilon\hat{s}_1$ using equation (2.14). The result is the equation

$$\sin((\theta_1+\theta_2)/2)\hat{s}_3 = \chi_1\hat{s}_1 + \chi_2\hat{s}_2 + \chi_{12}\hat{s}_1\times\hat{s}_2 + \chi_{112}\,\hat{s}_1\times(\hat{s}_1\times\hat{s}_2)$$

where the coefficients of the iterated commutators are,

$$\begin{aligned} \chi_1 &= \sin(\theta_1/2)\cos(\theta_2/2) - \chi_{112}, \\ \chi_2 &= \cos(\theta_1/2)\sin(\theta_2/2) + \chi_{112}, \\ \chi_{12} &= \sin(\theta_1/2)\sin(\theta_2/2) \end{aligned}$$

and

$$\begin{aligned}\chi_{112} &= \eta_1/(\vec{i}\cdot\vec{\nu}_2 - \vec{\nu}_1\cdot\vec{i}) \\ &= \eta_1/(p_2 - p_1).\end{aligned}$$

2.5.3 The IB^0 System

This is Hunt's 5th special system. The completion subalgebra is $so(2)\times\mathbb{R}$ the Lie algebra to the cylinder group. All twists in this screw system commute that is all commutators are zero, so the formula is simply,

$$s_3 = \frac{\theta_1}{2}\hat{s}_1 + \frac{\theta_2}{2}\hat{s}_2.$$

2.5.4 The IIA System

This Hunt's 1st special system. Completion group is $SO(3)$ if $p=0$ or $SE(2)$ is $p\neq 0$. Screw axes lie in a plane all passing through a single point. Constant pitch p.

When the two twists lie in this screw system all possible iterated Lie brackets lie in the Lie algebra to a rotation group $SO(3)$. This case was studied as a spherical triangle, see Figure 2.3. Now, the Baker-Campell-Hausdorff formula for $SO(3)$ will be computed.

Suppose $s_1 = \theta_1/2\hat{\omega}_1$ and $s_2 = \theta_2/2\hat{\omega}_2$ are elements of the Lie algebra to the rotation group $SO(3)$. Expressing these vectors as quarternions the product of exponentials gives,

$$e^{\frac{\theta_1}{2}\hat{\omega}_1}e^{\frac{\theta_2}{2}\hat{\omega}_2} = e^{\frac{\theta_3}{2}\hat{\omega}_3}.$$

Expanding the exponentials gives,

$$(\cos\frac{\theta_1}{2} + \sin\frac{\theta_1}{2}\hat{\omega}_1)(\cos\frac{\theta_2}{2} + \sin\frac{\theta_2}{2}\hat{\omega}_2) = (\cos\frac{\theta_3}{2} + \sin\frac{\theta_3}{2}\hat{\omega}_3).$$

Expanding the quaternion products and comparing the scalar and vector part of the equation gives the result equations,

$$\cos\frac{\theta_3}{2} = \cos\frac{\theta_1}{2}\cos\frac{\theta}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\hat{\omega}_1 \cdot \hat{\omega}_2 \tag{2.26}$$

and

$$\sin\frac{\theta_3}{2}\hat{\omega}_3 = \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\hat{\omega}_1 + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\hat{\omega}_2 + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\hat{\omega}_1 \times \hat{\omega}_2. \tag{2.27}$$

Note that, the Baker-Campell-Hausdorf formula for $SO(3)$ has been written explicitly using three independent twists s_1, s_2 and $s_1 \times s_2$. Geometricaly, the result equations (2.26) and (2.27) were found using spherical trigonometry in the equation (2.2) and (2.3). The twists $\hat{\omega}_1$, $\hat{\omega}_2$ and $\hat{\omega}_3$ form the spherical triangle, see McCarthy & Soh (2010) and Selig (2005).

2.5.5 The IIB Systems

Hunt's 2nd special system. Completion group is $SE(2)$ if $p = 0$, $H_p \ltimes \mathbb{R}^2$ otherwise. Screw axes parallel to each other and perpendicular to a fixed line (picket fence).

2.5.5.1 The completion group SE(2)

When the two twists lie in this screw system all possible iterated Lie brackets lie in the Lie algebra to the planar motion group $SE(2)$. Using the canonical basis the generators of $SE(2)$ are given by,

$$\tilde{s}_1 = \begin{pmatrix} \vec{i} \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{s}_2 = \begin{pmatrix} 0 \\ \vec{j} \end{pmatrix}.$$

So,

$$\tilde{s}_1 \times \tilde{s}_2 = \begin{pmatrix} 0 \\ \vec{k} \end{pmatrix} \quad \text{and} \quad \tilde{s}_1 \times (\tilde{s}_1 \times \tilde{s}_2) = \begin{pmatrix} 0 \\ -\vec{j} \end{pmatrix} = -\tilde{s}_2.$$

There are three non-zero basis element of this subalgebra. All other commutators are zero. Hence, any element of this subalgebra is written by,

$$s = \lambda_1 \tilde{s}_1 + \lambda_2 \tilde{s}_2 + \lambda_3 (\tilde{s}_1 \times \tilde{s}_2)$$

where λ_n, $n = 1 \cdots 3$ are the elements of real numbers.

Suppose $s_1 = \theta_1/2(\vec{i} + \varepsilon\vec{\nu}_1) = \theta_1/2\hat{s}_1$ and $s_2 = \theta_2/2(\vec{i} + \varepsilon\vec{\nu}_2) = \theta_2/2\hat{s}_2$ are two Lie algebra elements of planar motion group. Here $\vec{\nu}_1 = a_1\vec{j} + b_1\vec{k}$ and $\vec{\nu}_2 = a_2\vec{j} + b_2\vec{k}$ for some a_1, b_1 and a_2, b_2 in $\mathbb{R}$. Then the product of two exponentials is given by,

$$e^{\frac{\theta_1}{2}\hat{s}_1} e^{\frac{\theta_2}{2}\hat{s}_2} = (\alpha_3 + \varepsilon\beta_3) + (\gamma_3 + \varepsilon\delta_3)\hat{s}_3$$

where

$$\begin{aligned} \alpha_3 = &= \cos(\theta_1/2)\cos(\theta_2/2) - \sin(\theta_1/2)\sin(\theta_2/2)(\vec{i} \cdot \vec{i}) \\ &= \cos(\tfrac{\theta_1+\theta_2}{2}) \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} \beta_3 &= \alpha_1\beta_2 + \beta_1\alpha_2 - (\gamma_1\delta_2 + \delta_1\gamma_2)(\vec{i} \cdot i) - \gamma_1\gamma_2(\vec{i} \cdot \vec{\nu}_2 + \vec{\nu}_1 \cdot i) \\ &= 0. \end{aligned} \tag{2.29}$$

Hence, the rotation angle θ_3 and the pitch p_3 of the composite displacement can be found as follows,

$$\theta_3 = \theta_1 + \theta_2 \quad \text{and} \quad p_3 = 0.$$

Note that p_1, p_2, β_1, β_2 and δ_1, δ_2 are zero. So, η_1, η_2 and η_{12} are equal to zero. Hence the Baker-Campell-Hausdorff formula for $SE(2)$ is written by using three independent twists s_1, s_2 and $s_1 \times s_2$ such as,

$$\sin(\frac{\theta_1 + \theta_2}{2})\hat{s}_3 = \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\hat{s}_1 + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\hat{s}_2 + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\hat{s}_1 \times \hat{s}_2. \tag{2.30}$$

Geometrically, the twists s_1, s_2 and $s_1 \times s_2$ form the pole triangle, see Figure 2.2.

2.5.5.2 $H_p \ltimes \mathbb{R}^2$, 3-dimensional nonalgebraic group

The Lie algebra of this group consists of screws of pitch p with parallel axes together with infinite pitch screws along axes perpendicular to the first. Using the canonical basis the generator of this Lie algebra are given by,

$$\tilde{s}_1 = \begin{pmatrix} \vec{i} \\ p\vec{i} \end{pmatrix} \quad \text{and} \quad \tilde{s}_2 = \begin{pmatrix} 0 \\ \vec{j} \end{pmatrix}.$$

So,

$$\tilde{s}_1 \times \tilde{s}_2 = \begin{pmatrix} 0 \\ \vec{k} \end{pmatrix} \quad \text{and} \quad \tilde{s}_1 \times (\tilde{s}_1 \times \tilde{s}_2) = \begin{pmatrix} 0 \\ -\vec{j} \end{pmatrix} = -\tilde{s}_2.$$

There are three non-zero basis element of this subalgebra. All other commutators are zero. Hence, any element of this subalgebra is written by,

$$s = \lambda_1 \tilde{s}_1 + \lambda_2 \tilde{s}_2 + \lambda_3 (\tilde{s}_1 \times \tilde{s}_2)$$

where λ_n, $n = 1 \cdots 3$ are real numbers.

Suppose $s_1 = \theta_1/2(\vec{i} + \varepsilon\vec{\nu}_1) = \theta_1/2\hat{s}_1$ and $s_2 = \theta_2/2(\vec{i} + \varepsilon\vec{\nu}_2) = \theta_2/2\hat{s}_2$ are two Lie algebra elements of this motion group where $\vec{\nu}_1 = p\vec{i} + a_1\vec{j} + b_1\vec{k}$ and $\vec{\nu}_2 = p\vec{i} + a_2\vec{j} + b_2\vec{k}$ for some a_1, b_1 and a_2, b_2 in $\mathbb{R}$. Then the product of two exponentials is given by,

$$e^{\frac{\theta_1}{2}\hat{s}_1} e^{\frac{\theta_2}{2}\hat{s}_2} = (\alpha_3 + \varepsilon\beta_3) + (\gamma_3 + \varepsilon\delta_3)\hat{s}_3$$

where

$$\begin{aligned} \alpha_3 &= \cos(\theta_1/2)\cos(\theta_2/2) - \sin(\theta_1/2)\sin(\theta_2/2)(\vec{i} \cdot \vec{i}) \\ &= \cos(\tfrac{\theta_1+\theta_2}{2}) \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} \beta_3 &= \alpha_1\beta_2 + \beta_1\alpha_2 - (\gamma_1\delta_2 + \delta_1\gamma_2)(\vec{i}\cdot i) - \gamma_1\gamma_2(\vec{i}\cdot\vec{\nu}_2 + \vec{\nu}_1\cdot i) \\ &= -p(\tfrac{\theta_1+\theta_2}{2})\sin(\tfrac{\theta_1+\theta_2}{2}). \end{aligned} \tag{2.32}$$

Hence, the rotation angle θ_3 and the pitch p_3 of the composite displacement can be found as follows,

$$\theta_3 = \theta_1 + \theta_2 \quad \text{and} \quad p_3 = p.$$

Hence the Baker-Campell-Hausdorff formula for $H_p \ltimes \mathbb{R}^2$ is written by using three independent twists s_1, s_2 and $s_1 \times s_2$ such as,

$$\sin(\frac{\theta_1+\theta_2}{2})\hat{s}_3 = \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\hat{s}_1 + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\hat{s}_2 + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\hat{s}_1\times\hat{s}_2. \tag{2.33}$$

2.5.6 The IIC System

Hunt's 3rd special, completion group is $\mathbb{R}^2$. Generated by 2 translations, system consists of all translations in a plane. Again since all twists in this screw system commute the formula is simply,

$$s_3 = \frac{\theta_1}{2}\hat{s}_1 + \frac{\theta_2}{2}\hat{s}_2.$$

CHAPTER THREE
BAKER-CAMPEL-HAUSDORFF THEOREM LIKE FORMULA FOR CAYLEY MAPS

3.1 The Exponential and Cayley Maps

3.1.1 The exponential map

Suppose that $s = \vec{\omega} + \varepsilon\vec{v}$ is an element of the Lie algebra $se(3)$, a twist in other words, written as a dual quaternion. In Selig (2010b) a formula was found for the exponential map in terms of powers of s,

$$\begin{aligned} e^s &= \frac{1}{2}\big(2\cos|\vec{\omega}| + |\vec{\omega}|\sin|\vec{\omega}|\big) - \frac{1}{2|\vec{\omega}|}\big(|\vec{\omega}|\cos|\vec{\omega}| - 3\sin|\vec{\omega}|\big)s \\ &\quad + \frac{1}{2|\vec{\omega}|}\big(\sin|\vec{\omega}|\big)s^2 - \frac{1}{2|\vec{\omega}|^3}\big(|\vec{\omega}|\cos|\vec{\omega}| - \sin|\vec{\omega}|\big)s^3. \end{aligned} \quad (3.1)$$

Using the Gibbs relation to multiply the quaternions, the powers of $s = \vec{\omega} + \varepsilon\vec{v}$ can be expanded as,

$$\begin{aligned} s^2 &= -|\vec{\omega}|^2 - 2\varepsilon(\vec{\omega}\cdot\vec{v}), \\ s^3 &= -|\vec{\omega}|^2\vec{\omega} - \varepsilon\big(|\vec{\omega}|^2\vec{v} + 2(\vec{\omega}\cdot\vec{v})\vec{\omega}\big). \end{aligned}$$

Substituting these results into equation (4.1) gives the exponential map explicitly in terms of dual quaternions,

$$\begin{aligned} a_0 &= \cos|\vec{\omega}|, & c_0 &= \frac{-\sin|\vec{\omega}|}{|\vec{\omega}|}(\vec{\omega}\cdot\vec{v}), \\ \vec{a} &= \frac{\sin|\vec{\omega}|}{|\vec{\omega}|}\vec{\omega}, & \vec{c} &= \frac{\sin|\vec{\omega}|}{|\vec{\omega}|}\vec{v} + \frac{\vec{\omega}\cdot\vec{v}}{|\vec{\omega}|^3}(|\vec{\omega}|\cos|\vec{\omega}| - \sin|\vec{\omega}|)\vec{\omega}. \end{aligned} \quad (3.2)$$

The above gives a general result but a more specific result based on identifying the rotational and translational parts of the twist is useful. Suppose the twist is written in

terms of a unit screw,

$$\hat{s} = \big(\hat{\omega} + \varepsilon(\vec{r} \times \hat{\omega} + p\hat{\omega})\big)$$

where $\hat{\omega}$ is a unit vector in the direction of the screw axis, $\vec{r}$ is a point on the screw axis and p is the pitch of the screw, then a rotation of θ about this screw can be written as an exponential,

$$\begin{aligned} e^{\frac{\theta}{2}\hat{s}} = &\big(\cos(\theta/2) + \sin(\theta/2)\hat{\omega}\big) \\ &+ \varepsilon\big(-(p\theta/2)\sin(\theta/2) + \sin(\theta/2)\vec{r} \times \hat{\omega} + (p\theta/2)\cos(\theta/2)\hat{\omega}\big). \end{aligned} \quad (3.3)$$

3.1.2 The 4×4 *Cayley map*

Also in Selig (2010b) a formula was given for the Cayley map associated with the standard 4×4 representation of $SE(3)$. Writing the twist z as a dual quaternion $z = \vec{w} + \varepsilon\vec{u}$ the map is,

$$\mathrm{Cay}_4(z) = \frac{1}{2\sqrt{1 + |\vec{w}|^2}}\big((2 + |\vec{w}|^2) + 2z + z^2\big). \quad (3.4)$$

The Cayley map and its inverse can be viewed as a birational transformation between the six-dimensional projective space $\mathbb{P}^6$ and the Study quadric in $\mathbb{P}^7$. To see this introduce a homogenising variable w_0 so the homogeneous coordinates for the $\mathbb{P}^6$ will be $(w_0 : w_1 : w_2 : w_3 : u_1 : u_2 : u_3)$ and the coordinates in the $\mathbb{P}^7$ will be $(a_0 : a_1 : a_2 : a_3 : c_0 : c_1 : c_2 : c_3)$ as above. The homogeneous version of the Cayley map can now be written as,

$$\mathrm{Cay}_4(z) = \frac{1}{2\sqrt{w_0^2 + |\vec{w}|^2}}\big((2w_0^2 + |w|^2) + 2w_0 z + z^2\big).$$

Since the spaces are projective the overall factor $1/2\sqrt{1 + |\vec{w}|^2}$ can be ignored and the map is,

$$\begin{aligned} a_0 &= w_0^2, & c_0 &= -(\vec{w} \cdot \vec{u}), \\ \vec{a} &= w_0\vec{w}, & \vec{c} &= w_0\vec{u}. \end{aligned}$$

So the Cayley map is a quadratic transformation. The exceptional set, the set of elements not in the domain of the transformation, is the 4-dimensional non-singular quadric,

$$w_0 = 0,\ w_1u_1 + w_2u_2 + w_3u_3 = 0.$$

The vector space of twists can be identified with the projective points of the form $1 + \vec{w} + \varepsilon - \vec{w} \cdot \vec{u} + \vec{u}$, that is the open set with $w_0 \neq 0$.

3.1.3 The inverse 4×4 *Cayley map*

The inverse map, Cay_4^{-1} is given by a cubic polynomial,

$$\begin{aligned}\text{Cay}_4^{-1}(g) = \frac{1}{2a_0(a_0^2 + a_1^2 + a_2^2 + a_3^2)}\Big(&g^3 - 4a_0g^2 \\ &+ \big(4a_0^2 + 3(a_0^2 + a_1^2 + a_2^2 + a_3^2)\big)g - 4a_0(a_0^2 + a_1^2 + a_2^2 + a_3^2)\Big).\end{aligned}$$

If we assign $w_0 = 2a_0(a_0^2 + a_1^2 + a_2^2 + a_3^2)$, the common denominator, then the other coordinates of $z = \text{Cay}^{-1}(g)$ are given by expanding the polynomial in g and simplifying using the relation $a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0$ which defines the Study quadric. After cancelling the common factor, $2(a_0^2 + a_1^2 + a_2^2 + a_3^2)$, the result is,

$$\begin{aligned} w_0 &= a_0, & & \\ \vec{w} &= \vec{a}, & \vec{u} &= \vec{c}. \end{aligned}$$

So this is in fact a linear projection, and is clearly the inverse to the Cayley map above. The centre of this linear projection is just the point $(0:0:0:0:1:0:0:0)$ which, as a dual quaternion is simply ε.

3.1.4 The 6×6 *Cayley map*

An advantage of these algebraic methods is that we can derive similar relations for other Cayley maps. In Selig (2010b) the Cayley map Cay_6, for the adjoint

representation of $SE(3)$ and its inverse were described in terms of dual quaternions. Writing the twist z as a dual quaternion $z = \vec{w} + \varepsilon\vec{u}$ again, Cay_6 is given by the polynomial,

$$\text{Cay}_6(z) = \frac{1}{2(1+|\vec{w}|^2)^{3/2}}\big((2+3|\vec{w}|^2) + (2+3|\vec{w}|^2)z + z^2 + z^3\big).$$

and the homogeneous version of this Cayley map can be written as,

$$\text{Cay}_6(z) = \frac{1}{2(w_0^2+|\vec{w}|^2)^{3/2}}\big((2w_0^2+3|\vec{w}|^2)w_0 + (2w_0^2+3|\vec{w}|^2)z + w_0z^2 + z^3\big).$$

where w_0 is the homogenising variable. In this way the map can be viewed as a map from the projective space $\mathbb{P}^6$, with homogeneous coordinates, $(w_0 : w_1 : w_2 : w_3 : u_1 : u_2 : u_3)$ to Study quadric Q_s. Explicitly this is,

$$\begin{aligned} a_0 &= w_0(w_0^2+w_1^2+w_2^2+w_3^2), & c_0 &= -w_0(\vec{w}\cdot\vec{u}), \\ \vec{a} &= \vec{w}(w_0^2+w_1^2+w_2^2+w_3^2), & \vec{c} &= \vec{u}(w_0^2+w_1^2+w_2^2+w_3^2) - \vec{w}(\vec{w}\cdot\vec{u}) \end{aligned}$$

3.1.5 The inverse 6×6 *Cayley map*

In the homogeneous coordinates the inverse 6×6 Cayley map is given by quadratic polynomial

$$\text{Cay}_6^{-1}(g) = \frac{-1}{2a_0^2}\Big(g^2 - 4a_0g + (3a_0^2+a_1^2+a_2^2+a_3^2)\Big) \quad \text{where} \quad g \in Q_s.$$

If we assign $w_0 = -2a_0^2$, the common denominator, then the other coordinates of $z = \text{Cay}_6^{-1}(g)$ are given by expanding the polynomial in g and simplifying using the relation $a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0$ which defines the Study quadric.

$$\begin{aligned} w_0 &= a_0^2, \\ \vec{w} &= a_0\vec{a}, \qquad \vec{u} = a_0\vec{c} - c_0\vec{a}. \end{aligned}$$

This is a quadric transformation with exceptional set consisting of the 3-plane $a_0 = a_1 = a_2 = a_3 = 0$ and 5-plane $a_0 = c_0 = 0$. The 3-plane is the A-plane of ideal elements in the Study quadric, the points which do not correspond to any rigid-body transformation. The intersection of the 5-plane with the Study quadric is the set of half turns, that is rotations by π radians about some axis.

3.1.6 Relations between the exponential and Cayley maps

In this section, we will find the relations between twists corresponding to the same group element under the exponential and the Cayley maps.

Theorem 3.1.1. If the exponential of s and 4×4 Cayley map of z give the same group element in $SE(3)$, then the relation between s and z is given by,

$$z = \frac{\tan|\vec{\omega}|}{|\vec{\omega}|}\left(1 + \varepsilon p\left(\frac{|\vec{\omega}|}{\tan|\vec{\omega}|} - 1\right)\right)s$$

and

$$s = \left(\frac{\arctan|\vec{w}|}{|\vec{w}|} + \varepsilon\left(\frac{\vec{w}\cdot\vec{u}}{|\vec{w}|^2}\left(1 - \frac{\arctan|\vec{w}|}{|\vec{w}|}\right)\right)\right)z.$$

where, $s = \vec{\omega} + \varepsilon\vec{v}$ and $z = \vec{w} + \varepsilon\vec{u}$ lie in the Lie algebra to the SE(3) and the pitch $p = \vec{\omega}\cdot\vec{v}/|\vec{\omega}|^2$.

Proof. Here we look for a map from the Lie algebra to itself defined by the relation,

$$e^s = \text{Cay}_4(z).$$

That is, we associate twists s and z if the exponential of s and the Cayley map of z give the same group element. Since we have an explicit expression for inverse Cayley map we can rearrange this as,

$$z = \text{Cay}_4^{-1}(e^s),$$

which can be computed explicitly.

Using the formulas in (3.2) for the exponential, we ignore c_0 and divide by a_0 to get,

$$\vec{w} \;=\; \frac{\tan|\vec{\omega}|}{|\vec{\omega}|}\vec{\omega} \quad , \qquad \vec{u} \;=\; \frac{\tan|\vec{\omega}|}{|\vec{\omega}|}\vec{v} + p\Big(1 - \frac{\tan|\vec{\omega}|}{|\vec{\omega}|}\Big)\vec{\omega}$$

where as usual the pitch is, $p = \vec{\omega}\cdot\vec{v}/|\vec{\omega}|^2$. Now we get,

$$z = \frac{\tan|\vec{\omega}|}{|\vec{\omega}|}\left(1 + \varepsilon p\left(\frac{|\vec{\omega}|}{\tan|\vec{\omega}|} - 1\right)\right)s.$$

This looks more familiar in terms of the expression for the exponential given in equation (3.3),

$$z = \tan(\theta/2)\left(\hat{\omega} + \varepsilon\big(\vec{r}\times\hat{\omega} + \frac{p\theta/2}{\tan(\theta/2)}\hat{\omega}\big)\right)$$

This relation can be written using dual quaternions as,

$$z = \Big(\tan(\theta/2) + \varepsilon\big(p(\theta/2) - \tan(\theta/2)\big)\Big)\hat{s},$$

where $\hat{s} = \hat{\omega} + \varepsilon(\vec{r}\times\hat{\omega} + p\hat{\omega})$, as above.

Note that this map of twists to twist was refered to in Huang & Chen (1994) as a linearisation of twist space. Notice also that the pitch of the twist z is Parkin's measure Parkin (1992), later dubbed the ``quasi-pitch" or ``quatch".

Next we look at inverting this map, that is finding,

$$s = \log\big(\mathrm{Cay}_4(z)\big).$$

To do this it will be useful to write $\vec{w} = \lambda\hat{w}$ where $\lambda = |\vec{w}|$. Then, with $z = \vec{w} + \varepsilon\vec{u}$ we have,

$$\mathrm{Cay}_4(z) = (1 + \vec{w}) + \varepsilon(-\vec{w}\cdot\vec{u} + \vec{u}) = (1 + \lambda\hat{w}) + \varepsilon(-\lambda\hat{w}\cdot\vec{u} + \vec{u}).$$

This will be compared with the result given in (3.2) above for the exponential map. Now since the group $SE(3)$ is represented as dual quaternions parameterising a projective space, we may multiply the results above by any arbitrary non-zero number before

comparing them. So, to make the a_0 components agree, let us multiply the result of the Cayley map by $\cos|\vec{\omega}|$ to give,

$$\text{Cay}_4(z) = (\cos|\vec{\omega}| + \lambda\cos|\vec{\omega}|\hat{w}) + \varepsilon(-\lambda\cos|\vec{\omega}|\hat{w}\cdot\vec{u} + \cos|\vec{\omega}|\vec{u}).$$

Comparing the $\vec{a}$ components of the dual quaternions we see that,

$$\lambda\cos|\vec{\omega}|\hat{w} = \frac{\sin|\vec{\omega}|}{|\vec{\omega}|}\vec{\omega}.$$

From this we can see that the unit vectors $\hat{w}$ and $\hat{\omega}$ must be the same and that $\lambda = \tan|\vec{\omega}|$ or equivalently that $|\vec{\omega}| = \arctan(\lambda) = \arctan|\vec{w}|$. Turning to the c_0 component of the dual quaternion we get,

$$-\frac{\sin|\vec{\omega}|}{|\vec{\omega}|}(\vec{\omega}\cdot\vec{v}) = -\lambda\cos|\vec{\omega}|(\hat{w}\cdot\vec{u}) = -\sin|\vec{\omega}|(\hat{w}\cdot\vec{u}).$$

From this we get the relation $\hat{\omega}\cdot\vec{v} = \hat{w}\cdot\vec{u}$. The $\vec{c}$ component of the dual quaternion gives us,

$$\frac{\sin|\vec{\omega}|}{|\vec{\omega}|}\vec{v} + \frac{\hat{\omega}\cdot\vec{v}}{|\vec{\omega}|^2}(|\vec{\omega}|\cos|\vec{\omega}| - \sin|\vec{\omega}|)\vec{\omega} = \cos|\vec{\omega}|\vec{u}.$$

Rearranging this and substituting for $\vec{\omega}$ and $\hat{\omega}\cdot\vec{v}$, gives,

$$\vec{v} = \frac{\arctan(\lambda)}{\lambda}\vec{u} + \frac{\vec{w}\cdot\vec{u}}{\lambda^2}\left(1 - \frac{\arctan(\lambda)}{\lambda}\right)\vec{w}.$$

Putting these results together the effect of the map $\log(\text{Cay}_4(z))$ can be written as multiplication be a dual number,

$$s = \left(\frac{\arctan|\vec{w}|}{|\vec{w}|} + \varepsilon\left(\frac{\vec{w}\cdot\vec{u}}{|\vec{w}|^2}\left(1 - \frac{\arctan|\vec{w}|}{|\vec{w}|}\right)\right)\right)z.$$

where, as above, $s = \vec{\omega} + \varepsilon\vec{v}$ and $z = \vec{w} + \varepsilon\vec{u}$. □

Theorem 3.1.2. If the exponential of s and 6×6 Cayley map of z give the same group

element in $SE(3)$, then the relation between s and z is given by,

$$z = \left(\frac{\tan|\vec{w}|}{|\vec{w}|} + \varepsilon p \left(1 - \frac{\tan|\vec{w}|}{|\vec{w}|} + \tan^2|\vec{w}| \right) \right) s$$

and

$$s = \left(\frac{\arctan|\vec{w}|}{|\vec{w}|} + \varepsilon \left(\frac{\vec{w}\cdot\vec{u}}{|\vec{w}|^2} \left(\frac{1}{1+|\vec{w}|^2} - \frac{\arctan|\vec{w}|}{|\vec{w}|} \right) \right) \right) z$$

where, $s = \vec{\omega} + \varepsilon\vec{v}$ and $z = \vec{w} + \varepsilon\vec{u}$ lie in the Lie algebra to the SE(3) and the pitch $p = \vec{\omega}\cdot\vec{v}/|\vec{\omega}|^2$.

Theorem 3.1.3. If the 4×4 Cayley map of z_1 and 6×6 Cayley map of z_2 give the same group element in $SE(3)$, then the relation between z_1 and z_2 is given by,

$$z_1 = \left(1 - \varepsilon\left(\frac{\vec{w}_2\cdot\vec{u}_2}{1+|\vec{w}_2|^2} \right) \right) z_2$$

and

$$z_2 = \left(1 + \varepsilon(\vec{w}_2\cdot\vec{u}_2) \right) z_1$$

where, $z_1 = \vec{w}_1 + \varepsilon\vec{u}_1$ and $z_2 = \vec{w}_2 + \varepsilon\vec{u}_2$.

Proof of the Theorem (3.1.2) and Theorem (3.1.3) can be proved in a similar way to Theorem (3.1.1).

3.2 Formulas for Products of Group Elements

In this section we find formulas for the products of group elements given by Cayley maps. The idea is to find formulas in terms of iterated commutators in line with the classical Baker-Campbell-Hausdorff formula.

3.2.1 Baker-Campell-Hausdorff Like Formula for $SO(3)$ Using 4×4 Cayley Map

For the group of rotations, the product of group elements was studied by Mladenova (1990) using matrix representation of $SO(3)$. We will use quaternions to obtain Baker-

Campell-Hausdorff like formula for the group of rotations. Writing the Lie algebra and Lie group elements as quaternions make the computations simpler.

Let $\vec{w}_1$ and $\vec{w}_2$ be pure unit quaternions. These are the elements of the Lie algebra of the rotation group $SO(3)$. Then the product of group elements is given by,

$$\text{Cay}_4(\vec{w}_1)\,\text{Cay}_4(\vec{w}_2) = \big(1+\vec{w}_1\big)\big(1+\vec{w}_2\big)$$

Using the Gibbs relation to expand the quaternion products gives,

$$\begin{aligned}\text{Cay}_4(\vec{w}_3) &= 1 - \vec{w}_1\cdot\vec{w}_2 + \vec{w}_1 + \vec{w}_2 + \vec{w}_1\times\vec{w}_2\\ &= 1+\vec{w}_3\end{aligned}$$

where $\vec{w}_3$ is the corresponding Lie algebra element to the group elemet $\text{Cay}_4(\vec{w}_3) = \text{Cay}_4(\vec{w}_1)\,\text{Cay}_4(\vec{w}_2)$. After identifying the homogenous coordinates of the product with $1+\vec{w}_3$, the resulting twist $\vec{w}_3$ can be written,

$$\vec{w}_3 = \frac{1}{1-\vec{w}_1\cdot\vec{w}_2}\big(\vec{w}_1+\vec{w}_2+\vec{w}_1\times\vec{w}_2\big). \tag{3.5}$$

3.2.2 Baker-Campell-Hausdorff Like Formula for $SE(3)$ Using 4×4 Cayley Map

The Baker-Campell-Hausdorff like formula for $SE(3)$ using 4×4 Cayley map will be investigated. The formula is given by the theorem (3.2.1).

Theorem 3.2.1. If the product of the group elements $\text{Cay}_4(z_1)$ and $\text{Cay}_4(z_2)$ of $SE(3)$ gives the group element $\text{Cay}_4(z_3)$, then the corresponding Lie algebra element z_3 can be written in terms of z_1 and z_2 and their iterated products as follows:

$$\begin{aligned}z_3 = \frac{1}{1-\vec{w}_1\cdot\vec{w}_2}\Big((1-(\vec{w}_1\cdot\vec{w}_2)\alpha - |\vec{w}_2|^2\beta)z_1 + (1+|\vec{w}_1|^2\alpha + (\vec{w}_1\cdot\vec{w}_2)\beta)z_2\\ + z_1\times z_2 + \alpha(z_1\times(z_1\times z_2)) + \beta(z_2\times(z_1\times z_2))\Big)\end{aligned} \tag{3.6}$$

where

$$\alpha = \frac{(\vec{w}_2\cdot\vec{u}_2)(2(\vec{w}_1\cdot\vec{u}_1)+(\vec{w}_1\cdot\vec{u}_2+\vec{w}_2\cdot\vec{u}_1))}{4(\vec{w}_1\cdot\vec{u}_1)(\vec{w}_2\cdot\vec{u}_2)-(\vec{w}_1\cdot\vec{u}_2+\vec{w}_2\cdot\vec{u}_1)^2},$$

$$\beta = -\frac{(\vec{w}_1 \cdot \vec{u}_1)(2(\vec{w}_2 \cdot \vec{u}_2) + (\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1))}{4(\vec{w}_1 \cdot \vec{u}_1)(\vec{w}_2 \cdot \vec{u}_2) - (\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1)^2}$$

and the twists $z_1 = \vec{w}_1 + \varepsilon\vec{u}_1$ and $z_2 = \vec{w}_2 + \varepsilon\vec{u}_2$.

Proof. Here we seek a multiplication formula for twists related to the 4×4 Cayley map. Specifically we are considering the relation,

$$\text{Cay}_4(z_1)\,\text{Cay}_4(z_2) = \text{Cay}_4(z_3)$$

and looking for an explicit form for z_3 in terms of z_1 and z_2. We seek a generalisation of this to the case when the two twists are general elements of the Lie algebra to the group of rigid-body displacements $SE(3)$.

Writing $z_1 = \vec{w}_1 + \varepsilon\vec{u}_1$ and similar for z_2 we obtain,

$$\text{Cay}_4(z_1)\,\text{Cay}_4(z_2) = \big((1 + \vec{w}_1) + \varepsilon(-\vec{w}_1.\vec{u}_1 + \vec{u}_1)\big)\big((1 + \vec{w}_2) + \varepsilon(-\vec{w}_2.\vec{u}_2 + \vec{u}_2)\big).$$

Using the Gibbs relation to expand the quaternion products gives,

$$\begin{aligned} a_0 &= 1 - \vec{w}_1 \cdot \vec{w}_2, & c_0 &= -(\vec{w}_1 + \vec{w}_2) \cdot (\vec{u}_1 + \vec{u}_2), \\ \vec{a} &= \vec{w}_1 + \vec{w}_2 + \vec{w}_1 \times \vec{w}_2, & \vec{c} &= \vec{u}_1 + \vec{u}_2 - (\vec{w}_1 \cdot \vec{u}_1)\vec{w}_2 - (\vec{w}_2 \cdot \vec{u}_2)\vec{w}_1 \\ & & & \quad + \vec{w}_1 \times \vec{u}_2 + \vec{u}_1 \times \vec{w}_2. \end{aligned}$$

Applying the inverse Cayley map to this we obtain,

$$z_3 = \frac{1}{1 - \vec{w}_1 \cdot \vec{w}_2}\big(z_1 + z_2 + z_1 \times z_2 + \zeta + \xi\big). \tag{3.7}$$

The twists ζ and ξ have infinite pitch and are given by,

$$\zeta = -(\vec{w}_1 \cdot \vec{u}_1)\varepsilon z_2 \quad \text{and} \quad \xi = -(\vec{w}_2 \cdot \vec{u}_2)\varepsilon z_1,$$

where $\varepsilon z_1 = \varepsilon\vec{w}_1$ and $\varepsilon z_2 = \varepsilon\vec{w}_2$. This form of the result is essentially the result of Huang & Chen (1994), that the twist of a product of displacements can be written as a

sum of 5 twists. Here we seek a little more, we want to describe the two translational vectors ζ and ξ in terms of iterated commutators, in line with the Baker-Campbell-Hausdorff formula for Lie algebras.

Using the results of section 2.3.1 we can write,

$$z_3 = \frac{1}{1-\vec{w}_1\cdot\vec{w}_2}\Big((1-(\vec{w}_1\cdot\vec{w}_2)\alpha-|\vec{w}_2|^2\beta)z_1+(1+|\vec{w}_1|^2\alpha+(\vec{w}_1\cdot\vec{w}_2)\beta)z_2 \\ + z_1\times z_2+\alpha(z_1\times(z_1\times z_2))+\beta(z_2\times(z_1\times z_2))\Big) \quad (3.8)$$

where

$$\alpha = \frac{(\vec{w}_2\cdot\vec{u}_2)(2(\vec{w}_1\cdot\vec{u}_1)+(\vec{w}_1\cdot\vec{u}_2+\vec{w}_2\cdot\vec{u}_1))}{4(\vec{w}_1\cdot\vec{u}_1)(\vec{w}_2\cdot\vec{u}_2)-(\vec{w}_1\cdot\vec{u}_2+\vec{w}_2\cdot\vec{u}_1)^2}$$

and

$$\beta = -\frac{(\vec{w}_1\cdot\vec{u}_1)(2(\vec{w}_2\cdot\vec{u}_2)+(\vec{w}_1\cdot\vec{u}_2+\vec{w}_2\cdot\vec{u}_1))}{4(\vec{w}_1\cdot\vec{u}_1)(\vec{w}_2\cdot\vec{u}_2)-(\vec{w}_1\cdot\vec{u}_2+\vec{w}_2\cdot\vec{u}_1)^2}.$$

□

3.2.3 Baker-Campell-Hausdorff Like Formula for SE(3) Using 6×6 *Cayley Map*

For the group of rotations, the formula is the same with the formula for 4×4 Cayley Map, see (3.5). This similarity can be seen easily from the relation between Cayley maps, see section (3.1.6). The twists corresponding to the same rotations are equal.

The Baker-Campell-Hausdorff like formula for $SE(3)$ using 6×6 Cayley map will be given by the theorem (3.2.2).

Theorem 3.2.2. If the product of the group elements $\text{Cay}_6(z_1)$ and $\text{Cay}_6(z_2)$ of $SE(3)$ gives the group element $\text{Cay}_6(z_3)$, then the corresponding Lie algebra element z_3 can

be written in terms of z_1 and z_2 and their iterated products as follows:

$$z_3 = \frac{1}{1-\vec{w}_1\cdot\vec{w}_2}\big((1-(\vec{w}_1\cdot\vec{w}_2)\rho-|\vec{w}_2|^2\delta)z_1+(1+|\vec{w}_1|^2\rho+(\vec{w}_1\cdot\vec{w}_2)\delta)z_2 \\ + z_1\times z_2+\rho(z_1\times(z_1\times z_2))+\delta(z_2\times(z_1\times z_2)) \\ +\gamma\big((\vec{w}_1\cdot\vec{w}_2)z_1\times(z_1\times(z_1\times z_2))-|\vec{w}_1|^2 z_1\times(z_2\times(z_1\times z_2))\big)\big), \quad (3.9)$$

where,

$$\rho = -\frac{(\vec{w}_1\cdot\vec{u}_2+\vec{u}_1\cdot\vec{w}_2)(2(\vec{w}_2\cdot\vec{u}_2)+(\vec{w}_1\cdot\vec{u}_2+\vec{u}_1\cdot\vec{w}_2))}{(1-\vec{w}_1\cdot\vec{w}_2)(4(\vec{w}_1\cdot\vec{u}_1)(\vec{w}_2\cdot\vec{u}_2)-(\vec{w}_1\cdot\vec{u}_2+\vec{w}_2\cdot\vec{u}_1)^2)}$$

and

$$\delta = \frac{(\vec{w}_1\cdot\vec{u}_2+\vec{u}_1\cdot\vec{w}_2)(2(\vec{w}_1\cdot\vec{u}_1)+(\vec{w}_1\cdot\vec{u}_2+\vec{u}_1\cdot\vec{w}_2))}{(1-\vec{w}_1\cdot\vec{w}_2)(4(\vec{w}_1\cdot\vec{u}_1)(\vec{w}_2\cdot\vec{u}_2)-(\vec{w}_1\cdot\vec{u}_2+\vec{w}_2\cdot\vec{u}_1)^2)},$$

$$\gamma = \frac{\vec{w}_1\cdot\vec{u}_2+\vec{u}_1\cdot\vec{w}_2}{(1-\vec{w}_1\cdot\vec{w}_2)(|\vec{w}_1|^2(\vec{w}_1\cdot\vec{u}_2+\vec{u}_1\cdot\vec{w}_2)-2(\vec{w}_1\cdot\vec{u}_1)(\vec{w}_1\cdot\vec{w}_2))}.$$

Proof. The product of two group elements written as the image of a pair of twists under the Cayley map associated with the adjoint representation of the group it then,

$$\mathrm{Cay}_6(z_1)\,\mathrm{Cay}_6(z_2) = \\ \big(1+w_1+\varepsilon\big(-\frac{\vec{w}_1\cdot\vec{u}_1}{1+|\vec{w}_1|^2}(1+\vec{w}_1)+\vec{u}_1\big)\big) \\ (1+\vec{w}_2+\varepsilon(-\frac{\vec{w}_2\cdot\vec{u}_2}{1+|\vec{w}_2|^2}(1+\vec{w}_2)+\vec{u}_2)).$$

Using the Gibbs relation to expand the quaternion products gives,

$$a_0 = 1-\vec{w}_1\cdot\vec{w}_2,$$
$$\vec{a} = \vec{w}_1+\vec{w}_2+\vec{w}_1\times\vec{w}_2,$$
$$c_0 = -((\frac{\vec{w}_1\cdot\vec{u}_1}{1+|\vec{w}_1|^2}+\frac{\vec{w}_2\cdot\vec{u}_2}{1+|\vec{w}_2|^2})(1-\vec{w}_1\cdot\vec{w}_2)+\vec{u}_1\cdot\vec{w}_2+\vec{w}_1\cdot\vec{u}_2),$$
$$\vec{c} = -(\frac{\vec{w}_1\cdot\vec{u}_1}{1+|\vec{w}_1|^2}+\frac{\vec{w}_2\cdot\vec{u}_2}{1+|\vec{w}_2|^2})(\vec{w}_1+\vec{w}_2+\vec{w}_1\times\vec{w}_2)+\vec{u}_1+\vec{u}_2+\vec{w}_1\times\vec{u}_2+\vec{u}_1\times\vec{w}_2.$$

Applying the inverse 6×6 Cayley map to this gives

$$z_3 = \frac{1}{1 - \vec{w}_1 \cdot \vec{w}_2}\big(z_1 + z_2 + z_1 \times z_2 + \tau + \mu + \kappa\big), \tag{3.10}$$

where the twists τ, μ and κ have infinite pitch and are given by,

$$\tau = (\frac{\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1}{1 - \vec{w}_1 \cdot \vec{w}_2})\varepsilon\vec{w}_1 \quad , \quad \mu = (\frac{\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1}{1 - \vec{w}_1 \cdot \vec{w}_2})\varepsilon\vec{w}_2$$

and

$$\kappa = (\frac{\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1}{1 - \vec{w}_1 \cdot \vec{w}_2})\varepsilon\vec{w}_1 \times \vec{w}_2.$$

Using the relations given in (2.9), (2.10) and (2.11) above this can be written in terms of z_1, z_2 and their iterated products.

$$\begin{aligned} z_3 = \frac{1}{1 - \vec{w}_1 \cdot \vec{w}_2}\big((1 - (\vec{w}_1 \cdot \vec{w}_2)\rho - |\vec{w}_2|^2\delta)z_1 + (1 + |\vec{w}_1|^2\rho + (\vec{w}_1 \cdot \vec{w}_2)\delta)z_2 \\ + z_1 \times z_2 + \rho(z_1 \times (z_1 \times z_2)) + \delta(z_2 \times (z_1 \times z_2)) \\ + \gamma\big((\vec{w}_1 \cdot \vec{w}_2)z_1 \times (z_1 \times (z_1 \times z_2)) - |\vec{w}_1|^2 z_1 \times (z_2 \times (z_1 \times z_2))\big)\big), \end{aligned} \tag{3.11}$$

where

$$\rho = -\frac{(\vec{w}_1 \cdot \vec{u}_2 + \vec{u}_1 \cdot \vec{w}_2)(2(\vec{w}_2 \cdot \vec{u}_2) + (\vec{w}_1 \cdot \vec{u}_2 + \vec{u}_1 \cdot \vec{w}_2))}{(1 - \vec{w}_1 \cdot \vec{w}_2)(4(\vec{w}_1 \cdot \vec{u}_1)(\vec{w}_2 \cdot \vec{u}_2) - (\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1)^2)}$$

and

$$\delta = \frac{(\vec{w}_1 \cdot \vec{u}_2 + \vec{u}_1 \cdot \vec{w}_2)(2(\vec{w}_1 \cdot \vec{u}_1) + (\vec{w}_1 \cdot \vec{u}_2 + \vec{u}_1 \cdot \vec{w}_2))}{(1 - \vec{w}_1 \cdot \vec{w}_2)(4(\vec{w}_1 \cdot \vec{u}_1)(\vec{w}_2 \cdot \vec{u}_2) - (\vec{w}_1 \cdot \vec{u}_2 + \vec{w}_2 \cdot \vec{u}_1)^2)},$$

$$\gamma = \frac{\vec{w}_1 \cdot \vec{u}_2 + \vec{u}_1 \cdot \vec{w}_2}{(1 - \vec{w}_1 \cdot \vec{w}_2)(|\vec{w}_1|^2(\vec{w}_1 \cdot \vec{u}_2 + \vec{u}_1 \cdot \vec{w}_2) - 2(\vec{w}_1 \cdot \vec{u}_1)(\vec{w}_1 \cdot \vec{w}_2))}.$$

□

CHAPTER FOUR
ON PLANE-SYMMETRIC RIGID-BODY MOTIONS

We reflect a rigid body in successive planes of a one parameter family of planes to obtain a plane symmetric motion. To make this a rigid-body motion we begin by reflecting the body first in a fixed initial plane and then reflecting in the next plane of the family. We will begin this chapter by giving the definition of plane symmetric motions using the 4×4 standard homogeneous matrix representation of rigid body motions.

4.1 General Plane Symmetric Motions

To define plane symmetric motions, firstly we need to understand reflections in a plane. Consider a plane with unit normal $\vec{u}$ passing through a point B, see figure (Figure 4.1). Let $\vec{p}$ be any point in space and $\vec{p'}$ be the reflection of $\vec{p}$ in the given plane.

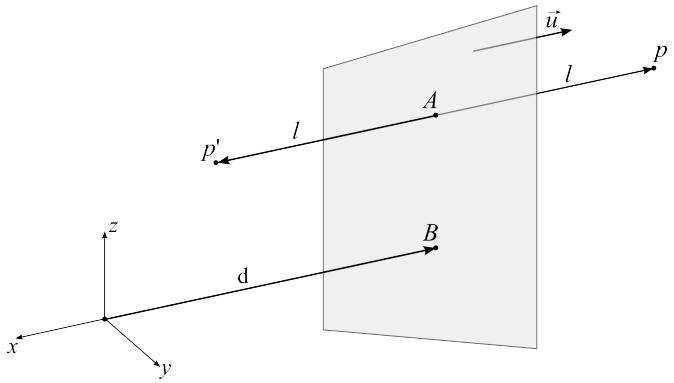


Figure 4.1 Reflection in a plane

The vectors $\vec{A} = \vec{p} - l\vec{u}$ and $\vec{B} = d\vec{u}$ where $\vec{A}$ is the midpoint of the line segment from $\vec{p}$ to $\vec{p'}$, $\vec{B}$ is the foot of the perpendicular drawn from the origin to the plane, the distance between the origin and the plane is d and l is the perpendicular distance of $\vec{p}$ to the plane. The vector from A to B can be expressed in terms of these quantities as

$d\vec{u} - (\vec{p} - l\vec{u})$. Now the equation of the plane is

$$(d\vec{u} - \vec{p} + l\vec{u}) \cdot \vec{u} = d - \vec{p} \cdot \vec{u} + l = 0.$$

Hence $l = \vec{p} \cdot \vec{u} - d$, so

$$\begin{aligned} \vec{p'} &= \vec{p} - 2(-d + \vec{p} \cdot \vec{u})\vec{u} \\ &= \vec{p} + 2d\vec{u} - 2(\vec{p} \cdot \vec{u})\vec{u} \\ &= \vec{p} + 2d\vec{u} - 2\vec{u}\vec{u}^t\vec{p}. \end{aligned}$$

Reflection in a plane with unit normal vector $\vec{u}$ can be represented by a 4×4 homogeneous matrix partitioned as,

$$M = \begin{pmatrix} I - 2\vec{u}\vec{u}^T & 2d\vec{u} \\ 0 & 1 \end{pmatrix}$$

where d is the perpendicular distance from the plane to the origin, see Figure 4.1. Notice that an arbitrary point $\vec{p}$ is transformed to the point $\vec{p'}$ by such a reflection and

$$M\tilde{p} = \begin{pmatrix} I - 2\vec{u}\vec{u}^T & 2d\vec{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{p} \\ 1 \end{pmatrix}.$$

Here $\tilde{p}$ indicates the extended vector; $\tilde{p} = (\vec{p}, 1)^T$. Successive reflections in a continuous one-parameter family of planes gives a plane symmetric motion. If a rigid body is being reflected then clearly the reflection in planes will change the orientation of the body. So, as an alternative description of a plane-symmetric motion an initial plane can be fixed, the motion is then given by first reflecting in the fixed plane and then reflecting in the successive planes in the family as before. If the fixed plane is the plane in the family associated with parameter value 0, then the rigid-body motion passes through the identity in the group of rigid-body displacements. It is also simple to see that subsequent displacements are pure rotations about lines in the fixed plane, or possibly translations perpendicular to the fixed plane.

In terms of 4×4 homogeneous matrices representing rigid-body displacements a plane symmetric motion can then be written as,

$$G(\mu) = \begin{pmatrix} I - 2\vec{u}(\mu)\vec{u}(\mu)^T & 2d(\mu)\vec{u}(\mu) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I - 2\vec{u}_0\vec{u}_0^T & 2d_0\vec{u}_0 \\ 0 & 1 \end{pmatrix},$$

where $\vec{u}_0$ and d_0 are the unit normal and perpendicular distance to the origin for the initial, fixed plane. For brevity, the parameter of the motion μ will not be written in the following .

The twist velocity of such a motion can be computed as follows, see (1.3),

$$S_d = \dot{G}G^{-1} = 2 \begin{pmatrix} \dot{\vec{u}}\vec{u}^T - \vec{u}\dot{\vec{u}}^T & \dot{d}\vec{u} - d\dot{\vec{u}} \\ 0 & 0 \end{pmatrix},$$

where

$$\dot{G} = 2 \begin{pmatrix} -\dot{\vec{u}}\vec{u}^T - \vec{u}\dot{\vec{u}}^T & \dot{d}\vec{u} + d\dot{\vec{u}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I - 2\vec{u}_0\vec{u}_0^T & 2d_0\vec{u}_0 \\ 0 & 1, \end{pmatrix}$$

and

$$G^{-1} = \begin{pmatrix} I - 2\vec{u}_0\vec{u}_0^T & 2d_0\vec{u}_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I - 2\vec{u}\vec{u}^T & 2d\vec{u} \\ 0 & 1 \end{pmatrix}.$$

Here the dot denotes differentiation with respect to the parameter μ. It is often more convenient to use a 6-vector representation of these Lie algebra elements, see equation 1.6. Note that since $\vec{u}$ is a unit vector, $\vec{u} \cdot \vec{u} = 1$ and hence $\vec{u} \cdot \dot{\vec{u}} = 0$. In partitioned form the velocity twist of a general plane symmetric motion is given by,

$$\mathbf{s}_d = 2 \begin{pmatrix} \vec{u} \times \dot{\vec{u}} \\ \dot{d}\vec{u} - d\dot{\vec{u}} \end{pmatrix} = 2 \begin{pmatrix} \vec{u} \times \dot{\vec{u}} \\ \vec{m} \times (\vec{u} \times \dot{\vec{u}}) \end{pmatrix} \tag{4.1}$$

where $\vec{m} = d\vec{u} + \frac{\dot{d}}{|\dot{\vec{u}}|^2}\dot{\vec{u}}$ is the reference point on the line. The reference point $\vec{m}$ can be found using Plücker coordinates of this line such that,

$$\vec{m} = \frac{\vec{\omega} \times \vec{v}}{|\vec{\omega}|^2}$$

where $\vec{\omega} = 2(\vec{u} \times \dot{\vec{u}})$ and $\vec{v} = 2(\dot{d}\vec{u} - d\dot{\vec{u}})$, notice that $|\vec{\omega}|^2 = 4|\dot{\vec{u}}|^2$. More generally, any point on the line can be written as $\vec{m} = d\vec{u} + \frac{\dot{d}}{|\dot{\vec{u}}|^2}\dot{\vec{u}} + \lambda\vec{u} \times \dot{\vec{u}}$, where λ is arbitrary. This shows,

Theorem 4.1.1. The velocity twist of a plane symmetric motion always has pitch 0.

Proof. The pitch of a motion is described as the ratio of linear and angular velocities, such that

$$p = \frac{\vec{\omega} \cdot \vec{v}}{|\vec{\omega}|^2}.$$

So the pitch of a plane symmetric motion is computed as,

$$p = \frac{(\vec{u} \times \dot{\vec{u}}) \cdot (\dot{d}\vec{u} - d\dot{\vec{u}})}{|\vec{u} \times \dot{\vec{u}}|^2}.$$

Since the vector $\vec{u} \times \dot{\vec{u}}$ is perpendicular to the vectors $\vec{u}$ and $\dot{\vec{u}}$, the pitch of the plane symmetric motions is always zero. □

Study described this type of motions as a Ribaucour motion, Study (1913), see also Selig & Carricato (2017). Since the velocity twist of a plane symmetric motion has zero pitch, we can easily write the fixed axode of the motion, but also it can be shown that,

Theorem 4.1.1. *The fixed axode of a plane-symmetric motion is a developable ruled surface.*

Proof. If a ruled surface has zero distribution parameter then it is developable surface.

Let assume a ruled surface is given in the form,

$$\vec{m} + \lambda(\vec{u} \times \dot{\vec{u}})$$

then the distribution parameter of the surface is given by,

$$\delta_d = \frac{\det\left(\dot{\vec{m}},\ (\vec{u} \times \dot{\vec{u}}),\ (\vec{u} \times \ddot{\vec{u}})\right)}{|\vec{u} \times \ddot{\vec{u}}|^2},$$

see for example (Willmore, 2013, ch. III). Expanding the determinant as a scalar triple product and remembering that $\vec{u} \cdot \dot{\vec{u}} = 0$, gives

$$\delta_d = \frac{(\dot{\vec{m}} \cdot \vec{u})(\vec{u} \times \dot{\vec{u}}) \cdot \ddot{\vec{u}}}{|\vec{u} \times \ddot{\vec{u}}|^2}.$$

Expanding $\dot{\vec{m}}$ finally gives,

$$\delta_d = \frac{\dot{d}}{|\vec{u} \times \ddot{\vec{u}}|^2}\big(\vec{u} \times (\dot{\vec{u}} \times \ddot{\vec{u}})\big)\big(1 + \frac{\vec{u} \cdot \dddot{\vec{u}}}{|\dot{\vec{u}}|^2}\big)$$

where $\dot{\vec{m}} = \dot{d}\vec{u} + \frac{\ddot{d}|\dot{\vec{u}}|^2 - 2\dot{d}(\dot{\vec{u}} \cdot \ddot{\vec{u}}) + d|\dot{\vec{u}}|^4}{|\dot{\vec{u}}|^4}\dot{\vec{u}} + \frac{\dot{d}}{|\dot{\vec{u}}|^2}\ddot{\vec{u}}$. But since $\vec{u} \cdot \dot{\vec{u}} = 0$, differentiating again gives $\dot{\vec{u}} \cdot \dot{\vec{u}} + \vec{u} \cdot \ddot{\vec{u}} = 0$ and hence $\delta_d = 0$. □

It is well known that there are just three possibilities, (Eisenhart, 1964, Chap. 1):

1. The planes are tangent to a cone.
2. The planes are tangent to a cylinder.
3. The fixed axode is the tangent developable to some curve.

If the planes are tangent to a cone all the planes will contain the vertex of the cone. The successive displacements will be rotations about lines in the initial plane passing through the vertex of the cone. So in this case the rigid-body motion will be a sequence of rotations about a fixed point, the vertex of the cone. That is, a spherical motion.

When the planes generating the motion are tangent to a general cylinder, the motion is clearly a planar motion.

The moving axode of a general plane-symmetric motion can be found in a similar fashion. The twist velocity of the motion in the moving coordinate frame is given by,

$$S_b = G^{-1}\dot{G} = G^{-1}S_d G.$$

This is a conjugation in the group of the fixed-frame velocity twist, hence the body-fixed velocity twist will have the same pitch as the fixed-frame velocity. That is, the body-fixed velocity twist will also consist of pure rotations and the body-fixed velocity twists can be thought as the lines generating the moving axode of the motion. As short computation shows that,

$$S_b = 2F_0 \begin{pmatrix} \vec{u}\dot{\vec{u}}^T - \dot{\vec{u}}\vec{u}^T & d\dot{\vec{u}} - \dot{d}\vec{u} \\ 0 & 0 \end{pmatrix} F_0$$

where

$$F_0 = \begin{pmatrix} I - 2\vec{u}_0\vec{u}_0^T & 2d_0\vec{u}_0 \\ 0 & 1 \end{pmatrix},$$

is the reflection in the initial plane.

Suppose the moving axode of the motion is given by,

$$S_b = 2 \begin{pmatrix} \Omega_b & \vec{v}_b \\ 0 & 0 \end{pmatrix}.$$

Then,

$$\Omega_b = (I - 2\vec{u}_0\vec{u}_0^T)(\vec{u}\dot{\vec{u}}^T - \dot{\vec{u}}\vec{u}^T)(I - 2\vec{u}_0\vec{u}_0^T) \tag{4.2}$$

and

$$\vec{v}_b = (I - 2\vec{u}_0\vec{u}_0^T)(2d_0(\vec{u}\dot{\vec{u}}^T - \dot{\vec{u}}\vec{u}^T)\vec{u}_0 + d\dot{\vec{u}} - \dot{d}\vec{u}). \tag{4.3}$$

Multiplying the matrices in (4.2) gives the matrix Ω_b and the corresponding vector $\vec{\omega}_b$ as follows,

$$\Omega_b = \vec{u}\dot{\vec{u}}^T - \dot{\vec{u}}\vec{u}^T - 2\dot{\vec{u}}^T\vec{u}_0(\vec{u}\vec{u}_0^T - \vec{u}_0\vec{u}^T) + 2\vec{u}_0^T\vec{u}(\dot{\vec{u}}\vec{u}_0^T - \vec{u}_0\dot{\vec{u}}^T)$$

and

$$\begin{aligned} \vec{\omega}_b =& \dot{\vec{u}} \times \vec{u} - 2(\dot{\vec{u}} \cdot \vec{u}_0)\vec{u}_0 \times \vec{u} + 2(\vec{u}_0 \cdot \vec{u})\vec{u}_0 \times \dot{\vec{u}} \\ =& \dot{\vec{u}} \times \vec{u} - 2\vec{u}_0 \times (\vec{u}_0 \times (\vec{u} \times \dot{\vec{u}})) \\ =& \vec{u} \times \dot{\vec{u}} - 2(\vec{u}_0 \cdot (\vec{u} \times \dot{\vec{u}}))\vec{u}_0. \end{aligned}$$

So, the 3×3 anti-symmetric matrix corresponding to the vector $\vec{\omega}_b$ can be written as,

$$\Omega_b = (I - 2\vec{u}_0\vec{u}_0^T)(\vec{u} \times \dot{\vec{u}}).$$

Similarly multiplying the matrices in (4.3) gives the vector $\vec{v}_b$ explicitly,

$$\begin{aligned}\vec{v}_b &= 2d_0((\dot{\vec{u}} \cdot \vec{u}_0)\vec{u} - (\vec{u}_0 \cdot \vec{u})\dot{\vec{u}}) + d\dot{\vec{u}} - \dot{d}\vec{u} - 2d(\dot{\vec{u}} \cdot \vec{u}_0)\vec{u}_0 + 2\dot{d}(\vec{u}_0 \cdot \vec{u})\vec{u}_0 \\ &= 2d_0\vec{u}_0 \times (\vec{u} \times \dot{\vec{u}}) - (I - 2\vec{u}_0\vec{u}_0^T)(\dot{d}\vec{u} - d\dot{\vec{u}}).\end{aligned}$$

As for the fixed-frame velocity, this can be cast into a 6-vector form,

$$\mathbf{s}_b = 2\begin{pmatrix} \mathcal{U}_0 & 0 \\ 2d_0U_0 & -\mathcal{U}_0 \end{pmatrix}\begin{pmatrix} \vec{u} \times \dot{\vec{u}} \\ \vec{m} \times (\vec{u} \times \dot{\vec{u}}) \end{pmatrix}. \tag{4.4}$$

Here $\mathcal{U}_0 = I - 2\vec{u}_0\vec{u}_0^T$ and the matrix U_0 is the 3×3 anti-symmetric matrix corresponding to $\vec{u}_0$, that is $U_0\vec{x} = \vec{u}_0 \times \vec{x}$ for any vector $\vec{x}$. As above, $\vec{m} = d\vec{u} + \frac{\dot{d}}{|\dot{\vec{u}}|^2}\dot{\vec{u}}$. From this we see that:

Theorem 4.1.2. *The moving axode of a plane-symmetric motion is the reflection of its fixed axode in the initial plane of the motion.*

Proof. Suppose

$$\mathbf{s} = \begin{pmatrix} \vec{\omega} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \vec{\omega} \\ \vec{c} \times \vec{\omega} \end{pmatrix} \tag{4.5}$$

is the Plücker coordinates of a line. Here $\vec{\omega}$ is a vector pointing along the direction of the line and the vector $\vec{v} = \vec{c} \times \vec{\omega}$ is the moment of the line. The vector $\vec{c}$ is the position vector of any point on the line. Consider a pair of points on the line $\vec{p_1} = \vec{c}$ and $\vec{p_2} = \vec{c} + \vec{\omega}$ to investigate the effect of a reflection F_0 on the Plücker coordinates of the line. Then reflecting these two points in the initial plane gives,

$$\vec{p_1'} = \vec{c} - 2(\vec{u}_0 \cdot \vec{c})\vec{u}_0 + 2d_0\vec{u}_0$$

and

$$\vec{p_2'} = \vec{c} + \vec{\omega} - 2(\vec{u}_0 \cdot \vec{c})\vec{u}_0 - 2(\vec{u}_0 \cdot \vec{\omega})\vec{u}_0 + 2d_0\vec{u}_0.$$

Now, the reflection vector of the direction vector ω in the initial plane can be written as,

$$\begin{aligned}\vec{\omega}' &= \vec{p_2'} - \vec{p_1'} \\ &= \vec{\omega} - 2(\vec{u}_0 \cdot \vec{\omega})\vec{u}_0.\end{aligned}$$

To find the moment vector of the reflected line, any point on the reflected line can be chosen, clearly $\vec{p_1'}$. Then the cross product of $\vec{\omega}'$ and $\vec{p_1'}$ gives the moment vector of the reflected line as follows,

$$\begin{aligned}\vec{p_1'} \times \vec{\omega}' &= \vec{c} \times \vec{\omega} - 2(\vec{u}_0 \cdot \vec{c})(\vec{u}_0 \times \vec{\omega}) - 2(\vec{u}_0 \cdot \vec{\omega})(c \times \vec{u}_0) + 2d_0(\vec{u}_0 \times \vec{\omega}) \\ &= \vec{c} \times \vec{\omega} - 2(-\vec{u}_0 \times ((\vec{c} \times \vec{u}_0) \times \vec{\omega}) - \vec{u}_0 \times (\vec{u}_0 \times (\vec{c} \times \vec{\omega}))) \\ &\quad -2\vec{u}_0 \times ((\vec{u}_0 \times \vec{c}) \times \vec{\omega}) + 2d_0(\vec{u}_0 \times \vec{\omega}).\end{aligned}$$

Hence,

$$\begin{aligned}\vec{p_1'} \times \vec{\omega}' =& \vec{c} \times \vec{\omega} - 2\vec{u}_0 \times (\vec{u}_0 \times (\vec{c} \times \vec{\omega})) + 2d_0(\vec{u}_0 \times \vec{\omega}) \\ =& -\vec{c} \times \vec{\omega} + 2(\vec{u}_0 \cdot (\vec{c} \times \vec{\omega}))\vec{u}_0 + 2d_0(\vec{u}_0 \times \vec{\omega}).\end{aligned}$$

Now, the effect of the reflection in the initial plane F_0 on the Plücker coordinates of the line **s** can be written by the 6×6 matrix,

$$\mathbf{s}' = \begin{pmatrix} \mathcal{U}_0 & 0 \\ 2d_0U_0 & -\mathcal{U}_0 \end{pmatrix} \mathbf{s}.$$

So the equation (4.4) can clearly be written as,

$$\mathbf{s}_b = \begin{pmatrix} \mathcal{U}_0 & 0 \\ 2d_0U_0 & -\mathcal{U}_0 \end{pmatrix} \mathbf{s}_d.$$

□

That is the moving axode of the motion is developable. Because reflections are isometries and isometries preserve the Gaussian curvature, also they preserve the

distrubition parameter for ruled surfaces. Moreover the moving axode has the same type as the fixed axode; if the family of planes has a fixed point then the axodes consist of a pair of cones. When the family of planes are parallel to a fixed direction, the axodes consist of a pair of cylinders. Otherwise the axodes are a pair of tangent developable ruled surfaces.

These results appear in (Bottema & Roth, 1990, ch. IX §8), the proofs above serve as an introduction to the notation and methods used in the rest of this thesis.

4.2 Reflections in the Osculating Planes of a Space Curve

It is well known that at each point on a regular curve, three planes can be found; the normal plane, osculating plane and rectifying plane. So as the parameter of the curve changes there are three 1-parameter families of planes determined by the curve. Each of these three families of planes can be taken as the continuous one-parameter family of planes determining a plane-symmetric motion.

To make it simpler to refer to these constructions the following definitions will be adopted.

Definition 4.2.1. The plane symmetric motion based on the osculating planes of a curve $\vec{\gamma}$ will be called the *osculating planes motion* of $\vec{\gamma}$. Likewise, the *normal planes motion* and the *rectifying planes motion* of $\vec{\gamma}$ will denote the plane-symmetric motions based on the normal planes and rectifying planes of $\vec{\gamma}$ respectively.

For the osculating planes motion, the instantaneous twist velocity is given by the intersection of consecutive planes, hence instantaneous rotations about the tangent lines to the original curve. The fixed axode is clearly the developable ruled surface generated by the tangent lines to the curve.

Consider a curve $\vec{\gamma}(\mu)$. For the osculating planes to this curve $\vec{u} = \vec{b}$ the unit binormal vector of the curve. The perpendicular distance to the origin can be found

from any point on the plane, for example the point $\vec{\gamma}$, so $d = \vec{\gamma} \cdot \vec{b}$. Substituting this into equation (4.1) gives the twist velocity as,

$$\mathbf{s}_d = -2\nu\tau \begin{pmatrix} \vec{b} \times \vec{n} \\ \vec{m} \times (\vec{b} \times \vec{n}) \end{pmatrix} = 2\nu\tau \begin{pmatrix} \vec{t} \\ \vec{m} \times \vec{t} \end{pmatrix},$$

where ν and τ are the speed and torsion of the curve respectively. As usual $\vec{t}$ and $\vec{n}$ denote the unit tangent and principal normal vector to the curve. The point $\vec{m}$ is a point on the axis of the instantaneous twist. From above we have that,

$$\vec{m} = d\vec{u} + \frac{\dot{d}}{|\dot{\vec{u}}|^2}\dot{\vec{u}} = (\vec{\gamma} \cdot \vec{b})\vec{b} + (\vec{\gamma} \cdot \vec{n})\vec{n}$$

A general point on the axis of the instantaneous twist is given by,

$$\vec{m}\,' = (\vec{\gamma} \cdot \vec{b})\vec{b} + (\vec{\gamma} \cdot \vec{n})\vec{n} + \lambda\vec{t},$$

where λ is arbitrary. Choosing $\lambda = (\vec{\gamma} \cdot \vec{t})$ shows that $\vec{\gamma}$ lies on this axis, since,

$$(\vec{\gamma} \cdot \vec{b})\vec{b} + (\vec{\gamma} \cdot \vec{n})\vec{n} + (\vec{\gamma} \cdot \vec{t})\vec{t} = \vec{\gamma},$$

as $\vec{t}$, $\vec{n}$ and $\vec{b}$ form and orthonormal system of unit vectors. This shows that the fixed axode consists of the tangent lines to the curve $\vec{\gamma}$. This confirms the general argument given at the beginning of this section, however the computation gives more information since it is clear that at each instant the magnitude of the instantaneous twist is $2\nu\tau$.

Note that this is really the general case for a tangent developable axode.

Theorem 4.2.2. *Any tangent developable ruled surface is the fixed axode of some plane-symmetric motion.*

Proof. For the planes generating the plane-symmetric motions all that is needed is to take the osculating planes to the cuspidal edge of the tangent developable ruled surface. □

4.3 Reflections in the Normal Planes of a Curve

It is well known that the envelope of the normal planes to a curve form a developable ruled surface. The cuspidal edge of this developable is known as the pole curve, see Forsyth (1912). Hence, the fixed axode of a plane-symmetric motion based on the normal planes to a curve $\vec{\gamma}$, will be the tangent lines to the pole curve of $\vec{\gamma}$.

In Selig & Carricato (2017) it was shown that a Bishop motion (or RMF motion) based on a curve $\vec{\gamma}$ has as its fixed axode the tangent lines to the pole curve of $\vec{\gamma}$. However, the normal planes symmetric motion and the Bishop motion are not the same. To understand this, the velocity twists of the two motions will be computed.

For the normal planes the relations $\vec{u} = \vec{t}$ and $d = \vec{\gamma} \cdot \vec{t}$ can be substituted into equation (4.1) to give,

$$\mathbf{s}_d = 2 \begin{pmatrix} \vec{t} \times (\nu\kappa)\vec{n} \\ \vec{m} \times \left(\vec{t} \times (\nu\kappa)\vec{n}\right) \end{pmatrix} = 2\nu\kappa \begin{pmatrix} \vec{b} \\ \vec{m} \times \vec{b} \end{pmatrix}.$$

Here, the point $\vec{m}$ is given by,

$$\vec{m} = (\vec{\gamma} \cdot \vec{t})\vec{t} + (\frac{1}{\kappa} + \vec{\gamma} \cdot \vec{n})\vec{n}.$$

Arguing as in the previous section, this can be written as,

$$\mathbf{s}_d = 2\nu\kappa \begin{pmatrix} \vec{b} \\ (\vec{\gamma} + \frac{1}{\kappa}\vec{n}) \times \vec{b} \end{pmatrix}. \tag{4.6}$$

The Bishop motion based on the curve $\vec{\gamma}$ will be given by a curve in the group $SE(3)$,

$$G = \begin{pmatrix} R & \vec{\gamma} \\ 0 & 1 \end{pmatrix}$$

where the rotation matrix R has columns given by the tangent and normal vectors of

the Bishop frame,

$$R = \left(\vec{t}\,\middle|\,\vec{n}_1\,\middle|\,\vec{n}_2\right).$$

To find the velocity twist the Bishop frame equations can be written as,

$$\begin{aligned}
\dot{\vec{t}} &= \nu(-k_2\vec{n}_1 + k_1\vec{n}_2) \times \vec{t}, \\
\dot{\vec{n}}_1 &= \nu(-k_2\vec{n}_1 + k_1\vec{n}_2) \times \vec{n}_1, \\
\dot{\vec{n}}_2 &= \nu(-k_2\vec{n}_1 + k_1\vec{n}_2) \times \vec{n}_2,
\end{aligned}$$

where the vector $-k_2\vec{n}_1 + k_1\vec{n}_2 = \kappa\vec{b}$ is the the binormal vector multiplied by the curvature, see Selig & Carricato (2017).

The velocity twist of this motion will be given by,

$$\dot{G}G^{-1} = \nu \begin{pmatrix} \kappa BR & \vec{t} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R^T & -R^T\vec{\gamma} \\ 0 & 1 \end{pmatrix} = \nu \begin{pmatrix} \kappa B & \vec{t} - \kappa\vec{b} \times \vec{\gamma} \\ 0 & 0 \end{pmatrix}$$

where B is the 3×3 anti-symmetric matrix corresponding to the binormal vector $\vec{b}$. The twist can be written as a partitioned 6-vector,

$$\mathbf{s}_d = \nu\kappa \begin{pmatrix} \vec{b} \\ (\vec{\gamma} + \frac{1}{\kappa}\vec{n}) \times \vec{b} \end{pmatrix}.$$

This proves the following,

Theorem 4.3.1. *Given a curve $\vec{\gamma}$ the Bishop motion based on the curve and the normal planes symmetric motion based on the curve have the same fixed axode, but the magnitude of the twist velocity of the normal planes motion is twice that of the Bishop motion.*

Notice also, that the moving axodes of the two motions are different. In theorem (4.1.2) it was shown that the moving axode of a plane symmetric motion is a reflection of its fixed axode in the initial plane of the motion. For a Bishop motion it is known that the moving axode is a sequence of lines lying in a plane, Selig & Carricato (2017).

4.4 Reflections in the Rectifying Planes of a Curve

The rectifying planes of a curve have normal vector $\vec{u} = \vec{n}$ and the distance to the origin of a rectifying plane is $d = \vec{\gamma} \cdot \vec{n}$. The velocity twist of a plane-symmetric motion based on such a family of planes is then,

$$\mathbf{s}_d = 2 \begin{pmatrix} \vec{n} \times \dot{\vec{n}} \\ \vec{m} \times (\vec{n} \times \dot{\vec{n}}) \end{pmatrix}$$

where,

$$\vec{n} \times \dot{\vec{n}} = \nu \vec{n} \times (-\kappa \vec{t} + \tau \vec{b}) = \nu(\tau \vec{t} + \kappa \vec{b}),$$

and points on the line are given by,

$$\vec{m} = d\vec{u} + \frac{\dot{d}}{|\dot{\vec{u}}|^2}\dot{\vec{u}} + \lambda \vec{u} \times \dot{\vec{u}} = (\vec{\gamma} \cdot \vec{n})\vec{n} + (\vec{\gamma} \cdot \dot{\vec{n}})\frac{\dot{\vec{n}}}{|\dot{\vec{n}}|^2} + \lambda \vec{n} \times \dot{\vec{n}}.$$

Notice that, $\vec{n}$ and $\dot{\vec{n}}$ are orthogonal, so if we choose the parameter λ to be,

$$\lambda = \left(\vec{\gamma} \cdot (\vec{n} \times \dot{\vec{n}})\right) \frac{\vec{n} \times \dot{\vec{n}}}{|\vec{n} \times \dot{\vec{n}}|^2}$$

then we can see that the curve $\vec{\gamma}$ lies on the fixed axode, that is, the velocity twist is given by,

$$\mathbf{s}_d = 2\nu \begin{pmatrix} \tau \vec{t} + \kappa \vec{b} \\ \vec{\gamma} \times (\tau \vec{t} + \kappa \vec{b}) \end{pmatrix}.$$

The cuspidal edge of the rectifying planes to a curve is known as the rectifying curve and it is well known that the original curve is a geodesic on the tangent developable to the rectifying curve, see (Forsyth, 1912, ch. I). Here, of course, the tangent developable is the fixed axode of the rectifying planes motion.

Suppose we let $\vec{\delta} = \frac{1}{\sqrt{\kappa^2+\tau^2}}(\tau \vec{t} + \kappa \vec{b})$. This is a unit vector in the direction of the Darboux vector to the curve. The velocity twist of a rectifying planes motion can be

written,

$$\mathbf{s}_d = \frac{2\nu}{\sqrt{\kappa^2+\tau^2}} \begin{pmatrix} \vec{\delta} \\ \vec{\gamma} \times \vec{\delta} \end{pmatrix}.$$

This shows that, the velocity twist of a rectifying planes motion is a line in the direction of the Darboux vector of the curve, passing through the curve. A Frenet-Serret motion, based on the same curve would have a velocity twist in the same direction but here the pitch of the twist is not zero and the axis of the twist does not meet the curve, see Selig & Carricato (2017).

This suggests another plane symmetric motion based on a curve. Imagine reflecting in the family of planes normal to the Darboux vector of the curve and passing through the curve at the current point on the curve. The fixed axode for this motion is then,

$$\mathbf{s}_d = 2 \begin{pmatrix} \vec{\delta} \times \dot{\vec{\delta}} \\ \vec{m} \times (\vec{\delta} \times \dot{\vec{\delta}}) \end{pmatrix},$$

see equation (4.1). Computations show that,

$$\dot{\vec{\delta}} = \left(\frac{\kappa\dot{\tau} - \tau\dot{\kappa}}{k^2+\tau^2}\right) \vec{n} \times \vec{\delta}, \qquad \vec{\delta} \times \dot{\vec{\delta}} = \left(\frac{\kappa\dot{\tau} - \tau\dot{\kappa}}{\kappa^2+\tau^2}\right) \vec{n}$$

and

$$\vec{m} = (\vec{\gamma}\cdot\vec{\delta})\vec{\delta} + \frac{1}{|\dot{\vec{\delta}}|^2}\frac{d(\vec{\gamma}\cdot\vec{\delta})}{d\mu}\dot{\vec{\delta}} + \lambda\vec{n} = \vec{\gamma} + \frac{\nu\tau}{\sqrt{k^2+\tau^2}}\dot{\vec{\delta}} + \lambda'\vec{n}$$

where λ' is arbitrary. This demonstrates the following:

Theorem 4.4.1. *Given a curve $\vec{\gamma}$, the twist velocity of the plane symmetric motion generated by the normal planes to the curve's Darboux vector $\vec{\delta}$, is given by,*

$$\mathbf{s}_d = 2\left(\frac{\kappa\dot{\tau} - \tau\dot{\kappa}}{k^2+\tau^2}\right) \begin{pmatrix} \vec{n} \\ (\vec{\gamma} + \vec{\zeta}) \times \vec{n} \end{pmatrix}$$

where,

$$\vec{\zeta} = \frac{\nu\tau(\kappa\dot{\tau} - \tau\dot{\kappa})}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\vec{n} \times \vec{\delta}$$

and, as usual, ν, κ and τ are the speed, curvature and torsion of the curve. So the lines of the fixed axode of this motion are parallel to the normals of the curve, but through a point displaced from the curve by $\vec{\zeta}$.

4.5 Example

As an example of a plane symmetric motion Bottema & Roth (1990), study the reflections in the osculating planes to a twisted cubic curve; $\vec{\gamma}(\mu) = (\mu^3, \mu^2, \mu)^T$. Here we look at the three plane symmetric motions defined by the osculating, normal and rectifying planes to this curve.

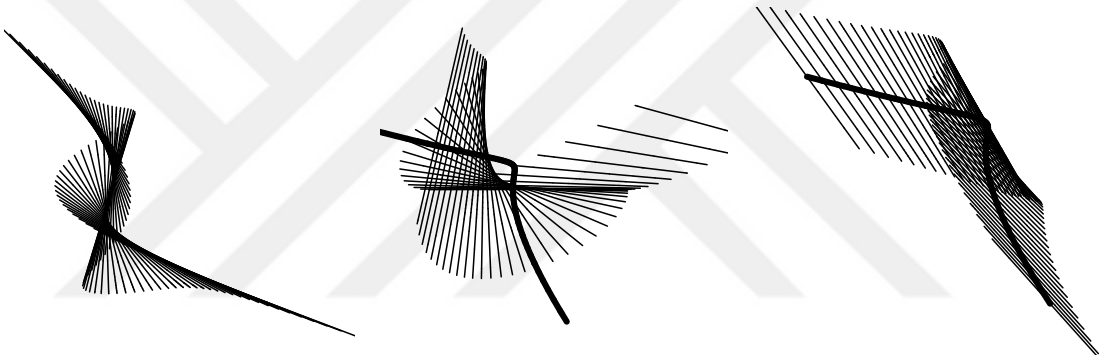


Figure 4.2 The fixed axodes of the osculating, normal and rectifying planes motions (left to right) associated to a twisted cubic curve.

The speed, curvature and torsion of the twisted cubic are respectively,

$$\nu = \sqrt{9\mu^4 + 4\mu^2 + 1},\ \kappa = 2\sqrt{\frac{9\mu^4 + 9\mu^2 + 1}{(9\mu^4 + 4\mu^2 + 1)^3}} \text{ and } \tau = \frac{-3}{9\mu^4 + 9\mu^2 + 1}.$$

Notice that these quantities satisfy the relation,

$$\nu^2\kappa^2\tau = -12.$$

In the following it is convenient to introduce the quantity $\sigma = \sqrt{9\mu^4 + 9\mu^2 + 1}$, so that $\tau = -3/\sigma^2$.

In terms of the above quantities the unit tangent, normal and binormal vectors to the curve are given by,

$$\vec{t}(\mu) = \frac{1}{\nu^2}\begin{pmatrix} 3\mu^2 \\ 2\mu \\ 1 \end{pmatrix}, \quad \vec{n}(\mu) = \frac{1}{\nu\sigma}\begin{pmatrix} 6\mu^3 + 3\mu \\ -9\mu^4 + 1 \\ -9\mu^3 - 2\mu \end{pmatrix} \quad \text{and} \quad \vec{b}(\mu) = \frac{1}{\sigma}\begin{pmatrix} -1 \\ 3\mu \\ -3\mu^2 \end{pmatrix}.$$

The 4×4 matrix representation of the osculating planes motion is given by,

$$G_o(\mu) = \begin{pmatrix} I - 2\vec{b}(\mu)\vec{b}^T(\mu) & 2d(\mu)\vec{b}\mu) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I - 2\vec{b}(0)\vec{b}^T(0) & 2d(0)\vec{b}(0) \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{\nu^2}\begin{pmatrix} 1 - 9\mu^2 - 9\mu^4 & 6\mu & -6\mu^2 & -2\mu^3 \\ -6\mu & 1 - 9\mu^2 + 9\mu^4 & 18\mu^3 & 6\mu^4 \\ 6\mu^2 & 18\mu^3 & 1 + 9\mu^2 - 9\mu^4 & -6\mu^5 \\ 0 & 0 & 0 & \nu^2 \end{pmatrix}$$

where $d(\mu) = -\mu^3/\sigma$. In Selig & Husty (2011), plane-symmetric motions are described as curves in a B-plane in the Study quadric. To see this we need to find the representation of the motion in dual quaternions, see Chapter 1. The dual quaternion representation of a rigid-body displacement is given by,

$$r + \varepsilon\frac{1}{2}rt = (a_0 + a_1 i + a_2 j + a_3 k) + \varepsilon(c_0 + c_1 i + c_2 j + c_3 k),$$

here g is the quaternion representing the rotation and t is the pure quaternion representing the translational part of the displacement. With $a_0, a_1, \ldots, c_3$ as homogeneous coordinates in a seven dimensional projective space, the Study quadric is the six dimensional quadric defined by the equation $a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0$, see Selig (2005).

To write group elements in terms of dual quaternions, we will use the map from the 4×4 representation of rigid body motions to the Study quadric, which was studied by

Selig (2010a). This map is given by,

$$a_0 = \frac{1}{2}\sqrt{Tr(R)+1}$$

and

$$A = \frac{1}{2\sqrt{Tr(R)+1}}(R - R^t),$$

then

$$c_0 + \vec{c} = \frac{1}{2}\vec{t}(a_0 + \vec{a})$$

where the matrix A is the corresponding skew-symmetric matrix to the vector $\vec{a} = a_1 i + a_2 j + a_3 k$.

The osculating planes motion based on the twisted cubic curve can be written as in terms of dual quaternions,

$$\begin{aligned} a_0 &= 1, & c_0 &= 0, \\ a_1 &= 0, & c_1 &= -\mu^3, \\ a_2 &= -3\mu^2, & c_2 &= 0, \\ a_3 &= -3\mu, & c_3 &= 0. \end{aligned}$$

Notice that this lies entirely in the B-plane determined by $a_1 = c_0 = c_2 = c_3 = 0$. Elements in this B-plane are all the pure rotations about axes lying in the yz-plane; the initial plane of the motion.

The velocity twist of the motion coincides with the fixed axode of the motion. Using Plücker coordinates this is given by,

$$\mathbf{s}_o = \begin{pmatrix} \vec{\omega}_o \\ \vec{v}_o \end{pmatrix} = \frac{6}{\nu^2} \begin{pmatrix} -3\mu^2 \\ -2\mu \\ -1 \\ \mu^2 \\ -2\mu^3 \\ \mu^4 \end{pmatrix}.$$

This is the example given in Bottema & Roth (1990) , the result there is slightly different to the above. The difference is accounted for by the slight difference in definition of these plane symmetric motions. Here, a plane symmetric motion is a reflection in a fixed plane followed by a reflection in the planes of a one-parameter family of planes, see section 4.1. This is done so that the motion is always a proper rigid-body motion.

For the normal planes motion on this twisted cubic, the 4×4 matrix representation of the normal planes motion is given by,

$$
\begin{aligned}
G_n(\mu) &= \begin{pmatrix} I - 2t(\vec{\mu})t(\vec{\mu})^t & 2d(\mu)t(\vec{\mu}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I - 2t(\vec{0})t(\vec{0})^t & 2d(0)t(\vec{0}) \\ 0 & 1 \end{pmatrix} \\
&= \tfrac{1}{\nu^2} \begin{pmatrix} -9\mu^4 + 4\mu^2 + 1 & -12\mu^3 & 6\mu^2 & -6\mu^3(3\mu^4 + 2\mu^2 + 1) \\ -12\mu^3 & 9\mu^4 - 4\mu^2 + 1 & 4\mu & -4\mu^2(3\mu^4 + 2\mu^2 + 1) \\ -6\mu^2 & -4\mu & -9\mu^4 - 4\mu^2 + 1 & -2(3\mu^4 + 2\mu^2 + 1) \\ 0 & 0 & 0 & \nu^2 \end{pmatrix}
\end{aligned}
$$

where $d(\mu) = (3\mu^5 + 2\mu^3 + \mu)/\sigma$. In terms of the dual quaternion representation of the normal planes motion is,

$$
\begin{aligned}
a_0 &= 1, & c_0 &= 0, \\
a_1 &= -2\mu, & c_1 &= 0, \\
a_2 &= 3\mu^2, & c_2 &= 0, \\
a_3 &= 0, & c_3 &= -3\mu^5 - 2\mu^3 - \mu.
\end{aligned}
$$

The B-plane this motion lies in is given by $a_3 = c_0 = c_1 = c_2 = 0$ and comprises the rotations about axes lying in the xy-plane.

The twist velocity of this motion is,

$$\mathbf{s}_n = \begin{pmatrix} \vec{\omega}_n \\ \vec{v}_n \end{pmatrix} = \frac{2}{\nu^2} \begin{pmatrix} -2 \\ 6\mu \\ -6\mu^2 \\ 27\mu^6 + 6\mu^4 - 3\mu^2 \\ 24\mu^5 + 8\mu^3 \\ 15\mu^4 + 6\mu^2 + 1 \end{pmatrix}.$$

Finally, the rectifying planes motion for the twisted cubic is given by,

$$\begin{aligned} G_r(\mu) &= \begin{pmatrix} I - 2n(\vec{\mu})n(\vec{\mu})^t & 2d(\mu)n(\vec{\mu}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I - 2n(\vec{0})n(\vec{0})^t & 2d(0)n(\vec{0}) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} R_r(\mu) & t_r(\mu) \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where

$$R_r(\mu) = \frac{9\mu^4 - 1}{\nu^2\sigma^2} \begin{pmatrix} 9\mu^4 + 5\mu^2 - 1 & -12\mu^3 - 6\mu & \frac{6\mu(9\mu^2+2)(2\mu^3+\mu)}{9\mu^4-1} \\ 12\mu^3 + 6\mu & \frac{81\mu^8 - 117\mu^6 - 90\mu^4 - 13\mu^2 + 1}{9\mu^4 - 1} & -18\mu^2 - 4\mu \\ \frac{6\mu(9\mu^2+2)(2\mu^3+\mu)}{9\mu^4-1} & 18\mu^2 + 4\mu & 9\mu^4 - 5\mu^2 - 1 \end{pmatrix}$$

and

$$t_r(\mu) = \frac{3\mu^4 + 6\mu^2 + 1}{\nu^2\sigma^2} \begin{pmatrix} 12\mu^5 + 6\mu^3 \\ -18\mu^6 + 2\mu^2 \\ -18\mu^5 - 4\mu^3 \end{pmatrix}$$

where the distance between the planes and the origin, $d(\mu) = -\mu^2(3\mu^4 + \mu^2 + 1)/\sigma\nu$.

The components of the motion on the Study quadric is given by,

$$\begin{aligned} a_0 &= 9\mu^4 - 1, & c_0 &= 0, \\ a_1 &= 9\mu^3 + 2\mu, & c_1 &= 0, \\ a_2 &= 0, & c_2 &= -3\mu^6 - 6\mu^4 - \mu^2, \\ a_3 &= 6\mu^3 + 3\mu, & c_3 &= 0. \end{aligned}$$

Its velocity twist is,

$$\mathbf{s}_r = \begin{pmatrix} \vec{\omega}_r \\ \vec{v}_r \end{pmatrix} = \frac{2}{\nu^2\sigma^2} \begin{pmatrix} -81\mu^6 - 54\mu^4 - 27\mu^2 - 2 \\ 30\mu^3 \\ -54\mu^6 - 81\mu^4 - 18\mu^2 - 3 \\ -54\mu^8 - 81\mu^6 - 48\mu^4 - 3\mu^2 \\ 54\mu^9 - 36\mu^5 - 24\mu^3 - 2\mu \\ 81\mu^8 + 84\mu^6 + 27\mu^4 + 2\mu^2 \end{pmatrix}.$$

The fixed axodes of these three motions are shown in Figure 4.2, the twisted cubic curve is also illustrated in these diagrams.

CHAPTER FIVE
ACCELERATION, INFLECTION POINTS AND BRESSE HYPERBOLOID OF PLANE SYMMETRIC MOTIONS

5.1 Acceleration

Let $\vec{p}_0$ be a point in the rigid-body when the value of the parameter is $\mu = 0$. Now, at subsequent values of the parameter the location of the point will be given by,

$$\begin{pmatrix} \vec{p}(\mu) \\ 1 \end{pmatrix} = G(\mu) \begin{pmatrix} \vec{p}_0 \\ 1 \end{pmatrix}.$$

Differentiating gives the velocity of the point,

$$\begin{pmatrix} \dot{\vec{p}}(\mu) \\ 0 \end{pmatrix} = \dot{G}(\mu) \begin{pmatrix} \vec{p}_0 \\ 1 \end{pmatrix} = \dot{G}(\mu)G^{-1}(\mu) \begin{pmatrix} \vec{p}(\mu) \\ 1 \end{pmatrix} = S_d \begin{pmatrix} \vec{p}(\mu) \\ 1 \end{pmatrix}.$$

The acceleration of the point can be found by differentiating again,

$$\begin{pmatrix} \ddot{\vec{p}}(\mu) \\ 0 \end{pmatrix} = (\dot{S}_d + S_d^2) \begin{pmatrix} \vec{p}(\mu) \\ 1 \end{pmatrix}.$$

Now, we will investigate the motion that the rigid-body is subject to is a plane-symmetric motion given by reflection in the osculating planes to some curve $\vec{\gamma}$.

Theorem 5.1.1. *At any instant of an osculating planes motion based on a space curve $\vec{\gamma}$ the acceleration centre of the motion is located at the current point on the curve $\vec{\gamma}$.*

Proof. The acceleration centre is the point whose acceleration is instantaneously zero. The acceleration centre of a plane symmetic motion generated by reflections in planes with unit normal vector $\vec{u}$ can be found using the results in Selig (2011) as follows,

$$a_c = \frac{d(\vec{u} \times \ddot{\vec{u}}) - \dot{d}(\vec{u} \times \dot{\vec{u}}) + \ddot{d}(\vec{u} \times \vec{u})}{\det(\vec{u}, \dot{\vec{u}}, \ddot{\vec{u}})}. \tag{5.1}$$

For osculating planes motion the vector $\vec{u} = \vec{b}$ and the distance $d = \vec{\gamma} \cdot \vec{b}$, then the first derivatives are,

$$\dot{\vec{u}} = -\nu\tau\vec{n} \quad \text{and} \quad \dot{d} = -\nu\tau(\vec{\gamma} \cdot \vec{n}),$$

and the second derivatives are,

$$\ddot{\vec{u}} = \nu^2\kappa\tau\vec{t} - (\dot{\nu}\tau + \nu\dot{\tau})\vec{n} - \nu^2\tau^2\vec{b}$$

and

$$\ddot{d} = \nu^2\kappa\tau(\vec{\gamma} \cdot \vec{t}) - (\dot{\nu}\tau + \nu\dot{\tau})(\vec{\gamma} \cdot \vec{n}) - \nu^2\tau^2(\vec{\gamma} \cdot \vec{b}).$$

Substituting these results into formula (5.1), the acceleration centre for osculating planes motion is,

$$a_c = (\vec{\gamma} \cdot \vec{t})\vec{t} + (\vec{\gamma} \cdot \vec{n})\vec{n} + (\vec{\gamma} \cdot \vec{b})\vec{b}.$$

That is, acceleration centre of osculating planes motion is on the curve γ. Note that $\det(\vec{u}, \dot{\vec{u}}, \ddot{\vec{u}}) = \nu^3\tau^2\kappa$ is not zero in general. □

Theorem 5.1.2. *The acceleration centre of a plane-symmetric motion based on the normal planes to a curve $\vec{\gamma}$, must lie on the polar curve of $\vec{\gamma}$.*

Proof. For normal planes motion the vector $\vec{u} = \vec{t}$ and the distance $d = \vec{\gamma} \cdot \vec{t}$, then the first derivatives are,

$$\dot{\vec{u}} = \nu\kappa\vec{n} \quad \text{and} \quad \dot{d} = \nu + \nu\kappa(\vec{\gamma} \cdot \vec{n}),$$

and the second derivatives are,

$$\ddot{\vec{u}} = -\nu^2\kappa^2\vec{t} + (\dot{\nu}\kappa + \nu\dot{\kappa})\vec{n} + \nu^2\tau\kappa\vec{b}$$

and

$$\ddot{d} = \dot{\nu} - \nu^2\kappa^2(\vec{\gamma} \cdot \vec{t}) + (\dot{\nu}\kappa + \nu\dot{\kappa})(\vec{\gamma} \cdot \vec{n}) + \nu^2\tau\kappa(\vec{\gamma} \cdot \vec{b}).$$

Substituting these results into formula (5.1), the acceleration centre for normal planes

motion is,

$$a_c = (\vec{\gamma} \cdot \vec{t})\vec{t} + (\vec{\gamma} \cdot \vec{n})\vec{n} + (\vec{\gamma} \cdot \vec{b})\vec{b} + (1/\kappa)\vec{n} - (\dot{\kappa}/\nu\tau\kappa^2)\vec{b}$$

That is,

$$a_c = \vec{\gamma} + \frac{1}{\kappa}\vec{n} - \frac{\dot{\kappa}}{\nu\tau\kappa^2}\vec{b}$$

is the polar curve of the curve $\vec{\gamma}$. Note that $\det(\vec{u}, \dot{\vec{u}}, \ddot{\vec{u}}) = \nu^3\kappa^2\tau$ is not zero in general.

□

Theorem 5.1.3. *For the motion based on the rectifying planes of $\vec{\gamma}$ the acceleration centre will lie on the rectifying curve of $\vec{\gamma}$.*

Proof. For rectifying planes plane symmetric motions the vector $\vec{u} = \vec{n}$ and the distance $d = \vec{\gamma} \cdot \vec{n}$, then the first derivatives are,

$$\dot{\vec{u}} = -\nu\kappa\vec{t} + \nu\tau\vec{b} \quad \text{and} \quad \dot{d} = -\nu\kappa(\vec{\gamma} \cdot \vec{t}) + \nu\tau(\vec{\gamma} \cdot \vec{b}),$$

and the second derivatives are,

$$\ddot{\vec{u}} = -(\dot{\nu}\kappa + \nu\dot{\kappa})\vec{t} - \nu^2(\kappa^2 + \tau^2)\vec{n} + (\dot{\nu}\tau + \nu\dot{\tau})\vec{b}$$

and

$$\ddot{d} = -(\dot{\nu}\kappa + \nu\dot{\kappa})(\vec{\gamma} \cdot \vec{t}) - \nu^2(\kappa^2 + \tau^2)(\vec{\gamma} \cdot \vec{n}) + (\dot{\nu}\tau + \nu\dot{\tau})(\vec{\gamma} \cdot \vec{b}) - \nu^2\kappa.$$

After computing the cross and dot products then applying formula (5.1), the acceleration center of rectifying planes motion can be found as follows,

$$a_c = (\vec{\gamma} \cdot \vec{t})\vec{t} + (\vec{\gamma} \cdot \vec{n})\vec{n} + (\vec{\gamma} \cdot \vec{b})\vec{b} - \nu\kappa\tau/(\kappa\dot{\tau} - \tau\dot{\kappa})\vec{t} - \nu\kappa^2/(\kappa\dot{\tau} - \tau\dot{\kappa})\vec{b}.$$

That is the acceleration centre is the rectifying curve of the curve $\vec{\gamma}$,

$$a_c = \vec{\gamma} + \lambda\vec{t} + \beta\vec{b} \tag{5.2}$$

where

$$\lambda = \frac{\nu\kappa\tau}{\tau\dot{\kappa} - \kappa\dot{\tau}} \quad \text{and} \quad \beta = \frac{\nu\kappa^2}{\tau\dot{\kappa} - \kappa\dot{\tau}}.$$

Note that, in general $\det(\vec{u}, \dot{\vec{u}}, \ddot{\vec{u}}) = \nu^2(\kappa\dot{\tau} - \tau\dot{\kappa})$ is not zero. □

For planes tangent to a cone, the vertex of the cone is the acceleration centre, in fact it is a fixed point for all of the motion. For a motion based on the tangent planes to a cylinder there is no acceleration centre.

5.2 Inflection Points

We will investigate the inflection points of osculating planes motion, normal planes motion and rectifying planes motion. *Inflection points* are the points whose velocity vectors and acceleration vectors are instantaneously parallel.

From Selig (2011), it is to be expected that the inflection points of the motion do not lie on a cubic curve but on a conic since the motion is always an instantaneous pure rotation. In fact, it can be shown that:

Theorem 5.2.1. *At any instant of an osculating planes motion based on a space curve $\vec{\gamma}$ the only point of inflection is the current point on the curve $\vec{\gamma}$.*

Proof. Without loss of generality we can write a point on the rigid body as,

$$\vec{p} = \vec{\gamma} + x\vec{t} + y\vec{n} + z\vec{b}$$

where x, y and z are variables and $\vec{t}$, $\vec{n}$ and $\vec{b}$ are the tangent, normal and binormal vectors to $\vec{\gamma}$. The velocity of a point is given by,

$$\dot{\vec{p}} = \vec{\omega} \times \vec{p} + \vec{v}. \tag{5.3}$$

For osculating planes motion $\vec{\omega} = 2\nu\tau\vec{t}$ and $\vec{v} = \vec{\gamma} \times \vec{\omega}$. Substituting this into the

equations above for the velocity and acceleration of the point gives,

$$\begin{aligned}\dot{\vec{p}} &= \vec{\omega} \times (\vec{p} - \vec{\gamma}) \\ &= 2\nu\tau(y\vec{b} - z\vec{n})\end{aligned}$$

and the acceleration is,

$$\ddot{\vec{p}} = \dot{\vec{\omega}} \times (\vec{p} - \vec{\gamma}) + \vec{\omega} \times (\dot{\vec{p}} - \dot{\vec{\gamma}}).$$

Explicitly,

$$\ddot{\vec{p}} = 2\Big(\nu^2\tau\kappa z\vec{t} - (2\nu^2\tau^2 y + \frac{d(\nu\tau)}{d\mu}z)\vec{n} + (\frac{d(\nu\tau)}{d\mu}y - \nu^2\tau\kappa x - 2\nu^2\tau^2 y)\vec{b}\Big).$$

Comparing the coefficients of the mutually orthogonal vectors $\vec{t}$, $\vec{n}$ and $\vec{b}$ it can be immediately seen that $z = 0$ since $\vec{t}$ only appears in the expression for $\ddot{\vec{p}}$. Similarly, from the coefficient of $\vec{n}$, $y = 0$. Then finally, looking at the coefficients of $\vec{b}$ it can be seen that $x = 0$. Hence, an osculating planes motion based on a space curve $\vec{\gamma}$ the only point of inflection is the current point on the curve $\vec{\gamma}$ at any instant. □

Theorem 5.2.2. *At any instant of an normal planes motion based on a space curve $\vec{\gamma}$ the only point of inflection is the current point on the polar curve of $\vec{\gamma}$.*

Proof. As in the proof of the above theorem, we can write any point on the rigid body as,

$$\vec{p} = \vec{\gamma} + x\vec{t} + y\vec{n} + z\vec{b}.$$

For normal planes motion $\vec{\omega} = 2\nu\kappa\vec{b}$ and $\vec{v} = (\vec{\gamma} + (1/\kappa)\vec{n}) \times \vec{\omega}$, so the velocity of the point is given by,

$$\begin{aligned}\dot{\vec{p}} &= \vec{\omega} \times \vec{p} + \vec{v} \\ &= \vec{\omega} \times (\vec{p} - \vec{\gamma} - (1/\kappa)\vec{n}) \\ &= 2\nu((1 - \kappa y)\vec{t} + \kappa x\vec{n}).\end{aligned}$$

and the acceleration is

$$\ddot{\vec{p}} = \dot{\vec{\omega}} \times (x\vec{t} + y\vec{n} + z\vec{b} - (1/\kappa)\vec{n}) + \vec{\omega} \times \big(\dot{\vec{p}} - \dot{\vec{\gamma}} - \frac{d}{d\mu}((1/\kappa)\vec{n})\big)$$

and explicitly,

$$\begin{aligned}\ddot{\vec{p}} = 2\big((-\nu\kappa(2\nu\kappa x + \dot{\kappa}/\kappa^2) - \frac{d(\nu\kappa)}{d\mu}(y - 1/\kappa) - \nu^2\kappa\tau z)\vec{t} \\ + (\frac{d(\nu\kappa)}{d\mu}x + \nu^2(\kappa - \kappa^2 y))\vec{n} + (\nu^2\kappa\tau x)\vec{b}\big).\end{aligned}$$

where $\dot{\vec{\omega}} = 2(-\nu^2\kappa\tau\vec{n} + d/d\mu(\nu\kappa)\vec{b})$. At the inflection points $\dot{\vec{p}}$ and $\ddot{\vec{p}}$ must be parallel, so we will start comparing the coefficients of the binormal vector $\vec{b}$. It can be easily seen $x = 0$ since the binormal vector $\vec{b}$ only appears in the expression for $\ddot{\vec{p}}$. Then the coefficient of normal vectors $\vec{n}$ must be zero, so $y = 1/\kappa$. Finally, comparing the coefficients of the tangent vector $\vec{t}$ gives $z = -\dot{\kappa}/\nu\kappa^2\tau$. Hence at any instant the inflection point of the normal planes motion is the current point on the following curve,

$$\vec{\gamma} + \frac{1}{\kappa}\vec{n} - \frac{\dot{\kappa}}{\nu\kappa^2\tau}\vec{b}.$$

Note that this curve is the polar curve of the curve $\vec{\gamma}$. □

Theorem 5.2.3. *At any instant of an rectifying planes motion based on a space curve $\vec{\gamma}$ the only point of inflection is the current point on the rectifying curve of $\vec{\gamma}$.*

Proof. Let

$$\vec{p} = \vec{\gamma} + x\vec{t} + y\vec{n} + z\vec{b}$$

be any point on the rigid body. For rectifying planes motion the angular velocity vector $\vec{\omega} = 2\nu(\tau\vec{t} + \kappa\vec{b})$ and the linear velocity vector $\vec{v} = \vec{\gamma} \times \vec{\omega}$. The velocity of the point $\vec{p}$ is given by,

$$\begin{aligned}\dot{\vec{p}} &= \vec{\omega} \times (\vec{p} - \vec{\gamma}) \\ &= 2\nu(\tau\vec{t} + \kappa\vec{b}) \times (x\vec{t} + y\vec{n} + z\vec{b}) \\ &= 2\nu\big(- \kappa y\vec{t} + (\kappa x - \tau z)\vec{n} + \tau y\vec{b}\big),\end{aligned}$$

and the acceleration of the point $\vec{p}$ is given by,

$$\ddot{\vec{p}} = \dot{\vec{\omega}} \times (\vec{p} - \vec{\gamma}) + \vec{\omega} \times (\dot{\vec{p}} - \dot{\vec{\gamma}}).$$

Explicitly,

$$\begin{aligned}\ddot{\vec{p}} = 2\big((-2\nu^2\kappa^2 x - \frac{d(\nu\kappa)}{d\mu}y + 2\nu^2\kappa\tau z)\vec{t} \\ +(\frac{d(\nu\kappa)}{d\mu}x - 2\nu^2(\kappa^2 + \tau^2)y - \frac{d(\nu\tau)}{d\mu}z - \nu^2\kappa)\vec{n} \\ +(2\nu^2\kappa\tau x + \frac{d(\nu\tau)}{d\mu}y - 2\nu^2\tau^2 z)\vec{b}\big).\end{aligned}$$

Comparing the coefficients of tangent vector $\vec{t}$, normal vector $\vec{n}$ and binormal vector $\vec{b}$ is tidiying but luckily we have the solution curve such that,

$$\vec{\gamma} + \frac{\nu\kappa\tau}{\tau\dot{\kappa} - \kappa\dot{\tau}}\vec{t} + \frac{\nu\kappa^2}{\tau\dot{\kappa} - \kappa\dot{\tau}}\vec{b}.$$

Note that this curve is the same rectifiying curve of the curve $\vec{\gamma}$ with the acceleration centre of the rectifying planes motion, see equation (5.2). □

5.3 Bresse Hyperboloid

In this section the Bresse hyperboloid for these motions are studied. The points on this surface are points in the moving body where, instantaneously the velocity and acceleration are perpendicular.

Theorem 5.3.1. *At any instant of an osculating planes motion based on a space curve $\vec{\gamma}$, the points with perpendicular velocity and acceleration lie on an elliptical cone whose*

vertex is the current point on the curve $\vec{\gamma}$.

Proof. As in the proof of above theorems, write an arbitrary point on the rigid body as,

$$\vec{p} = \vec{\gamma} + x\vec{t} + y\vec{n} + z\vec{b}$$

so that,

$$\dot{\vec{p}} = 2\nu\tau(y\vec{b} - z\vec{n})$$

and

$$\ddot{\vec{p}} = 2\Big(\nu^2\tau\kappa z\vec{t} - (2\nu^2\tau^2 y + \frac{d(\nu\tau)}{d\mu}z)\vec{n} + (\frac{d(\nu\tau)}{d\mu}y - \nu^2\tau\kappa x - 2\nu^2\tau^2 y)\vec{b}\Big).$$

Now

$$\dot{\vec{p}} \cdot \ddot{\vec{p}} = 4\nu\tau\Big(\frac{d(\nu\tau)}{d\mu}(y^2 + z^2) - \nu^2\tau\kappa xy\Big).$$

Setting the above equation to zero gives a homogeneous degree 2 equation in $x,$ and z. This can be written as,

$$y^2 + z^2 - \alpha xy = 0$$

where $\alpha = \nu^2\tau\kappa/\frac{d(\nu\tau)}{d\mu}$. This can be rearranged to read,

$$(y - \frac{\alpha x}{2})^2 + z^2 = \Big(\frac{\alpha x}{2}\Big)^2.$$

Clearly $x = y = z = 0$ is a solution corresponding to the point $\vec{\gamma}$ on the curve. Because the equation is homogenous, every point on a line joining this point to another solution is also a solution. So the solutions form a cone with vertex $\vec{\gamma}$. Intersecting the cone with a plane parallel to the normal plane to the curve at γ means fixing the value of x. The intersection is a circle with centre located at $\vec{\gamma} + x\vec{t} + \frac{\alpha x}{2}\vec{n}$. □

Theorem 5.3.2. *At any instant of a normal planes motion based on a space curve $\vec{\gamma}$, the points with perpendicular velocity and acceleration are,*

$$\begin{array}{rll} \textit{Hyperbolic paraboloid or hyperbolic cylinder}, & \textit{if} & (\nu\kappa)\frac{d(\nu\kappa)}{d\mu} = 0 \\ \textit{Hyperboloid or cone}, & \textit{otherwise}. & \end{array} \tag{5.4}$$

where ν, κ and τ are the speed, the curvature and the torsion of the curve $\vec{\gamma}$.

Proof. An arbitrary point on the rigid body can be written as,

$$\vec{p} = \vec{\gamma} + x\vec{t} + y\vec{n} + z\vec{b}$$

so that,

$$\dot{\vec{p}} = 2\nu((1-\kappa y)\vec{t} + \kappa x\vec{n}).$$

and the acceleration is

$$\begin{aligned}\ddot{\vec{p}} = 2\big((-\nu\kappa(2\nu\kappa x + \dot{\kappa}/\kappa^2) - \frac{d(\nu\kappa)}{d\mu}(y - 1/\kappa) - \nu^2\kappa\tau z)\vec{t} \\ + (\frac{d(\nu\kappa)}{d\mu}x + \nu^2(\kappa - \kappa^2 y))\vec{n} + (\nu^2\kappa\tau x)\vec{b}\big).\end{aligned}$$

Now

$$\dot{\vec{p}} \cdot \ddot{\vec{p}} = 4\Big(\frac{d(\nu\kappa)}{d\mu}\nu\kappa x^2 + \frac{d(\nu\kappa)}{d\mu}\nu\kappa y^2 + \nu^3\kappa^2\tau yz - \frac{d(\nu^2\kappa)}{d\mu}y - \nu^3\tau\kappa z + \nu\dot{\nu}\Big).$$

Setting the above equation to zero gives,

$$\frac{d(\nu\kappa)}{d\mu}\nu\kappa x^2 + \frac{d(\nu\kappa)}{d\mu}\nu\kappa y^2 + \nu^3\kappa^2\tau yz - \frac{d(\nu^2\kappa)}{d\mu}y - \nu^3\tau\kappa z + \nu\dot{\nu} = 0. \tag{5.5}$$

To classify this quadratic surface, it is necessary to find the eigenvalues of the following symmetric matrix, for the classification of quadratics in detailed see Zwillinger (2003),

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & f \\ 0 & f & 0 \end{pmatrix}$$

where $a = \frac{d(\nu\kappa)}{d\mu}\nu\kappa$ and $f = \nu^3\kappa^2\tau/2$. Then

$$\det(A - \lambda I) = (a - \lambda)(\lambda^2 - a\lambda - f^2).$$

Hence, the eigenvalues of the matrix A can be found as follows

$$\lambda_1 = \frac{d(\nu\kappa)}{d\mu}\nu\kappa \quad \text{and} \quad \lambda_{2,3} = \big(\frac{d(\nu\kappa)}{d\mu}\nu\kappa \pm \sqrt{\frac{d(\nu\kappa)}{d\mu}^2 \nu^2\kappa^2 + \nu^6\kappa^4\tau^2}\big)/2.$$

If $\lambda_1 = 0$, that is $\nu\kappa$ is a constant or the curve $\vec{\gamma}$ is a straight line, then

$$\lambda_1 = 0, \quad \lambda_2 > 0 \quad \text{and} \quad \lambda_3 < 0.$$

So, the signature of the matrix A is $(n_1, n_2, n_3) = (1, 1, 1)$ where n_1, n_2, n_3 represent the number of strictly possitive eigenvalues, the number of strictly negative eigenvalues and the number of zero eigenvalues respectively. In this case, equation (5.5) represents hyperbolic paraboloid or hyperbolic cylinder.

If $\lambda_1 > 0$, that is the derivative of $\nu\kappa$ is possitive, then

$$\lambda_1 > 0, \quad \lambda_2 > 0 \quad \text{and} \quad \lambda_3 < 0.$$

So, the signature of the matrix A is $(2, 1, 0)$.

If $\lambda_1 < 0$, then

$$\lambda_1 < 0, \quad \lambda_2 > 0 \quad \text{and} \quad \lambda_3 < 0.$$

Hence, the signature of the matrix A is $(1, 2, 0)$. In these two cases the equation (5.5) represents one or two sheeted hyperboloid or cone. □

Theorem 5.3.3. *At any instant of a rectifying planes motion based on a space curve $\vec{\gamma}$, the points with perpendicular velocity and acceleration represent*

$$\begin{aligned} &\textit{Hyperbolic paraboloid or hyperbolic cylinder}, && \textit{if} \quad \sigma = 0 \quad \textit{and} \quad \tfrac{d(\nu\tau)}{d\mu} \neq 0 \\ &\textit{Eliptic paraboloid or eliptic cylinder}, && \textit{if} \quad \sigma \neq 0 \quad \textit{and} \quad \tfrac{d(\nu\tau)}{d\mu} = 0 \\ &\textit{Hyperboloid or cone}, && \textit{if} \quad \sigma \neq 0 \quad \textit{and} \quad \tfrac{d(\nu\tau)}{d\mu} \neq 0 \end{aligned} \tag{5.6}$$

where ν, κ and τ are the speed, the curvature and the torsion of the curve $\vec{\gamma}$ and $\sigma = (\nu\kappa)\frac{d(\nu\kappa)}{d\mu} + (\nu\tau)\frac{d(\nu\tau)}{d\mu}$.

Proof. Let

$$\vec{p} = \vec{\gamma} + x\vec{t} + y\vec{n} + z\vec{b}$$

be any point on the rigid body. Then the velocity of a point is given by,

$$\dot{\vec{p}} = 2\nu\big(- \kappa y\vec{t} + (\kappa x - \tau z)\vec{n} + \tau y\vec{b}\big),$$

and the acceleration of the point $\vec{p}$ is given by explicitly,

$$\begin{aligned}\ddot{\vec{p}} = 2\big((-2\nu^2\kappa^2 x - \frac{d(\nu\kappa)}{d\mu}y + 2\nu^2\kappa\tau z)\vec{t} \\ +(\frac{d(\nu\kappa)}{d\mu}x - 2\nu^2(\kappa^2+\tau^2)y - \frac{d(\nu\tau)}{d\mu}z - \nu^2\kappa)\vec{n} \\ +(2\nu^2\kappa\tau x + \frac{d(\nu\tau)}{d\mu}y - 2\nu^2\tau^2 z)\vec{b}\big).\end{aligned}$$

The points where the velocity and acceleration are perpendicular instantaneously are given by the following equation, that is $\dot{p}\cdot\ddot{p} = 0$,

$$\begin{aligned}\frac{d(\nu\kappa)}{d\mu}\nu\kappa x^2 + (\frac{d(\nu\kappa)}{d\mu}\nu\kappa + \frac{d(\nu\tau)}{d\mu}\nu\tau)y^2 + \frac{d(\nu\tau)}{d\mu}\nu\tau z^2 \\ - (\frac{d(\nu\kappa)}{d\mu}\nu\tau + \frac{d(\nu\tau)}{d\mu}\nu\kappa)xz - \nu^3\kappa^2 x + \nu^3\kappa\tau z = 0.\end{aligned} \tag{5.7}$$

To classify this quadratic surface, it is necessary to find the eigenvalues of the given symmetric matrix,

$$A = \begin{pmatrix} \frac{d(\nu\kappa)}{d\mu}\nu\kappa & 0 & -(\frac{d(\nu\kappa)}{d\mu}\nu\tau + \frac{d(\nu\tau)}{d\mu}\nu\kappa)/2 \\ 0 & \frac{d(\nu\kappa)}{d\mu}\nu\kappa + \frac{d(\nu\tau)}{d\mu}\nu\tau & 0 \\ -(\frac{d(\nu\kappa)}{d\mu}\nu\tau + \frac{d(\nu\tau)}{d\mu}\nu\kappa)/2 & 0 & \frac{d(\nu\tau)}{d\mu}\nu\tau \end{pmatrix}.$$

Then the eigenvalues of A can be found as follows,

$$\lambda_1 = \frac{d(\nu\kappa)}{d\mu}\nu\kappa + \frac{d(\nu\tau)}{d\mu}\nu\tau$$

and

$$\lambda_{2,3} = \big(\frac{d(\nu\kappa)}{d\mu}\nu\kappa + \frac{d(\nu\tau)}{d\mu}\nu\tau \pm \sqrt{(\frac{d(\nu\kappa)}{d\mu}\nu\kappa + \frac{d(\nu\tau)}{d\mu}\nu\tau)^2 + (\frac{d(\nu\tau)}{d\mu}\nu\kappa - \frac{d(\nu\kappa)}{d\mu}\nu\tau)^2}\big)/2. \tag{5.8}$$

If $\lambda_1 = 0$, that is

$$\frac{d(\nu\kappa)}{d\mu}\nu\kappa + \frac{d(\nu\tau)}{d\mu}\nu\tau = 0.$$

This differential equation can be written as follows,

$$\frac{d((\nu\kappa)^2/2)}{d\mu} + \frac{d((\nu\tau)^2/2)}{d\mu} = 0.$$

So, the solution of this differantial equation is

$$(\nu\kappa)^2 + (\nu\tau)^2 = c_1$$

where c_1 is a possitive constant. Note that if $c_1 = 0$ then $\nu\kappa$ and $\nu\tau$ are equal to zero. So, this case does not specify a quadratic surface. Hence c_1 must be possitive. Rearranging the terms of the below equation gives,

$$\frac{d(\nu\tau)}{d\mu}\nu\kappa - \frac{d(\nu\kappa)}{d\mu}\nu\tau = 2\frac{d(\nu\tau)}{d\mu}(c_1 + \nu^2\tau^2).$$

If the derivative of $\nu\tau$ is different than zero then the eigenvalues of the matrix A are given by,

$$\lambda_1 = 0, \quad \lambda_2 > 0 \quad \text{and} \quad \lambda_3 < 0.$$

Hence, the signature of the matrix A is $(1, 1, 1)$. In this case the equation (5.7) represents hyperbolic paraboloid or hyperbolic cylinder. Note that, if the derivative of $\nu\tau$ is equal to zero then the equation (5.7) is vanishing.

If $\lambda_1 \neq 0$ and the derivative of $\nu\tau$ is zero then there are two possibilities,

$$\lambda_1 > 0 \quad \text{then} \quad \lambda_2 > 0 \quad \text{and} \quad \lambda_3 = 0,$$

$$\lambda_1 < 0 \quad \text{then} \quad \lambda_2 = 0 \quad \text{and} \quad \lambda_3 < 0.$$

So, the signature of the matrix A is $(2, 0, 1)$ and $(0, 2, 1)$ respectively. In these cases, the equation (5.7) represents eliptic paraboloid or eliptic cylinder.

If $\lambda_1 \neq 0$ and the derivative of $\nu\tau$ is not zero then there are two possibilities,

$$\lambda_1 > 0 \quad \text{then} \quad \lambda_2 > 0 \quad \text{and} \quad \lambda_3 < 0,$$

$$\lambda_1 < 0 \quad \text{then} \quad \lambda_2 > 0 \quad \text{and} \quad \lambda_3 < 0.$$

So, the signature of the matrix A is $(2, 1, 0)$ and $(1, 2, 0)$ respectively. In these cases, the equation (5.7) represents one or two sheeted hyperboloid or cone.

□

CHAPTER SIX
MOTIONS BASED ON CURVES WITH MATES AND LINE SYMMETRIC MOTIONS

6.1 Motions Based on Curves with Mates

Many special types of curves have been defined in the literature by the property that they can be used to define another curve, sometimes called a partner curve, mate, offset of parallel curve. In this section some relations between plane symmetric motions based on such curves and their mates are investigated.

Bertrand curves are curves such that at any point the principal normals to the curve and its partner curve coincide, (Willmore, 2013, p.29). Notice that this means that the rectifying planes to the two curves are parallel. This leads to the following theorem,

Theorem 6.1.1. *Let $G_r(\mu)$ and $\tilde{G}_r(\mu)$ be the rectifying planes motion based on a Bertrand curve and its mate respectively, then the product, $G_r(\mu)\tilde{G}_r^{-1}(\mu)$, is always a pure translation.*

Proof. Write,

$$G_r = \begin{pmatrix} I - 2\vec{n}\vec{n}^T & 2d\vec{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I - 2\vec{n}_0\vec{n}_0^T & 2d_0\vec{n}_0 \\ 0 & 1 \end{pmatrix}$$

for the motion associated with the Bertrand curve and

$$\tilde{G}_r = \begin{pmatrix} I - 2\vec{n}\vec{n}^T & 2\tilde{d}\vec{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I - 2\vec{n}_0\vec{n}_0^T & 2\tilde{d}_0\vec{n}_0 \\ 0 & 1 \end{pmatrix}$$

for the partner curve. The dependence on the parameter μ has been suppressed for

brevity. Inverting the second matrix and multiplying then gives,

$$G_r\tilde{G}_r^{-1} = \begin{pmatrix} I - 2\vec{n}\vec{n}^T & 2d\vec{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 2(d_0 - \tilde{d}_0)\vec{n}_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I - 2\vec{n}\vec{n}^T & 2\tilde{d}\vec{n} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & \vec{q} \\ 0 & 1 \end{pmatrix}$$

where,

$$\vec{q} = 2(d - \tilde{d})\vec{n} + 2(d_0 - \tilde{d}_0)(I - 2\vec{n}\vec{n}^T)\vec{n}_0.$$

□

Mannheim curves, are also partner curves. Here the principal normal line of the Mannheim curve coincides with the binormal line to the partner curve at the corresponding parameter value, see Liu & Wang (2008). The computations above, proving theorem (6.1.1) will also show the following,

Theorem 6.1.2. *Let $G_r(\mu)$ and $\tilde{G}_o(\mu)$ be the rectifying planes motion based on a Mannheim curve and the osculating planes motion based on its mate respectively, then the product, $G_r(\mu)\tilde{G}_b^{-1}(\mu)$, is always a pure translation.*

In Menninger (2013), Menninger describes a pair of transformations from one curve to another. These are named the successor transformation and the predecessor transformation, one is the inverse of the other. The tangent vector to a curve is the normal vector to its successor curve and so the normal vector to a curve is the tangent vector to its predecessor. This gives another theorem in the same vein as the previous two, and with an almost identical proof:

Theorem 6.1.3. *Let $G_n(\mu)$ and $\tilde{G}_r(\mu)$ be the normal planes motion based on a curve and the osculating planes motion based on its successor curve respectively, then the product, $G_r(\mu)\tilde{G}_b^{-1}(\mu)$, is always a pure translation.*

Note that we could also describe $\tilde{G}_r(\mu)$ as the rectifying planes motion of a curve and $G_n(\mu)$ as the normal planes motion of its predecessor curve. Clearly, composing

these plane symmetric motion as in the theorems of this section will produce a motion that is composed of pure translations. Such motions are well known in robotics, but under several different names. In Selig & Husty (2011), it was shown that such motions are represented by curves in the Study quadric which lie entirely in the B-plane $a_1 = a_2 = a_3 = c_0 = 0$. These motions can also be described as point-symmetric motions, that is motions consisting of successive reflections in the points of a curve in space.

6.2 Line Symmetric Motions

Some of the ideas introduced above can be used to investigate line symmetric motions. These are rigid-body motions generated by reflecting a rigid body in successive lines of a ruled surface, see Bottema & Roth (1990) for example. A reflection in a line is simply a rotation of π radians about the line. Such a rotation can be generated by reflecting in a pair of orthogonal planes that intersect along the line. So for example, we can reflect in the normal and osculating planes of a space curve to get the line symmetric motion generated by the normal lines to the curve.

Consider a curve $\vec{\gamma}$ and let,

$$M = \begin{pmatrix} I - 2\vec{b}\vec{b}^T & 2(\vec{\gamma}\cdot\vec{b})\vec{b} \\ 0 & 1 \end{pmatrix}$$

represent reflection in the osculating planes of the curve and

$$N = \begin{pmatrix} I - 2\vec{t}\vec{t}^T & 2(\vec{\gamma}\cdot\vec{t})\vec{t} \\ 0 & 1 \end{pmatrix}$$

represent reflection in the normal planes to the curve. Reflection in both planes gives a π-rotation about the normal line to the curve given by MN. In Selig & Husty (2011) a line symmetric motion generated by a ruled surface was defined as a π-rotation about the initial generator of the surface, at the parameter value $\mu = 0$, followed by π-rotations about successive generators of the ruled surface. This was

done to ensure that the curve in the $SE(3)$ representing the motion passes through the identity of the group. In the Study quadric the identity is the dual quaternion 1. In the following, omitting or including this initial π-rotation makes no difference.

Theorem 6.2.1. *The twist velocity of the line symmetric motion generated by the normal lines to a curve $\vec{\gamma}$ is the sum of the twist velocities of the osculating planes motion and the normal planes motion associated with the curve.*

Proof. The line symmetric motion is given by,

$$G = MNM_0N_0$$

where M and N are the reflections in the osculating planes and normal planes, as above; and M_0 and N_0 are the reflections in the initial osculating and normal planes to the motion. The twist velocity is given, as usual, by $\dot{G}G^{-1}$, where the dot denotes differentiation with respect to the parameter μ. Substituting for G and remembering that reflections are self-inverse gives,

$$\dot{G}G^{-1} = (\dot{M}NM_0N_0 + M\dot{N}M_0N_0)N_0M_0NM = \dot{M}M + M\dot{N}NM.$$

Now $\dot{N}N$ is the twist velocity of the normal planes motion so is represented by a pitch zero twist about a line normal to the osculating plane of the curve at the current point. The conjugation $M\dot{N}NM$, reflects the line $\dot{N}N$ in the osculating plane and hence doesn't change $\dot{N}N$. That is, $M\dot{N}NM = \dot{N}N$. Hence we have the result,

$$\dot{G}G^{-1} = \dot{M}M + \dot{N}N.$$

□

Remember that from the sections 4.2 and 4.3 the twist velocity of osculating planes

motion and the twist velocity of normal planes motion are given by,

$$2\nu\tau \begin{pmatrix} \vec{t} \\ \vec{\gamma} \times \vec{t} \end{pmatrix} \quad \text{and} \quad 2\nu\kappa \begin{pmatrix} \vec{b} \\ (\vec{\gamma} + \frac{1}{\kappa}\vec{n}) \times \vec{b} \end{pmatrix}.$$

Hence it can be seen that the twist velocity of the line symmetric motion generated by the normal lines to a curve $\vec{\gamma}$ are given, as a 6-vector, by,

$$\mathbf{s}_d = 2\nu\tau \begin{pmatrix} \vec{t} \\ \vec{\gamma} \times \vec{t} \end{pmatrix} + 2\nu\kappa \begin{pmatrix} \vec{b} \\ (\vec{\gamma} + \frac{1}{\kappa}\vec{n}) \times \vec{b} \end{pmatrix} = 2\nu \begin{pmatrix} \tau\vec{t} + \kappa\vec{b} \\ \vec{\gamma} \times (\tau\vec{t} + \kappa\vec{b}) + \vec{t} \end{pmatrix}.$$

The pitch h, of this twist is easily found to be,

$$h = \frac{\vec{\omega} \cdot \vec{v}}{|\vec{\omega}|^2} = \frac{(\tau\vec{t} + \kappa\vec{b}) \cdot (\vec{\gamma} \times (\tau\vec{t} + \kappa\vec{b}) + \vec{t})}{(\tau\vec{t} + \kappa\vec{b}) \cdot (\tau\vec{t} + \kappa\vec{b})} = \frac{\tau}{\kappa^2 + \tau^2}.$$

To find the axis of this twist ℓ, that is, the generator line of the fixed axode, we subtract h times the first three components of the twist from the last three components, this gives,

$$\ell = \begin{pmatrix} \tau\vec{t} + \kappa\vec{b} \\ \vec{\gamma} \times (\tau\vec{t} + \kappa\vec{b}) + \frac{\kappa}{\kappa^2+\tau^2}(\kappa\vec{t} - \tau\vec{b}) \end{pmatrix} = \begin{pmatrix} \tau\vec{t} + \kappa\vec{b} \\ (\vec{\gamma} + \frac{\kappa}{\kappa^2+\tau^2}\vec{n}) \times (\tau\vec{t} + \kappa\vec{b}) \end{pmatrix}.$$

Note, that we are now treating the components of this 6-vector as Plücker coordinates. Since these are homogeneous coordinates we may ignore overall multiplicative factors. Hence, we have shown the following:

Theorem 6.2.2. *The Plücker coordinates of the fixed axode of the line symmetric motion generated by the normal lines to a curve $\vec{\gamma}$ are given by,*

$$\ell = \begin{pmatrix} \vec{\delta} \\ (\vec{\gamma} + \frac{\kappa}{\kappa^2+\tau^2}\vec{n}) \times \vec{\delta} \end{pmatrix}.$$

This shows that the fixed axode consists of lines parallel to the Darboux vector $\vec{\delta}$ of the curve passing through a point displaced from the current point on the curve by $\frac{\kappa}{\kappa^2+\tau^2}$ along the normal to the curve.

This method can also be used to investigate the line symmetric motion generated by reflections about the binormal lines to a curve. In this case the two plane symmetric motions which combine to produce this motion are reflections in the normal and rectifying planes. Let N be as above, the 4×4 matrix representing a reflection in the normal plane to the curve $\vec{\gamma}$, and let

$$C = \begin{pmatrix} I - 2\vec{n}\vec{n}^T & 2(\vec{\gamma} \cdot \vec{n})\vec{n} \\ 0 & 1 \end{pmatrix}$$

be the matrix representing a reflection in the rectifying plane of $\vec{\gamma}$. Writing the line symmetric motion as,

$$G = CNC_0N_0.$$

It is clear from the proof of theorem (6.2.1), that the twist velocity of the motion is,

$$\dot{G}G^{-1} = \dot{C}C + C\dot{N}NC.$$

This time however, the twist $\dot{N}N$ must be reflected in the rectifying plane of the curve. To compute the reflection of the twist velocity of normal planes motion in the rectifying plane, we will use the 4×4 matrix representation of velocity twist of plane symmetric motions, which was computed in section 4.1 as,

$$S_d = 2 \begin{pmatrix} \dot{\vec{u}}\vec{u}^T - \vec{u}\dot{\vec{u}}^T & \dot{d}\vec{u} - d\dot{\vec{u}} \\ 0 & 0 \end{pmatrix},$$

for the unit normal vector of the planes $\vec{u}$. For normal planes motion it can be computed as,

$$\dot{N}N = 2\nu\kappa \begin{pmatrix} \vec{n}\vec{t}^T - \vec{t}\vec{n}^T & (\vec{\gamma} \cdot \vec{n})\vec{t} - (\vec{\gamma} \cdot \vec{t})\vec{n} + \frac{1}{\kappa}\vec{t} \\ 0 & 0 \end{pmatrix}.$$

Now the reflection of $\dot{N}N$ in the osculating plane is computed as,

$$C\dot{N}NC = 2\nu\kappa \begin{pmatrix} \vec{t}\vec{n}^T - \vec{n}\vec{t}^T & -(\vec{\gamma} \cdot \vec{n})\vec{t} + (\vec{\gamma} \cdot \vec{t})\vec{n} + \frac{1}{\kappa}\vec{t} \\ 0 & 0 \end{pmatrix}.$$

It can be seen easily, the corresponding vector to the scew symmetric matrix $(\vec{t}\vec{n}^T - \vec{n}\vec{t}^T)$ is equal to $\vec{n} \times \vec{t} = -\vec{b}$. A straightforward computation reveals that the 6-vector representation of the twist $C\dot{N}NC$ is

$$2\nu\kappa \begin{pmatrix} -\vec{b} \\ (-\vec{\gamma} + \frac{1}{\kappa}\vec{n}) \times \vec{b} \end{pmatrix}.$$

Hence the velocity twist of the line symmetric motion generated by the binormal lines to a curve $\vec{\gamma}$ is given by,

$$\mathbf{s}_d = 2\nu \begin{pmatrix} \tau\vec{t} + \kappa\vec{b} \\ \vec{\gamma} \times (\tau\vec{t} + \kappa\vec{b}) \end{pmatrix} + 2\nu\kappa \begin{pmatrix} -\vec{b} \\ (-\vec{\gamma} + \frac{1}{\kappa}\vec{n}) \times \vec{b} \end{pmatrix} = 2\nu\tau \begin{pmatrix} \vec{t} \\ \vec{\gamma} \times \vec{t} + \frac{1}{\tau}\vec{t} \end{pmatrix}.$$

The pitch of the motion is

$$p = \frac{\vec{t} \cdot (\vec{\gamma} \times \vec{t} + 1/\tau\vec{t})}{|\vec{t}|^2} = \frac{1}{\tau}.$$

So the axis of this twist ℓ, that is, the generator line of the fixed axode is

$$\ell = 2\nu\tau \begin{pmatrix} \vec{t} \\ \vec{\gamma} \times \vec{t} \end{pmatrix}.$$

We can ignore the coefficient $2\nu\tau$, since these are homogeneous coordinates. From these computations have the following:

Theorem 6.2.3. *The pitch of the twist velocity of a line symmetric motion generated by the binormal lines to a curve $\vec{\gamma}$ is $p = 1/\tau$ and the fixed axode of the motion is generated by the tangent lines to the curve.*

Finally we will investigate the line symmetric motion generated by reflections about the tangent lines to a curve. In this case the two plane symmetric motions which combine to produce this motion are reflections in the osculating and rectifying planes. The reflections in the rectifying plane and osculating plane of $\vec{\gamma}$, C and M be as

above, Writing the line symmetric motion generated by tangent lines of $\vec{\gamma}$ as,

$$G = CMC_0M_0.$$

It is clear from the proof of theorem (6.2.1), that the twist velocity of the motion is,

$$\dot{G}G^{-1} = \dot{C}C + C\dot{M}MC.$$

This time however, the twist $\dot{M}M$ must be reflected in the rectifying plane of the curve.

The 4×4 matrix representation of velocity twist of osculating planes motion can be computed as,

$$\dot{M}M = 2\nu\tau \begin{pmatrix} \vec{b}\vec{n}^T - \vec{n}\vec{b}^T & (\vec{\gamma}\cdot\vec{b})\vec{n} - (\vec{\gamma}\cdot\vec{n})\vec{b} \\ 0 & 0 \end{pmatrix},$$

and the reflection of the twist velocity of osculating planes motion in the rectifying plane,

$$C\dot{M}MC = 2\nu\tau \begin{pmatrix} \vec{n}\vec{b}^T - \vec{b}\vec{n}^T & (\vec{\gamma}\cdot\vec{n})\vec{b} - (\vec{\gamma}\cdot\vec{b})\vec{n} \\ 0 & 0 \end{pmatrix}.$$

So the 6-vector representation of the reflected twist $C\dot{M}MC$ is

$$2\nu\tau \begin{pmatrix} -\vec{t} \\ \vec{\gamma}\times -\vec{t} \end{pmatrix}.$$

Hence the velocity twist of the line symmetric motion generated by the tangent lines to a curve $\vec{\gamma}$ is given by,

$$\mathbf{s}_d = 2\nu \begin{pmatrix} \tau\vec{t} + \kappa\vec{b} \\ \vec{\gamma}\times(\tau\vec{t} + \kappa\vec{b}) \end{pmatrix} - 2\nu\tau \begin{pmatrix} \vec{t} \\ \vec{\gamma}\times\vec{t} \end{pmatrix} = 2\nu\kappa \begin{pmatrix} \vec{b} \\ \vec{\gamma}\times\vec{b} \end{pmatrix}.$$

The pitch of the motion is

$$p = \frac{(\vec{\gamma}\times\vec{b})\cdot\vec{b}}{|\vec{b}|^2} = 0.$$

Hence the generator line ℓ of the fixed axode is equal to s_d. From these computations have the following:

Theorem 6.2.4. *The pitch of the twist velocity of a line symmetric motion generated by the tangent lines to a curve $\vec{\gamma}$ is zero and the fixed axode of the motion is generated by parallel lines to the binormal vector of the curve.*

CHAPTER SEVEN
CONCLUSION

Two rotational displacements and their composition are studied as a spherical triangle. Then Campell-Baker-Hausdorff formula for $SO(3)$ is obtained by using dual quaternions. This case is also a special type of $SE(3)$. Then two spatial displacements and their product is investigated as a screw triangle and the Campell-Baker-Hausdorff formula for $SE(3)$ is obtained by using dual quaternions in terms of iterated commutators. Some relations between iterated commutators in $SE(3)$ are given. It would be interesting to see the relations between iterated commutators in $SE(3)$ or to see which of them are linearly independent. In the last section of this chapter, special types of 2-system of screws are investigated by using Gibson & Hunt (1990) classification. The planar case, pole triangle, is also studied in these special cases. It might be useful to investigate Lie triple system of screws using the tools developed in this chapter.

The product of two group elements of $SE(3)$ is investigated by using Cayley maps. The relations between the 4×4 Cayley map, the 6×6 Cayley map and the exponential map are obtained. The Campell-Baker-Hausdorff like formula for the 4×4 Cayley map and the 6×6 Cayley map are derived.

The latest work is independent from the above studies. The plane symmetric motions are investigated. The definition of plane symmetric motions is given and the axodes of these motions are obtained as developable ruled surfaces. Then plane symmetric motions generated by reflections in the osculating, normal and rectifying planes of a space curve are investigated. It is shown that any ruled surface that is a tangent developable surface is the fixed axode for some plane symmetric motion. So, in some sense, the case of the motion based on the osculating planes to a curve is the general case. A connection is shown between the plane symmetric motion based on the normal planes to a curve and the Bishop motion for the same curve. The fixed axodes of the motions are the same, however the motions are different. This demonstrates that the fixed axode is not sufficient to determine a rigid-body motion. As well as looking at

the motion based on the rectifying planes to a curve a forth case, given by the reflection in the normal planes to the curve's Darboux vector, is studied. Then, in particular the motion based on the osculating, normal and rectifying planes of a twisted cubic curve are studied as an example. Then the acceleration properties and inflection points of these plane symmetric motions are obtained. It is shown that the acceleration centre and inflection point of a motion based on the osculating planes to a curve is the current point on the curve and the acceleration centres and inflection points of motions based on the normal planes and rectifying planes to a curve are the polar curve of the curve and the rectifying curve of the given curve respectively. It might be useful to investigate higher order properties of these motions. Then particular types of curves that have partner curves are studied. In these cases there are relations between the motions determined by the curve and the motion determined by its partner curve. Finally, line-symmetric motions are investigated by thinking of a reflection in a line as a composite of reflections in orthogonal planes meeting at the line in question. In this way the twist velocity and fixed axodes of line symmetric motions generated by reflections in the tangent, normal and binormal lines of a space curve are found. The acceleration and higher order properties of line symmetric motions could be investigated using the tools developed in this thesis.

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