DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

PRODUCT TYPE OSCILLATION TESTS FOR PARTIAL DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

by
Büşra ÖZSAVAŞ

March, 2021
İZMİR

PRODUCT TYPE OSCILLATION TESTS FOR PARTIAL DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**PRODUCT TYPE OSCILLATION TESTS FOR PARTIAL DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS**" completed by **BÜŞRA ÖZSAVAŞ** under supervision of **PROF. DR. BAŞAK KARPUZ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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PRODUCT TYPE OSCILLATION TESTS FOR PARTIAL DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

ABSTRACT

In this thesis, we will review some of the most well known the results in the oscillation theory of ordinary difference equations and partial difference equations. Later, we present new oscillation tests for the partial difference equations. Our main result improves some of the well-known results in the literature. We also provide a numerical example where not all previous results in the literature give an answer.

Keywords: Difference equations, ordinary difference equations, partial difference equations, oscillation, nonoscillation

DEĞİŞKEN KATSAYILI KISMİ FARK DENKLEMLER İÇİN ÇARPIM TİPLİ SALINIM TESTLERİ

ÖZ

Bu çalışmada, adi fark denklemlerinin ve kısmi fark denklemlerinin salınımına ilişkin sonuçları gözden geçirilmektedir. Daha sonra, kısmi fark denklemleri için yeni salınım testleri sunulmaktadır. Ana sonucumuz, literatürde iyi bilinen bazı sonuçları geliştirmektedir. Ayrıca, literatürdeki önceki tüm sonuçların cevap vermediği sayısal bir örnek sunulmaktadır.

Anahtar kelimeler: Fark denklemleri, adi fark denklemleri, kısmi fark denklemleri, salınım, salınımsızlık

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CHAPTER ONE
INTRODUCTION

Researchers who conduct scientific studies in technological inventions that make our daily lives easier, in the investigation of various natural phenomena, physical events, and economic fluctuations describe the problems they encounter with mathematical models. These models are expressed by mathematical equations. If the independent variable in the problem is continuous and we will express the relationships between rates of change of various quantities, the model is expressed with differential equations. In the case that the independent variable is not continuous, the model is expressed with difference equations. In this case, one can said that difference equations are a discrete analogue of differential equations.

Definition 1.0.1. *The equation*

$$u(m+\tau) = f\big(m, u(m+\tau-1), u(m+\tau-2), \cdots, u(m)\big)$$

for a given function $f \in \mathrm{C}(\mathbb{N}_0 \times \mathbb{R}^\tau, \mathbb{R})$*, where* $\mathbb{N}_0 := \{0, 1, \cdots\}$*, and unknown quantities* $u(m)$*,* $m = 0, 1, \cdots$ *is called a difference equation of order* τ*.*

Definition 1.0.2. *The equation*

$$a_\tau(m)u(m+\tau)+a_{\tau-1}(m)u(m+\tau-1)+\cdots+a_0(m)u(m) = \beta(m), \quad a_0(m)a_\tau(m) \neq 0$$

is called linear difference equation of order τ*.*
If the coefficients $a_i(m)$ *(*$i = 0, 1, \cdots, \tau$*) depend on* m*, the equation is called an equation with variable. If the coefficients* $a_i(m)$ *(*$i = 0, 1, \cdots, \tau$*) does* ***not*** *depend on* m*, the equation is called an equation with constant coefficients.*
If $\beta(m) \equiv 0$*, then the equation is homogenous. If* $\beta(m) \not\equiv 0$*, then the equation is nonhomogeous.*

Difference equations appear mostly in some real world problems and approaching numerically to solutions of differential equations with finite difference schemes. For instance, consider the transport equation

$$z_t + az_x + bz = 0 \quad \text{for } (x,t) \in \mathbb{R} \times [0,\infty), \tag{1.1}$$

where $a, b \in \mathbb{R}$ (Evans, 2010, § 2.1). In real world applications, it is not always the case that changes happen instantaneously but involves some response delay. That is, z_t and z_x in general depend on $z(x, t-\beta)$, $z(x-\alpha, t)$, or $z(x-\alpha, t-\beta)$ rather than $z(x,t)$. Thus, we modify (1.1) as in the following form

$$z_t + az_x + bz(x-\alpha, t-\beta) = 0, \tag{1.2}$$

where $\alpha \geq 0$ and $\beta \geq 0$. Using the forward difference approximation for (1.2), we can write

$$\frac{z(x,t+k_0) - z(x,t)}{k_0} + a\frac{z(x+h_0,t) - z(x,t)}{h_0} + bz(x-\alpha, t-\beta) = 0,$$

where $k_0, h_0 > 0$. Supposing that $z(x,t) \approx u(m,n)$, $z(x,t+k_0) \approx u(m,n+1)$, $z(x+h_0,t) \approx u(m+1,n)$ and $z(x-\alpha,t-\beta) \approx u(m-\tau, n-\sigma)$, we obtain the linear partial difference equation

$$ak_0u(m+1,n) + h_0u(m,n+1) - (ak_0+h_0)u(m,n) + bk_0h_0u(m-\tau, n-\sigma) = 0.$$

Although the existence and uniqueness of solutions can be proved for many classes of equations, it is not possible to obtain the solution with a simple notation or to clearly state it. For such equations it is important to at least determine how the solutions behave. Two important areas that examine how solutions behave are stability theory and oscillation theory. While the stability theory examines whether small changes in the initial data will cause small changes in the solution or whether the solution can converge to any equilibrium value according to the initial data, the oscillation theory examines whether the sign of the solutions will change according to the initial values.

In this thesis, we will review the results on the oscillation of ordinary difference equation

$$u(m+1) - u(m) + \sum_{k=1}^{r} p_k(m)u(m - \tau_k) = 0, \quad m \geq 0,$$

where $r \in \mathbb{N}$, $\{p_k(m)\} \subset \mathbb{R}_0^+ := [0, \infty)$ and $\tau_k \in \mathbb{N}_0 := \{0, 1, \cdots\}$ and partial difference equations

$$u(m+1, n) + u(m, n+1) - u(m, n) + \sum_{k=1}^{r} p_k(m, n)u(m - \tau_k, n - \sigma_k) = 0, \quad m, n \geq 0,$$

where $r \in \mathbb{N}$, $\{p_k(m, n)\} \subset \mathbb{R}_0^+$ and $\tau_k, \sigma_k \in \mathbb{N}_0$.
In Chapter 2, we present some theorems in the literature about oscillation of ordinary difference equations. Firstly, we will give the most fundamental result obtained in Győri & Ladas (1989) regarding the oscillation of ordinary difference equations. After, we give more general results that that extends this result. Also, we will compare these results with examples.

In Chapter 3, we investigate the partial difference equations. Firstly, we will give necessary and sufficient condition for the oscillation of partial difference equations. (Zhang & Liu, 1997a, Theorem 3.1). Then, we will present results obtained in (Zhang et al., 1995, Theorem 3.4) and (Zhang et al., 1995, Theorem 3.6) that give the oscillation conditions of the partial difference equation. After, we will give the results that Zhang and Liu improved these results.

In Chapter 4, we will improve the results for the oscillation of the partial difference equations. Firstly, we prove a simple but important result which generalizes (Zhang et al., 1995, Theorem 3.1). Then, we introduce the notion of path, which forms the snipe of this study, and we prove our new theorems improving (Zhang & Liu, 1997b, Theorem 2.1) and give several corollaries. Also, we give an example that none of the results in the literature can deliver an answer.

CHAPTER TWO
ORDINARY DIFFERENCE EQUATIONS

In this chapter, we will consider the ordinary difference equation

$$u(m+1) - u(m) + \sum_{k=1}^{r} p_k(m)u(m-\tau_k) = 0, \quad m \geq 0, \tag{2.1}$$

where $r \in \mathbb{N}$, $\{p_k(m)\} \subset \mathbb{R}_0^+ := [0, \infty)$ and $\tau_k \in \mathbb{N}_0 := \{0, 1, \cdots\}$.

Definition 2.0.1. *A **solution** of* (2.1) *is a sequence* $\{u(m)\}$ *defined for* $m \geq -\tau^*$*, where* $\tau^* := \max_k\{\tau_k\}$*, which satisfies* (2.1) *for* $m = 0, 1, \cdots$.

One can easily see that if $\{u(m)\}$ is prescribed on $\{-\tau^*, -\tau^* + 1, \cdots, 0\}$, then it is uniquely defined on $\{-\tau^*, -\tau^* + 1, \cdots\}$.

Definition 2.0.2. *A solution of* (2.1) *is called **eventually positive** if* $u(m) > 0$ *for all large* m*, while it is called **eventually negative** if* $u(m) < 0$ *for all large* m*. A solution of* (2.1) *is called **oscillatory** if it is neither eventually positive nor eventually negative.*

First, we will consider the autonomous difference equation

$$u(m+1) - u(m) + \sum_{k=1}^{r} p_k u(m-\tau_k) = 0, \quad m \geq 0, \tag{2.2}$$

where $r \in \mathbb{N}$, $p_k \in \mathbb{R}$ and $\tau_k \in \mathbb{N}_0$. Equation (2.2) is associated with the characteristic equation

$$\lambda - 1 + \sum_{k=1}^{r} \frac{p_k}{\lambda^{\tau_k}} = 0. \tag{2.3}$$

In Győri & Ladas (1989), the most fundamental result on the oscillation of ordinary difference equations is obtained.

Theorem 2.0.1 (Győri & Ladas, 1989, Theorem 2)**.** *Every solution of the autonomous equation* (2.2) *is oscillatory if and only if the characteristic equation* (2.3) *has no roots in* $\mathbb{R}^+$.

Proof. Assume that every solution of equation (2.2) is oscillatory. Suppose, for the sake of contradiction, that the characteristic equation (2.3) has a positive root λ. But then $u(m) = \lambda^m$ is a nonoscillatory solution of equation (2.2), which is a contradiction.

Assume for the sake of contradiction, that equation (2.2) has a nonoscilatory solution $\{u(m)\}$. As the opposite of a solution of equation (2.2) is also a solution, we may assume that $\{u(m)\}$ is eventually positive. Then eventually,

$$u(m+1) - u(m) = -\sum_{k=1}^{r} p_k u(m-\tau_k) < 0$$

and so $\{u(m)\}$ is eventually decreasing. If $\tau_k = 0$ for all $k = 1, 2, \cdots, r$, then from (2.2) it follows that

$$u(m+1) - \left(1 - \sum_{k=1}^{r} p_k\right) u(m) = 0, \quad m = 0, 1, 2, \cdots$$

and so $\lambda_0 = 1 - \sum_{k=1}^{r} p_k > 0$. But in this case equation (2.3) has the positive root λ_0, which is impossible. Hence, we will suppose that $\tau_k > 0$ for at least one $k \in \{1, 2, \cdots, r\}$. Define the set

$$F(u) := \{\lambda \in (-\infty, \infty) : \; u(m+1) - \lambda u(m) \leq 0, \quad m \gg 0\}$$

with the convention that all the inequalities in this proof, which involve m, are assumed to be true for all sufficiently large m.

Clearly,

$$1 \in F(u) \quad \text{and} \quad (-\infty, 0] \cap F(u) = \emptyset.$$

Therefore, the proof will be complete if we establish the following claim: There is at least a positive number α such that

$$\lambda \in F(u) \Rightarrow \lambda - \alpha \in F(u). \tag{2.4}$$

To this end, let us first define the number α. Set

$$F(\lambda) := \lambda - 1 + \sum_{k=1}^{r} p_k \lambda^{-\tau_k}, \quad \lambda \in \mathbb{R}\backslash\{0\}$$

and observe that $F(\infty) = \infty$ and $F(0^+) = \infty$. As $F(\lambda) = 0$ has no positive roots, it follows that

$$\alpha := \min_{\lambda>0}\{F(\lambda)\}$$

exists and is positive. Clearly,

$$-1 + \sum_{k=1}^{r} p_k \lambda^{-\tau_k} \geq \alpha - \lambda, \quad \lambda > 0. \tag{2.5}$$

Now, let $\lambda \in F(u)$. Then, for every $k = 1, 2, \cdots, r$ and for m sufficiently large,

$$u(m+1) \leq \lambda u(m) \quad \text{or} \quad u(m-1) \leq \frac{1}{\lambda} u(m)$$

and by induction

$$u(m - \tau_k) \geq \lambda^{-\tau_k} u(m). \tag{2.6}$$

Hence, from (2.2), (2.5) and (2.6), we see that

$$\begin{aligned}
0 &= u(m+1) - u(m) + \sum_{k=1}^{r} p_k(m) u(m - \tau_k) \\
&\geq u(m+1) - u(m) + \sum_{k=1}^{r} p_k(m) \lambda^{-\tau_k} u(m) \\
&= u(m+1) + \left(-1 + \sum_{k=1}^{r} p_k \lambda^{-\tau_k}\right) u(m) \\
&\geq u(m+1) + (\alpha - \lambda) u(m) \\
&= u(m+1) - (\lambda - \alpha) u(m),
\end{aligned}$$

which proves (2.4). The proof of the Theorem 2.0.1 is complete. □

Remark 1. *Note that if* $p_k \in \mathbb{R}^+$ *and* $\lambda \geq 1$*, then* (2.3) *cannot hold, and*

$$\inf_{\lambda \in (0,1)} \left\{ \frac{1}{(1-\lambda)\lambda^{\tau}} \right\} = \frac{(\tau+1)^{\tau+1}}{\tau^{\tau}},$$

where $\tau \in \mathbb{N}_0$ and we use the convention that $0^0 := 1$.

Corollary 2.0.1.1 (Ladas et al., 1989a, Remark 1)**.** *Assume $p_k \in \mathbb{R}^+$ and*

$$\sum_{k=1}^{r} p_k \frac{(\tau_k + 1)^{\tau_k+1}}{\tau_k^{\tau_k}} > 1.$$

Then, every solution of the autonomous equation (2.2) *is oscillatory.*

For the autonomous equation with a single delay term

$$u(m+1) - u(m) + pu(m-\tau) = 0, \quad m \geq 0, \tag{2.7}$$

where $p \in \mathbb{R}$ and $\tau \in \mathbb{N}$, we can give the following more general result.

Theorem 2.0.2 (Győri & Ladas, 1991, Theorem 7.2.1)**.** *Every solution of the autonomous equation* (2.7) *is oscillatory if and only if*

$$p > \frac{\tau^\tau}{(\tau+1)^{\tau+1}}.$$

Proof. We can get the proof by computing the extreme value of characteristic eqaution and by applying Theorem 2.0.1. Set

$$F(\lambda) := \lambda - 1 + p\lambda^{-\tau}, \quad \lambda \in \mathbb{R}\backslash\{0\}.$$

Then,

$$F'(\lambda) = 1 - p\tau\lambda^{-\tau-1}$$

and

$$F'(\lambda_0) = 1 - p\tau\lambda_0^{-\tau-1} = 0,$$

where $\lambda_0 := (p\tau)^{\frac{1}{\tau+1}}$.

Also,

$$F''(\lambda) = p(\tau+1)\tau\lambda^{-\tau-2} \quad \text{for} \quad \lambda > 0.$$

Clearly, λ_0 is only critical point of F in $(0, \infty)$ and F has a minimum at λ_0. Also,

$F(0^+) = \infty$ and $F(\infty) = \infty$. Therefore, $F(\lambda)$ has a global minimum in $(0, \infty)$ at the point λ_0.

By the Theorem 2.0.1, every solution of the autonomous equation (2.7) is oscillatory if and only if $F(\lambda_0) > 0$. We simply compute that

$$\begin{aligned} F(\lambda_0) &= \lambda_0 - 1 + p\lambda_0^{-\tau} \\ &= \lambda_0\left(1 - \frac{1}{\lambda_0} + p\lambda_0^{-\tau-1}\right) \\ &= \lambda_0\left(1 - \frac{1}{\lambda_0} + p\big((p\tau)^{\frac{1}{\tau+1}}\big)^{-\tau-1}\right) \\ &= \lambda_0\left(1 - \frac{1}{\lambda_0} + \frac{1}{\tau}\right) \\ &= \lambda_0\left(\frac{\tau+1}{\tau} - \frac{1}{\lambda_0}\right). \end{aligned}$$

Thus, $F(\lambda_0) > 0$ if and only if $\lambda_0 > \frac{\tau}{\tau+1}$ if and only if $p\tau = \lambda_0^{\tau+1} > \left(\frac{\tau}{\tau+1}\right)^{\tau+1}$ if and only if $p > \frac{\tau^\tau}{(\tau+1)^{\tau+1}}$, as desired. □

Now, we consider the difference equation with a single delay term and a variable coefficient

$$u(m+1) - u(m) + p(m)u(m-\tau) = 0, \quad m \geq 0, \tag{2.8}$$

where $\{p(m)\} \subset \mathbb{R}_0^+$ and $\tau \in \mathbb{N}_0$, we can give the following more general result, which extends Theorem 2.0.2.

Theorem 2.0.3 (Erbe & Zhang, 1989, Theorem 3.1, Theorem 3.2)**.**

(i) Assume that

$$\liminf_{m\to\infty} p(m) > \frac{\tau^\tau}{(\tau+1)^{\tau+1}}. \tag{2.9}$$

Then, every solution of (2.8) *is oscillatory.*

(ii) Assume that

$$p(m) \leq \frac{\tau^\tau}{(\tau+1)^{\tau+1}} \quad \textit{for all large } m. \tag{2.10}$$

Then, (2.8) *possesses an eventually positive solution.*

Proof. (i) Suppose that there is a solution $u(m)$ of (2.8) with $u(m) > 0$ for $m \geq M_1$. Setting $w(m) := \frac{u(m)}{u(m+1)}$ and dividing inequality $u(m+1) - u(m) + p(m)u(m-\tau) \leq 0$ by $u(m)$ and rearranging we have

$$\frac{1}{w(m)} \leq 1 - p(m)w(m-\tau)\cdots w(m-1), \qquad m \geq M_1 + \tau. \tag{2.11}$$

From (2.9), $p(m) > 0$ for $m \geq M_2$ and setting $M := \max\{M_2, M_1 + \tau\}$ it follows that $u(m)$ is weakly decreasing for $m \geq M$ and so $w(m) \geq 1$. Also (2.9) and (2.11) imply that $w(m) < 0$ for arbitrarily large m. If we set $\ell := \liminf_{m\to\infty} w(m)$, then from (2.11) we get

$$\limsup_{m\to\infty} \frac{1}{w(m)} = \frac{1}{\ell} \leq 1 - \liminf_{m\to\infty}\big[p(m)w(m-\tau)\cdots w(m-1)\big]. \tag{2.12}$$

Since $\liminf_{m\to\infty}[p(m)w(m-\tau)\cdots w(m-1)] \geq \alpha\ell^{\tau}$, where $\alpha := \liminf_{m\to\infty} p(m)$, we have $\frac{1}{\ell} \leq 1 - \alpha\ell^{\tau}$ or

$$\alpha \leq h(\ell), \quad h(\lambda) = \frac{\lambda - 1}{\lambda^{\lambda+1}},\ \lambda \geq 1. \tag{2.13}$$

Now, it is easy to see that $\max_{\lambda\geq 1}\{h(\lambda)\} = \frac{\tau^{\tau}}{(\tau+1)^{\tau+1}}$ and hence we obtain $\alpha \leq \frac{\tau^{\tau}}{(\tau+1)^{\tau+1}}$, contradicting (2.9). This completes the proof.

(ii) We show that

$$\frac{1}{w(m)} = 1 - p(m)w(m-\tau)\cdots w(m-1) \tag{2.14}$$

has a positive solution. To see this, define

$$s(m) := \begin{cases} \dfrac{\tau+1}{\tau}, & m = M-\tau, \cdots, M-1 \\ \dfrac{1}{1 - p(m)s(m-\tau)\cdots s(m-1)}, & m = M, M+1, \cdots. \end{cases} \tag{2.15}$$

From (2.15), it follows that $s(m) < 1$ and $s(M) < \frac{\tau+1}{\tau}$ so we define

$$s(M+1) = \frac{1}{1 - p(M+1)s(M+1-\tau)\cdots s(M)} < \frac{\tau+1}{\tau} \tag{2.16}$$

and by induction $1 < s(M+1), s(M+2), \cdots < \frac{\tau+1}{\tau}$ so that the sequence $\{s(m)\}$, $m = M, M+1, \cdots$, is a solution of (2.14). Next, defining

$$u(m) := \begin{cases} 1, & m = M-\tau, \cdots, M \\ \dfrac{u(m-1)}{s(m-1)}, & m = M+1, M+2, \cdots, \end{cases}$$

it follows that $\{u(m)\}$ satisfies (2.8). □

Remark 2. *Note that for $\tau \in \mathbb{N}$, we have*

$$1 > \frac{1}{\tau+1}\left(\frac{\tau}{\tau+1}\right)^{\tau} = \frac{\tau^{\tau}}{(\tau+1)^{\tau+1}}.$$

For autonomous equations the following theorem is not useful but it is new for (2.8), i.e., for equations with variable coefficients.

Theorem 2.0.4 ((Erbe & Zhang, 1989, Theorem 2.1))**.** *Assume that there exists an increasing sequence $\{\xi_k\} \subset \mathbb{N}$ such that*

$$p(\xi_k) \geq 1 \quad \textit{for all } k. \tag{2.17}$$

Then, every solution of (2.8) *is oscillatory.*

Proof. Assume that there exist an increasing sequence $\{\xi_k\}$ such that $1 - p(\xi_k) \leq 0$ for all k, i.e., $p(\xi_k) \geq 1$ for all k. Now, assume that $\{u(m)\}$ is an eventually positive solution of (2.8). Then, there exists $m_1 \geq m_0$ such that $u(m) > 0$ and $u(m-\tau) > 0$ for $m = m_1, m_1+1, \cdots$. From (2.8), we see that $u(m+1) - u(m) \leq 0$ for $m = m_1, m_1+1, \cdots$, i.e., $u(m+1) < u(m)$ for $m = m_1, m_1+1, \cdots$. Clearly, $\xi_k \geq m_1 + \tau$ for all $k \gg 0$. It follows from (2.8) that

$$\begin{aligned} 0 &> u(\xi_k+1) - u(\xi_k) + p(\xi_k)u(\xi_k) \\ &= u(\xi_k+1) - [1 - p(\xi_k)]u(\xi_k) > 0 \end{aligned}$$

for all $k \gg 0$, which is a contradiction. □

Remark 3. *Theorem 2.0.3 (i) and Theorem 2.0.4 are* ***not*** *comparable.*

Example 1. Set the difference equation

$$u(m+1) - u(m) + \left\{ \begin{matrix} 1, & m = \text{even} \\ \frac{1}{9}, & m = \text{odd} \end{matrix} \right\} u(m-2) = 0 \quad \text{for } m = 0, 1, \cdots. \quad (2.18)$$

For this equation, we compute that

$$\liminf_{m\to\infty} p(m) = \frac{1}{9} \not> \frac{2^2}{3^3}.$$

showing that Theorem 2.0.3 (i) is **not** applicable here. Fortunately,

$$p(2k) = 1 \quad \text{for } k = 1, 2, \cdots,$$

and thus, by Theorem 2.0.4, all solutions of (2.18) oscillate.

Example 2. Set the difference equation

$$u(m+1) - u(m) + \left\{ \begin{matrix} \frac{5}{27}, & m = \text{even} \\ \frac{2}{9}, & m = \text{odd} \end{matrix} \right\} u(m-2) = 0 \quad \text{for } m = 0, 1, \cdots. \quad (2.19)$$

For this equation, we see that

$$p(m) \leq \frac{2}{9} < 1 \quad \text{for all } m,$$

showing that Theorem 2.0.4 is **not** applicable here. Fortunately,

$$\liminf_{m\to\infty} p(m) = \frac{5}{27} > \frac{4}{27} = \frac{2^2}{3^3}.$$

and thus, by Theorem 2.0.3 (i), every solution of (2.19) oscillates.

The following theorem extends Theorem 2.0.3 (i) to (2.1).

Theorem 2.0.5 (Tang & Deng, 1998, Theorem 4.1)**.** *Assume that*

$$\liminf_{m\to\infty} \sum_{k=1}^{r} \frac{(\tau_k+1)^{\tau_k+1}}{\tau_k^{\tau_k}} p_k(m) > 1. \tag{2.20}$$

Then, all solutions of (2.1) *are oscillatory.*

We need a lemma for the proof of Theorem 2.0.5.

Lemma 2.0.6. *Assume that* $\{u(m)\}$ *is eventually positive solution of* (2.1)*, and*

$$\liminf_{m\to\infty} \sum_{k=1}^{r} p_k(m) > 0. \tag{2.21}$$

Then, the set

$$F(u) = \{\lambda > 0 : u(m+1) - \lambda u(m) \leq 0, \quad m \gg 0\} \tag{2.22}$$

is nonempty and is bounded.

Proof. Since $\{u(m)\}$ is the eventually positive solution of (2.1), without loss of generality, we can set $u(m) > 0$, for $m \geq -\tau$, where $\tau := \max_k\{\tau_k\}$. By equation (2.1), there is $u(m+1) - u(m) = -\sum_{k=1}^{r} p_k(m)u(m-\tau_k) \leq 0$, then $\{u(m)\}$ is monotonically decreasing. From (2.21), there exist a sufficiently large positive integer M and a positive number $\alpha \in (0,1)$ such that

$$\sum_{k=1}^{r} p_k(m) > \alpha, \quad m \geq M. \tag{2.23}$$

Then, by (2.23) and from the monotonicity of $\{u(m)\}$, we have

$$u(m+1) - u(m) \leq -\sum_{k=1}^{r} p_k(m)u(m) \leq -\alpha u(m).$$

So, we have

$$u(m+1) - (1-\alpha)u(m) \leq 0, \quad m \geq M.$$

This shows that $(1-\alpha) \in F(u)$, that is, the set $F(u)$ is not empty, and because $\lambda \leq 1 - \frac{u(m+1)}{u(m)} < 1$, $m = M, M+1, \cdots$, for any $\lambda \in F(u)$, therefore $F(u) \subset (0,1)$. □

Proof of Theorem 2.0.5. Suppose to contrary that equation (2.1) has a nonoscillatory solution. Without loss of generality, we may suppose that (2.1) has an eventually positive solution $\{u(m)\}$, that is $u(m) > 0$ for $m \geq -\tau$, where $\tau := \max_k\{\tau_k\}$. Note that (2.20) implies (2.21). By Lemma 2.0.6, the set $F(u)$ defined in (2.22) satisfies $F(u) \neq \emptyset$ and $\sup\{F(u)\} < \infty$. For any $\lambda \in F(u)$, we get

$$u(m) \leq (1-\lambda)^{\tau_k} u(m-\tau_k), \quad k = 1, 2, \cdots, r. \tag{2.24}$$

Substituting (2.24) into (2.1), we obtain,

$$u(m+1) - u(m) + \sum_{k=1}^{r} \frac{p_k(m)}{(1-\lambda)^{\tau_k}} u(m) \leq 0$$

or equivalently

$$u(m+1) - \left(1 - \sum_{k=1}^{r} \frac{p_k(m)}{(1-\lambda)^{\tau_k}}\right) u(m) \leq 0.$$

That is,

$$u(m+1) - \left(1 - \inf_{m \geq M_0} \left\{\sum_{k=1}^{r} \frac{p_k(m)}{(1-\lambda)^{\tau_k}}\right\}\right) u(m) \leq 0.$$

So

$$\inf_{m \geq M_0} \left\{\sum_{k=1}^{r} \frac{p_k(m)}{(1-\lambda)^{\tau_k}}\right\} \in F(u).$$

Since $\lambda_0 := \sup\{F(u)\}$ exists, we can write

$$\inf_{m \geq M_0} \left\{\frac{1}{\lambda_0} \sum_{k=1}^{r} \frac{p_k(m)}{(1-\lambda)^{\tau_k}}\right\} \leq 1.$$

As $\lambda \to \lambda_0$, we get

$$\inf_{m \geq M_0} \left\{\sum_{k=1}^{r} \frac{p_k(m)}{\lambda_0 (1-\lambda_0)^{\tau_k}}\right\} \leq 1 \tag{2.25}$$

We compute that

$$\min_{0\le\lambda\le 1}\left\{\frac{1}{\lambda(1-\lambda)^{\tau_k}}\right\}=\frac{(\tau_k+1)^{\tau_k+1}}{\tau_k^{\tau_k}}\le\frac{1}{\lambda_0(1-\lambda_0)^{\tau_k}}.$$

From (2.25), we get

$$\inf_{m\ge M_0}\left\{\sum_{k=1}^{r}\frac{(\tau_k+1)^{\tau_k+1}}{\tau_k^{\tau_k}}p_k(m)\right\}\le\inf_{m\ge M_0}\left\{\sum_{k=1}^{r}\frac{p_k(m)}{\lambda_0(1-\lambda)^{\tau_k}}\right\}\le 1.$$

This contradiction completes the proof. □

Theorem 2.0.3 can be regarded as a pointwise result, and the following result improves Theorem 2.0.3 by using the arithmetic mean of the coefficient p over consecutive τ-terms.

Theorem 2.0.7 (Ladas et al., 1989b, Theorem 1)**.** *Assume that*

$$\liminf_{m\to\infty}\sum_{i=m-\tau}^{m-1}p(i)>\left(\frac{\tau}{\tau+1}\right)^{\tau+1}. \tag{2.26}$$

Then, all solutions of (2.8) *are oscillatory.*

Proof. Suppose that the equation (2.8) has a nonoscillatory solution $\{u(m)\}$.

We can suppose that $\{u(m)\}$ is eventually positive since the opposite of a solution of equation (2.8) is also a solution. Then

$$u(m+1)-u(m)=-p(m)u(m-\tau)\le 0,$$

and $\{u(m)\}$ is decreasing sequence of positive numbers. Then

$$u(m-\tau)>u(m)$$

and so

$$u(m+1)-u(m)+p(m)u(m)\le 0,$$

or equivalently

$$p(m) \leq 1 - \frac{u(m+1)}{u(m)}.$$

Then,

$$\frac{1}{\tau}\sum_{i=m-\tau}^{m-1} p(i) \leq \frac{1}{\tau}\sum_{i=m-\tau}^{m-1}\left(1 - \frac{u(i+1)}{u(i)}\right). \tag{2.27}$$

Let

$$\alpha := \frac{\tau^\tau}{(\tau+1)^{\tau+1}}. \tag{2.28}$$

Then, from (2.26), it is easily seen that we can take a constant β such that,

$$\alpha < \beta \leq \frac{1}{\tau}\sum_{i=m-\tau}^{m-1} p(i), \quad m \gg 0. \tag{2.29}$$

Thus, in view of (2.27),

$$\beta \leq \frac{1}{\tau}\sum_{i=m-\tau}^{m-1}\left(1 - \frac{u(i+1)}{u(i)}\right), \quad m \gg 0. \tag{2.30}$$

By using (2.30) and the inequality between the arithmetic and geometric means, we get

$$\begin{aligned}\beta &\leq \frac{1}{\tau}\sum_{i=m-\tau}^{m-1}\left(1 - \frac{u(i+1)}{u(i)}\right) = 1 - \frac{1}{\tau}\sum_{i=m-\tau}^{m-1}\frac{u(i+1)}{u(i)} \\ &\leq 1 - \left(\prod_{i=m-\tau}^{m-1}\frac{u(i+1)}{u(i)}\right)^{1/\tau} = 1 - \left(\frac{u(m)}{u(m-\tau)}\right)^{1/\tau}, \quad m \gg 0,\end{aligned}$$

that is,

$$\left(\frac{u(m)}{u(m-\tau)}\right)^{1/\tau} \leq 1 - \beta, \quad m \gg 0. \tag{2.31}$$

Especially, this implies $0 < \beta < 1$.

Now, observe that

$$\max_{0\leq\lambda\leq 1}\left\{(1-\lambda)\lambda^{1/\tau}\right\} = \frac{\tau}{(\tau+1)^{1+\frac{1}{\tau}}} = \alpha^{\frac{1}{\tau}},$$

where α is the positive constant defined by (2.28). Therefore,

$$1 - \lambda \leq \alpha^{\frac{1}{\tau}}\lambda^{-\frac{1}{\tau}} \quad \text{for} \quad 0 < \lambda \leq 1$$

and (2.31) yields

$$\frac{\beta}{\alpha}u(m) \le u(m-\tau) \quad \text{for all large} \quad m \tag{2.32}$$

By using (2.32) in equation (2.8) and then by repeating the above arguments, we find that

$$\left(\frac{\beta}{\alpha}\right)^2 u(m) \le u(m-\tau) \quad \text{for all large} \quad m$$

and, by induction, for every $k = 1, 2, \cdots$ there is at least an integer m_k such that for $m \ge m_k$,

$$\left(\frac{\beta}{\alpha}\right)^k u(m) \le u(m-\tau). \tag{2.33}$$

Next observe that because of (2.29), for m sufficiently large,

$$\sum_{i=m-\tau}^{m} p(i) \ge \sum_{i=m-\tau}^{m-1} p(i) \ge \tau\beta.$$

Hence, for m sufficiently large,

$$\sum_{i=m-\tau}^{m} p(i) \ge M, \tag{2.34}$$

where $M := \tau\beta > 0$. Choose n such that

$$\left(\frac{\beta}{\alpha}\right)^n > \left(\frac{2}{M}\right)^2. \tag{2.35}$$

This is possible because from (2.29), $\beta > \alpha$. Then for m sufficiently large, say for $m \ge m_0$, (2.33) is satisfied for specific n, which was chosen in (2.35), also (2.29) and (2.34) hold, $\{u(m)\}$ is decreasing for $m \ge m_0$. Now in view of (2.34) and for $m \ge m_0 + \tau$, there exists an integer m^* with $m - \tau \le m^* \le m$ such that

$$\sum_{i=m-\tau}^{m^*} p(i) \ge \frac{M}{2} \quad \text{and} \quad \sum_{i=m^*}^{m} p(i) \le \frac{M}{2}.$$

From equation (2.8) and the decreasing nature of $\{u(m)\}$, we have

$$\begin{aligned}
u(m^*-1)-u(m-\tau) &= \sum_{i=m-\tau}^{m^*}(u(i+1)-u(i)) \\
&= -\sum_{i=m-\tau}^{m^*}p(i)u(i-\tau) \\
&\leq -\left(\sum_{i=m-\tau}^{m^*}p(i)\right)u(m^*-\tau) \\
&\leq -\frac{M}{2}u(m^*-\tau)
\end{aligned}$$

hence

$$\frac{M}{2}u(m^*-\tau) \leq u(m-\tau). \tag{2.36}$$

Similarly

$$\begin{aligned}
u(m-1)-u(m^*) &= \sum_{i=m^*}^{m}(u(i+1)-u(i)) \\
&= -\sum_{i=m^*}^{m}p(i)u(i-\tau) \\
&\leq -\left(\sum_{i=m^*}^{m}p(i)\right)u(m-\tau) \\
&\leq -\frac{M}{2}u(m-\tau)
\end{aligned}$$

and so

$$\frac{M}{2}u(m-\tau) \leq u(m^*). \tag{2.37}$$

From (2.36) and (2.37) we find

$$\left(\frac{M}{2}\right)^2 u(m^*-\tau) \leq u(m^*),$$

which in view of (2.33) yields

$$\left(\frac{\beta}{\alpha}\right)^n \leq \frac{u(m^*-\tau)}{u(m^*)} \leq \left(\frac{2}{M}\right)^2.$$

This contradicts (2.35) and so the proof of the theorem is complete. □

Example 3. Consider the difference equation

$$u(m+1)-u(m)+\begin{Bmatrix} \frac{27}{256}, & m=3k \\ \frac{13}{128}, & m=3k+1 \\ \frac{29}{256}, & m=3k+2 \end{Bmatrix} u(m-3)=0 \quad \text{for } m=0,1,\cdots. \tag{2.38}$$

We compute that

$$\liminf_{m\to\infty} p(m)=\frac{13}{128}\not>\frac{27}{256}=\frac{3^3}{4^4}$$

and

$$\liminf_{m\to\infty}\sum_{i=m-3}^{m-1}\begin{Bmatrix} \frac{27}{256}, & i=3k \\ \frac{13}{128}, & i=3k+1 \\ \frac{29}{256}, & i=3k+2 \end{Bmatrix}=\frac{41}{128}>\frac{81}{256}=\left(\frac{3}{4}\right)^4.$$

Thus, Theorem 2.0.7 applies to (2.38) but Theorem 2.0.3 does **not**.

Remark 4. *Since the condition of Theorem 2.0.7 reduces to that of Theorem 2.0.2, it follows that Theorem 2.0.7 is sharp.*

Remark 5. *The condition of Theorem 2.0.7 improves Theorem 2.0.3 (i) since*

$$\tau\liminf_{m\to\infty} p(m)=\sum_{i=m-\tau}^{m-1}\liminf_{m\to\infty} p(i)\leq\liminf_{m\to\infty}\sum_{i=m-\tau}^{m-1} p(i).$$

The following theorem improves Theorem 2.0.4 by using sum of consecutive $(\tau+1)$-terms of a single one.

Theorem 2.0.8 (See (Erbe & Zhang, 1989, Theorem 2.5))**.** *Assume that there exists an increasing sequence* $\{\xi_k\}\subset\mathbb{N}$ *such that*

$$\sum_{i=\xi_k-\tau}^{\xi_k} p(i)\geq 1 \quad \textit{for all } k. \tag{2.39}$$

Then, every solution of (2.8) *is oscillatory.*

Proof. Let $\{u(m)\}$ be a positive solution of (2.8). Summing (2.8) from $(\xi_k-\tau)$ to ξ_k, we have

$$u(\xi_k+1)-u(\xi_k-\tau)+\sum_{i=\xi_k-\tau}^{\xi_k} p(i)u(i-\tau)=0.$$

Hence, we obtain

$$u(\xi_k+1)-u(\xi_k-\tau)+u(\xi_k-\tau)\sum_{i=\xi_k-\tau}^{\xi_k} p(i)\le 0. \tag{2.40}$$

Rearranging (2.40) we have,

$$u(\xi_k+1)-u(\xi_k-\tau)\left(1-\sum_{i=\xi_k-\tau}^{\xi_k} p(i)\right)\le 0$$

and hence

$$\sum_{i=\xi_k-\tau}^{\xi_k} p(i)\le 1. \tag{2.41}$$

This contradicts (2.39) and so proof of theorem is complete. □

Remark 6. *Theorem 2.0.7 and Theorem 2.0.8 are* ***not*** *comparable.*

Example 4. Consider the difference equation

$$u(m+1)-u(m)+\left\{\begin{array}{ll}\frac{27}{256}, & m=3k\\ \frac{26}{256}, & m=3k+1\\ \frac{29}{256}, & m=3k+2\end{array}\right\}u(m-2)=0 \quad \text{for } m=0,1,\cdots. \tag{2.42}$$

Then,

$$\liminf_{m\to\infty}\sum_{i=m-3}^{m-1}\left\{\begin{array}{ll}\frac{27}{256}, & i=3k\\ \frac{13}{128}, & i=3k+1\\ \frac{29}{256}, & i=3k+2\end{array}\right\}=\frac{41}{128}>\frac{81}{256}=\left(\frac{3}{4}\right)^4$$

and

$$\sum_{i=m-\tau}^{m}\left\{\begin{array}{ll}\frac{27}{256}, & i=3k\\ \frac{26}{256}, & i=3k+1\\ \frac{29}{256}, & i=3k+2\end{array}\right\}=\left\{\begin{array}{ll}\frac{109}{256}, & m=3k\\ \frac{27}{64}, & m=3k+1\\ \frac{111}{256}, & m=3k+2\end{array}\right\}<1 \quad \text{for all } m.$$

Thus, Theorem 2.0.7 applies to (2.38) but Theorem 2.0.8 does **not**.

Example 5. Set the difference equation

$$u(m+1)-u(m)+\left\{\begin{array}{ll}\frac{1}{9}, & m=2k\\ \frac{20}{27}, & m=4k+1\\ \frac{4}{27}, & m=4k+3\end{array}\right\}u(m-2)=0 \quad \text{for } m=0,1,\cdots. \tag{2.43}$$

Then,

$$\liminf_{m\to\infty}\sum_{i=m-2}^{m-1}\left\{\begin{array}{ll}\frac{1}{9}, & i=2k\\ \frac{20}{27}, & i=4k+1\\ \frac{4}{27}, & i=4k+3\end{array}\right\}=\left\{\begin{array}{ll}\frac{7}{27}, & i=4k,4k+1\\ \frac{23}{27}, & i=4k+2,4k+3\end{array}\right\}\not>\frac{8}{27}=\left(\frac{2}{3}\right)^3.$$

Theorem 2.0.7 does **not** apply. Fortunately,

$$\sum_{i=m-2}^{m}\left\{\begin{array}{ll}\frac{1}{9}, & i=2k\\ \frac{20}{27}, & i=4k+1\\ \frac{4}{27}, & i=4k+3\end{array}\right\}=\left\{\begin{array}{ll}1, & i=2\ell+1\\ \frac{10}{27}, & i=4k\\ \frac{26}{27}, & i=4k+2\end{array}\right\},$$

which shows that Theorem 2.0.8 holds with $\{\xi_k\}=\{2k+1\}$.

Remark 7. *In view of Theorem 2.0.3 (ii), Ladas conjectured if* (2.8) *possesses an eventually positive solution under the condition*

$$\sum_{i=m-\tau}^{m-1}p(i)\leq\left(\frac{\tau}{\tau+1}\right)^{\tau+1}\quad \textit{for all large } m.$$

For the next result, we need to define the set

$$\Lambda:=\{\lambda>0:\ 1-\lambda p(m)>0,\quad m\gg 0\}.$$

Remark 8. *Note that the condition in Theorem 2.0.4 is equivalent to saying* $1\notin\Lambda$ *or* $\Lambda\subset(0,1)$.

The following theorem improves Theorem 2.0.7.

Theorem 2.0.9 (Yu et al., 1994, Theorem 1, Theorem 4)**.** *(i) Assume that*

$$\limsup_{m\to\infty} \sup_{\lambda\in\Lambda}\left\{\lambda \prod_{i=m-\tau}^{m-1} [1-\lambda p(i)]\right\} < 1. \tag{2.44}$$

Then, every solution of (2.8) *is oscillatory.*

(ii) Assume that there exists $\lambda_0 \in \Lambda$ *such that*

$$\lambda_0 \prod_{i=m-\tau}^{m-1} [1-\lambda_0 p(i)] \geq 1 \quad \textit{for all large } m. \tag{2.45}$$

Then, (2.8) *possesses an eventually positive solution.*

Proof. (i) Let $\{u(m)\}$ be an eventually positive solution of equation (2.8). Then, we have

$$u(m+1) - u(m) = -p(m)u(m-\tau) \leq 0,$$

and so $\{u(m)\}$ is an eventually nonincreasing sequence of positive numbers. It follows from (2.8) that

$$u(m+1) - [1-p(m)]u(m) \leq 0.$$

From Erbe & Zhang (1989) it suffices to assume $p(m) < 1$. Thus, we obtain

$$1 \in F(u) := \{\lambda \geq 0 : u(m+1) - [1-\lambda p(m)]u(m) \leq 0\}.$$

Obviously, $F(u) \subset \Lambda$. It is easy to show that Λ is bounded and $\lambda_0 := \sup\{F(u)\} \in \Lambda$. For any $\lambda \in F(u)$, we have

$$u(m) \leq \left(\prod_{i=m-\tau}^{m-1} [1-\lambda p(i)]\right) u(m-\tau).$$

Substituting this into (2.8), we have

$$u(m+1) - \left(1 - \frac{p(m)}{\prod_{i=m-\tau}^{m-1}[1-\lambda p(i)]}\right) u(m) \leq 0.$$

Further

$$u(m+1)-\left(1-\frac{p(m)}{\sup_{m\geq M}\left\{\prod_{i=m-\tau}^{m-1}[1-\lambda p(i)]\right\}}\right)u(m)\leq 0,$$

which yields that

$$\frac{1}{\sup_{m\geq M}\left\{\prod_{i=m-\tau}^{m-1}[1-\lambda p(i)]\right\}}\in F(u)\quad\text{for}\quad\lambda\in F(u)$$

and so

$$\lambda_0\sup_{m\geq M}\left\{\prod_{i=m-\tau}^{m-1}[1-\lambda p(i)]\right\}\geq 1\quad\text{for}\quad\lambda\in F(u).$$

Let $\lambda\to\lambda_0$, we obtain

$$\sup_{m\geq M}\left\{\lambda_0\prod_{i=m-\tau}^{m-1}[1-\lambda_0 p(i)]\right\}\geq 1,$$

which yields

$$\sup_{m\geq M}\left\{\sup_{\lambda\in F(u)}\left\{\lambda\prod_{i=m-\tau}^{m-1}[1-\lambda p(i)]\right\}\right\}\geq 1,$$

which contradicts (2.44) and so the proof of (i) is complete.

(ii) We shall construct a positive solution of equation (2.8). To this end we choose a positive integer M such that $M\geq\tau$ and

$$\lambda_0\prod_{i=m-\tau}^{m-1}[1-\lambda_0 p(i)]\geq 1,\quad m=M,M+1,\cdots.$$

Define

$$x(m):=\begin{cases}1, & m=M-\tau,\cdots,M-1\\ \dfrac{1}{\lambda_0\prod_{i=m-\tau}^{m-1}[1-\lambda_0 x(i)p(i)]}, & m=M,M+1,\cdots.\end{cases}$$

Then,

$$x(M)=\frac{1}{\lambda_0\prod_{i=M-\tau}^{M-1}[1-\lambda_0 x(i)p(i)]}=\frac{1}{\lambda_0\prod_{i=M-\tau}^{M-1}[1-\lambda_0 p(i)]}\leq 1.$$

In general, by induction, we obtain

$$x(m) = \frac{1}{\lambda_0 \prod_{i=m-\tau}^{m-1}[1-\lambda_0 x(i)p(i)]} \leq 1, \quad m = M, M+1, \cdots.$$

Thus, $\{x(m)\}$ is defined. Also, we define

$$z(m) := 1 - \lambda_0 x(m)p(m), \quad m = M, M+1, \cdots.$$

Then, $z(m) > 0$ for $m \geq M - \tau$ and

$$z(m) = 1 - \frac{p(m)}{\prod_{i=m-\tau}^{m-1} z(i)}, \quad m = M, M+1, \cdots. \tag{2.46}$$

Define

$$u(m) := \begin{cases} 1 & , m = M-1 \\ \displaystyle\prod_{i=M-\tau}^{M-1} z(i) & , m = M, M+1, \cdots. \end{cases}$$

Then we have by (2.46)

$$\frac{u(m+1)}{u(m)} - 1 + p(m)\frac{u(m-\tau)}{u(m)} = 0.$$

That is,

$$u(m+1) - u(m) + p(m)u(m-\tau) = 0.$$

Thus, we obtain a positive solution $\{u(m)\}$ of equation (2.8). The proof of (ii) is complete. □

Remark 9. *Theorem 2.0.9 is sharp, i.e., Theorem 2.0.9 becomes a criterion when* $\{p(m)\}$ *be a* τ*-periodic sequence. More precisely, if* $\{p(m)\}$ *be* τ*-periodic, then every solution of* (2.8) *is oscillatory if and only if*

$$\max_{\lambda\in\Lambda}\left\{\lambda \prod_{i=1}^{\tau}[1-\lambda p(i)]\right\} < 1,$$

where

$$\Lambda = \begin{cases} (0, \frac{1}{p^*}), & p^* := \max\{p(1), p(2), \cdots, p(\tau)\} > 0 \\ (0, \infty), & p^* = 0. \end{cases}$$

Remark 10. *By the inequality of arithmetic and geometric means, we obtain*

$$\lambda \prod_{i=m-\tau}^{m-1} [1 - \lambda p(i)] \leq \lambda \left(\frac{1}{\tau} \sum_{i=m-\tau}^{m-1} [1 - \lambda p(i)] \right)^{\tau} = \lambda \left(1 - \frac{\lambda}{\tau} \sum_{i=m-\tau}^{m-1} p(i) \right)^{\tau}$$
$$\leq \frac{\tau^{\tau}}{(\tau+1)^{\tau+1}} \frac{1}{\frac{1}{\tau} \sum_{i=m-\tau}^{m-1} p(i)} = \left(\frac{\tau}{\tau+1} \right)^{\tau+1} \frac{1}{\sum_{i=m-\tau}^{m-1} p(i)}$$

for $\lambda \in \Lambda$ and all large m, where we have applied some calculus to reach the second line of the inequality. Thus, the condition in Theorem 2.0.7 naturally implies that of Theorem 2.0.9 (i).

Example 6. Consider the difference equation

$$u(m+1) - u(m) + \left\{ \begin{matrix} \frac{27}{512}, & m = 4k \\ \frac{135}{1024}, & m \neq 4k \end{matrix} \right\} u(m-3) = 0 \quad \text{for } m = 0, 1, \cdots. \tag{2.47}$$

Then,

$$\sum_{i=m-3}^{m-1} \left\{ \begin{matrix} \frac{27}{512}, & i = 4k \\ \frac{135}{1024}, & i \neq 4k \end{matrix} \right\} = \left\{ \begin{matrix} \frac{405}{1024}, & m = 4k \\ \frac{81}{256}, & m \neq 4k \end{matrix} \right\} \leq \frac{81}{256} = \left(\frac{3}{4} \right)^4.$$

However,

$$\begin{aligned}
&\sup_{0<\lambda<\frac{1024}{135}} \left\{\lambda \prod_{i=m-3}^{m-1} \left[1-\lambda \begin{Bmatrix} \frac{27}{512}, & i=4k \\ \frac{135}{1024}, & i\neq 4k \end{Bmatrix}\right]\right\} \\
&= \sup_{0<\lambda<\frac{1024}{135}} \begin{Bmatrix} \lambda\big(1-\frac{135}{1024}\lambda\big)^3, & m=4k \\ \lambda\big(1-\frac{135}{1024}\lambda\big)^2\big(1-\frac{27}{512}\lambda\big), & m\neq 4k \end{Bmatrix} \\
&= \begin{Bmatrix} \lambda\big(1-\frac{135}{1024}\lambda\big)^3\big|_{\lambda=\frac{256}{135}}, & m=4k \\ \lambda\big(1-\frac{135}{1024}\lambda\big)^2\big(1-\frac{27}{512}\lambda\big)\big|_{\lambda=\frac{64}{135}(19-\sqrt{201})}, & m\neq 4k \end{Bmatrix} \\
&= \begin{Bmatrix} \frac{4}{5}\approx 0.80, & m=4k \\ \frac{1}{900}\big(1833-67\sqrt{201}\big)\approx 0.98, & m\neq 4k \end{Bmatrix} < 1,
\end{aligned}$$

which by Theorem 2.0.9 (i) shows that every solution of (2.47) is oscillatory. Therefore, Ladas' conjecture mentioned in Remark 7 is wrong.

CHAPTER THREE
PARTIAL DIFFERENCE EQUATIONS

In this chapter, we will consider the partial difference equation

$$u(m+1,n)+u(m,n+1)-u(m,n)+\sum_{k=1}^{r}p_k(m,n)u(m-\tau_k,n-\sigma_k)=0,\quad m,n\geq 0, \tag{3.1}$$

where $r\in\mathbb{N}$, $\{p_k(m,n)\}\subset\mathbb{R}_0^+$ and $\tau_k,\sigma_k\in\mathbb{N}_0$.

Definition 3.0.1. *A* ***solution*** *of* (3.1) *is a double sequence* $\{u(m,n)\}$ *of reals defined for* $m\geq-\tau^*$ *and* $n\geq-\sigma^*$*, where* $\tau^*:=\max_k\{\tau_k\}$ *and* $\sigma^*:=\max_k\{\sigma_k\}$*, which satisfies* (3.1) *for* $m,n\geq 0$.

Let us rewrite (3.1) in the form

$$u(m+1,n)=u(m,n)-u(m,n+1)-\sum_{k=1}^{r}p_k(m,n)u(m-\tau_k,n-\sigma_k),\quad m,n\geq 0. \tag{3.2}$$

Letting $\Psi:=\{(m,n):\ m\geq-\tau^*\text{ and }n\geq-\sigma^*\}\backslash\mathbb{N}\times\mathbb{N}_0$, we can iterate (3.2) and write unique solution of (3.1) provided $\{u(m,n)\}$ is prescribed on Ψ.

On the other hand, we may also rewrite (3.1) in the form

$$u(m,n+1)=u(m,n)-u(m+1,n)-\sum_{k=1}^{r}p_k(m,n)u(m-\tau_k,n-\sigma_k),\quad m,n\geq 0. \tag{3.3}$$

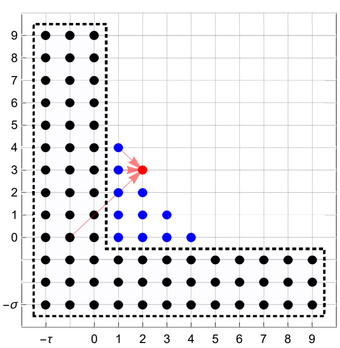


Figure 3.1 Black points denote Ψ, blue points denote the calculated points of $u(m,n)$, and red point is the currently computed point of $u(m,n)$

Letting $\Upsilon := \{(m,n) : \ m \geq -\tau^* \text{ and } n \geq -\sigma^*\}\backslash\mathbb{N}_0 \times \mathbb{N}$, we can iterate (3.3) and write solution of (3.1) provided $\{u(m,n)\}$ is prescribed on Υ.

Definition 3.0.2. *A solution* $\{u(m,n)\}$ *of* (3.1) *is said to be **eventually positive** if* $u(m,n) > 0$ *for all large* m,n*. Similarity, a solution* $\{u(m,n)\}$ *of* (3.1) *is said to be **eventually negative** if* $u(m,n) < 0$ *for all large* m,n*. A solution of* (3.1) *is said to be **oscillatory** if it is neither eventually positive nor eventually negative.*

Definition 3.0.3. *A solution* $\{u(m,n)\}$ *of* (3.1) *is called **proper** if there is at least* $M,\lambda,\mu \in \mathbb{R}^+$ *such that* $|u(m,n)| \leq M\lambda^m\mu^n$ *for all* $m,n \geq 0$*.*

It can be proved easily that if the initial condition satisfies either

$$|u(m,n)| \leq K\lambda^m\mu^n \text{ for all } (m,n) \in \Psi \text{ or } |u(m,n)| \leq K\lambda^m\mu^n \text{ for all } (m,n) \in \Upsilon$$

for some $K,\lambda,\mu \in \mathbb{R}^+$, then the corresponding solution $\{u(m,n)\}$ of the autonomous equation

$$u(m+1,n)+u(m,n+1)-u(m,n)+\sum_{k=1}^{r} p_k u(m-\tau_k,n-\sigma_k) = 0, \quad m,n \geq 0, \tag{3.4}$$

where $r \in \mathbb{N}$, $p_k \in \mathbb{R}$ and $\tau_k,\sigma_k \in \mathbb{N}_0$, is proper.

We associate (3.4) with the characteristic equation

$$\lambda + \mu - 1 + \sum_{k=1}^{r} \frac{p_k}{\lambda^{\tau_k}\mu^{\sigma_k}} = 0, \quad \lambda,\mu \in \mathbb{R}. \tag{3.5}$$

Theorem 3.0.1 (Zhang & Liu, 1997a, Theorem 3.1)**.** *Every proper solution of the autonomous equation* (3.4) *is oscillatory if and only if the characteristic equation* (3.5) *has no roots in* $\mathbb{R}^+ \times \mathbb{R}^+$.

We need two lemmas for the proof of Theorem 3.0.1. Computing directly we can show the following lemmas.

Lemma 3.0.2 (Zhang & Zhou, 2007, Lemma 1.4)**.** *Assume that there exist positive* *constants M_1, M, N such that*

$$|f(m,n)| \leq M_1 r_1^m r_2^n, \quad m \geq M, n \geq N.$$

Then, z*-transform of $\{f(m,n)\}$ exists in the region $|z_1| > r_1$ and $|z_2| > r_2$.*

Lemma 3.0.3 (Zhang & Zhou, 2007, Lemma 1.5)**.** *We have the following identities.*

(i)

$$\mathrm{z}\{f(m-\tau, n-\sigma)\} = z_1^{-\tau} z_2^{-\sigma} F(z_1, z_2),$$

where $F(z_1, z_2) := \mathrm{z}\{f(m,n)\}$.

(ii)

$$\sum_{i=0}^{\infty} F(\tau+i, z_2) z_1^{-i} = z_1^{\tau} \left(F(z_1, z_2) - \sum_{m=0}^{\tau-1} F(m, z_2) z_1^{-m} \right),$$

where

$$F(\tau+i, z_2) = \sum_{i=0}^{\infty} F(\tau+i, n) z_2^{-n}.$$

(iii)

$$\sum_{i=0}^{\infty} \sum_{n=0}^{\sigma-1} F(\tau+i, n) z_1^{-i} z_2^{-n} = z_1^{\tau} \Bigg(\sum_{m=0}^{\infty} \sum_{n=0}^{\sigma-1} f(m,n) z_1^{-m} z_2^{-n} - \sum_{m=0}^{\tau-1} \sum_{n=0}^{\sigma-1} f(m,n) z_1^{-m} z_2^{-n} \Bigg).$$

(iv)

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\sigma-1} f(m,n) z_1^{-m} z_2^{-n} = \sum_{i=0}^{\sigma-1} F(z_1, i) z_2^{-i},$$

where

$$F(z_1, n) = \sum_{m=0}^{\infty} f(m,n) z_1^{-m}.$$

(v)

$$\text{z}\{f(m+\tau,n+\sigma)\} = z_1^{\tau} z_2^{\sigma}\Bigg(F(z_1,z_2) - \sum_{m=0}^{\tau-1} F(m,z_2) z_1^{-m} \\ - \sum_{n=0}^{\sigma-1} F(z_1,n) z_2^{-n} + \sum_{m=0}^{\tau-1}\sum_{n=0}^{\sigma-1} f(m,n) z_1^{-m} z_2^{-n}\Bigg).$$

Proof of Theorem 3.0.1. Necessity. Suppose that characteristic equation (3.5) has a positive root (λ_0, μ_0). Then, it is easy to show that $u(m,n)$ with $u(m,n) = \lambda_0^m \mu_0^n$ is a proper positive solution of (3.4), a contradiction.

Sufficiency. Assume that (3.5) has no positive roots. Let $u(m,n)$ be a proper positive solution of (3.4). Then, $|u(m,n)| \leq K\lambda^m \mu^n$ holds. Let $\alpha := \max\{\lambda, \mu\}$, then

$$|u(m,n)| \leq K\alpha^{m+n}, \quad (m,n) \in \mathbb{N}_0^2. \tag{3.6}$$

Hence, by Lemma 3.0.2 for $|z_1|, |z_2| > \alpha$, the z-transform of $u(m,n)$ defined by

$$\text{z}\{u(m,n)\} = F(z_1,z_2) = \sum_{m,n=0}^{\infty} u(m,n) z_1^{-m} z_2^{-n} \tag{3.7}$$

exists. By taking the image of both sides of (3.4) under z-transform and by using Lemma 3.0.3, we get

$$\phi(z_1,z_2)F(z_1,z_2) = \psi(z_1,z_2), \quad |z_1|, |z_2| > \alpha, \tag{3.8}$$

where

$$\phi(z_1,z_2) = z_1 + z_2 - 1 + \sum_{k=1}^{r} p_k z_1^{\tau_k} z_2^{\sigma_k}, \tag{3.9}$$

$$\psi(z_1,z_2) = z_1 \sum_{n=0}^{\infty} u(0,n) z_2^{-n} + z_2 \sum_{m=0}^{\infty} u(m,0) z_1^{-m}.$$

We rewrite (3.8) in the form

$$\phi(z_1^{-1}, z_2^{-1})F(z_1^{-1}, z_2^{-1}) = \psi(z_1^{-1}, z_2^{-1}). \tag{3.10}$$

Set

$$w(z_1, z_2) := F(z_1^{-1}, z_2^{-1}) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u(m,n) z_1^m z_2^n. \tag{3.11}$$

(3.11) has radius of convergence r_1 and r_2, i.e., (3.10) holds for $|z_1| < r_1$ and $|z_2| < r_2$. Equivalently, (3.8) holds for $|z_1| < \frac{1}{r_1}$ and $|z_2| < \frac{1}{r_2}$. It is known that a power series with positive coefficients having radius of convergence r_i, $i = 1, 2$ has a singularity at $z_1 = r_1$ and $z_2 = r_2$. For $(z_1, z_2) \in (0,\infty) \times (0,\infty)$, according to the condition $\phi(z_1, z_2) \neq 0$. Thus, $\phi(\frac{1}{r_1}, \frac{1}{r_2}) \neq 0$, and hence

$$w(z_1, z_2) = \frac{\psi(\frac{1}{z_1}, \frac{1}{z_2})}{\phi(\frac{1}{z_1}, \frac{1}{z_2})} \tag{3.12}$$

is analytic in the region $|z_1 - r_1| < \rho_1$ and $|z_2 - r_2| < \rho_2$. This is a contradiction for the singularity of $w(z_1, z_2)$ at $z_1 = r_1$ and $z_2 = r_2$. Therefore, we must have $r_1 = \infty$ and $r_2 = \infty$, i.e., (3.8) holds for $|z_1| > 0$ and $|z_2| > 0$, which leads to $u(m,n) = 0$ for all large m, n. Otherwise, the left hand side of (3.8) is **not** equal to the right hand side. This contradiction completes the proof. □

Corollary 3.0.3.1 (Zhang & Liu, 1997a, Theorem 3.2)**.** *Assume $p_k \in \mathbb{R}^+$ and*

$$\sum_{k=1}^{r} p_k \frac{(\tau_k + \sigma_k + 1)^{\tau_k+\sigma_k+1}}{\tau_k^{\tau_k}\sigma_k^{\sigma_k}} > 1. \tag{3.13}$$

Then, every proper solution of the autonomous equation (3.4) *is oscillatory.*

Proof. Suppose that (3.13) is valid. We will prove that the characteristic equation (3.5) has no positive roots. For $\lambda + \mu \geq 1$, (3.5) has no positive roots obviously. For $\lambda + \mu < 1$, we write ϕ defined in (3.9) as

$$\phi(\lambda, \mu) = (1 - \lambda - \mu)\Big(-1 + \sum_{k=1}^{r} p_k \frac{\lambda^{-\tau_k}\mu^{-\sigma_k}}{1 - \lambda - \mu}\Big).$$

Let

$$f(\lambda, \mu) := \frac{\lambda^{-\tau_k}\mu^{-\sigma_k}}{1 - \lambda - \mu}. \tag{3.14}$$

It is easy to show that the function f obtains its minimum value at $\lambda_0 := \frac{\tau_k}{\tau_k+\sigma_k+1}$ and

$\mu_0 := \frac{\sigma_k}{\tau_k+\sigma_k+1}$, i.e.,

$$\min_{0<\lambda+\mu<1}\{f(\lambda,\mu)\} = f(\lambda_0,\mu_0) = \frac{(\tau_k+\sigma_k+1)^{\tau_k+\sigma_k+1}}{\tau_k^{\tau_k}\sigma_k^{\sigma_k}}.$$

Thus, we get

$$\phi(\lambda,\mu) \geq (1-\lambda-\mu)\bigg(-1+\sum_{k=1}^{r} p_k \frac{(\tau_k+\sigma_k+1)^{\tau_k+\sigma_k+1}}{\tau_k^{\tau_k}\sigma_k^{\sigma_k}}\bigg) > 0$$

for $0 < \lambda+\mu < 1$. This means that (3.5) has no positive roots. All proper solution of (3.4) oscillate by Theorem 3.0.1, □

Consider the equation

$$u(m+1,n)+u(m,n+1)-u(m,n)+pu(m-\tau,n-\sigma) = 0, \quad m,n \geq 0, \tag{3.15}$$

where $p \in \mathbb{R}$ and $\tau, \sigma \in \mathbb{N}_0$.

Corollary 3.0.3.2 (Zhang & Liu, 1997a, Theorem 3.3)**.** *Every proper solution of the autonomous equation* (3.15) *is oscillatory if and only if*

$$p\frac{(\tau+\sigma+1)^{\tau+\sigma+1}}{\tau^\tau\sigma^\sigma} > 1. \tag{3.16}$$

Proof. Assume that (3.16) does not hold. The characteristic equation of (3.15) is

$$\phi(\lambda,\mu) := \lambda+\mu-1+p\lambda^{-\tau}\mu^{-\sigma} = 0. \tag{3.17}$$

Clearly,

$$\phi\bigg(\frac{\tau+1}{\tau+\sigma+1}, \frac{\sigma}{\tau+\sigma+1}\bigg) > 0 \tag{3.18}$$

and

$$\phi\bigg(\frac{\tau}{\tau+\sigma+1}, \frac{\sigma}{\tau+\sigma+1}\bigg) = \frac{1}{\tau+\sigma+1}\bigg(-1+p\frac{(\tau+\sigma+1)^{(\tau+\sigma+1)}}{\tau^\tau\sigma^\sigma}\bigg) \leq 0.$$

Since ϕ is continuous, by the Intermediate Value Theorem, there exist

$\lambda_0 \in \left[\frac{\tau}{\tau+\sigma+1}, \frac{\tau+1}{\tau+\sigma+1}\right)$ and $\mu_0 = \frac{\sigma}{\tau+\sigma+1}$ such that $\phi(\lambda_0, \mu_0) = 0$. This means that (3.15) has a proper positive solution. By Theorem 3.0.1, (3.15) is not oscillatory. This completes the proof. □

Consider the equation

$$u(m+1,n)+u(m,n+1)-u(m,n)+p(m,n)u(m-\tau,n-\sigma) = 0, \quad m,n \geq 0, \tag{3.19}$$

where $\{p(m,n)\} \subset \mathbb{R}_0^+$ and $\tau, \sigma \in \mathbb{N}_0$.

Lemma 3.0.4 (Zhang et al., 1995, Lemma 2.2)**.** *For $m \geq M$ and $n \geq N$, the following is satisfied:*

$$\begin{aligned} &\sum_{i=M}^{m}\sum_{j=N}^{n}[u(i+1,j)+u(i,j+1)-u(i,j)] \\ &= \sum_{i=M+1}^{m+1}\sum_{j=N+1}^{n} u(i,j) + \sum_{i=M+1}^{m} u(i,n+1) + u(m+1,N) - u(M,N). \end{aligned} \tag{3.20}$$

Proof. The left hand side of (3.20) is equal to

$$\begin{aligned} &\sum_{i=M}^{m}\sum_{j=N}^{n} u(i+1,j) + \sum_{i=M}^{m} u(i,n+1) - \sum_{i=M}^{m} u(i,N) \\ &= \sum_{i=M}^{m}\sum_{j=N+1}^{n} u(i+1,j) + \sum_{i=M}^{m} u(i+1,N) + \sum_{i=M}^{m} u(i,n+1) - \sum_{i=M}^{m} u(i,N), \end{aligned}$$

which is equal to the right hand side of (3.20). □

Lemma 3.0.5 (Zhang et al., 1995, Lemma 2.5)**.** *Suppose that $\{u(m,n)\}$ is an eventually positive solution of* (3.19)*. And assume that there exists a positive number β such that for all large m and n, we have*

$$\sum_{i=m-\tau}^{m-1}\sum_{j=n-\sigma}^{n-1} p(i,j) \geq \beta. \tag{3.21}$$

Then for all large s and t,

$$\frac{u(s-\tau,t-\sigma)}{u(s+1,t+1)} \leq \Big(\frac{4}{\beta^2}\Big)^2.$$

Proof. Let $\{u(m,n)\}$ be an eventually positive solution of (3.19). Then, we have

$$u(m+1,n)+u(m,n+1) = -p(m,n)u(m-\tau,n-\sigma) \leq 0$$

and so $u(m+1,n) \leq u(m,n)$ and $u(m,n+1) \leq u(m,n)$. Then, $\{u(m,n)\}$ is decreasing in m and n. Then, in view of (3.20),

$$\begin{aligned} u(M,N) &\geq u(m+1,N) + \sum_{i=M}^{m}\sum_{j=N}^{n} p(i,j)u(i-\tau,j-\sigma) \\ &\geq u(m+1,N) + \beta u(m-\tau,n-\sigma). \end{aligned} \tag{3.22}$$

In view of (3.22), we have

$$u(m-\tau,t) \geq u(s+1,t) + \frac{\beta}{2}u(s-\tau,t), \tag{3.23}$$

$$u(s+1,t) \geq u(m+1,t) + \frac{\beta}{2}u(m-\tau,t). \tag{3.24}$$

Then, in view of (3.23) and (3.24) we have respectively

$$u(m-\tau,t) \geq \frac{\beta}{2}u(s-\tau,t),$$

and

$$\frac{\beta}{2}u(m-\tau,t) \geq u(s+1,t).$$

Thus,

$$\frac{u(s-\tau,t)}{u(s+1,t)} \leq \frac{4}{\beta^2}$$

for all large s and t. Similarly,

$$\frac{u(s,t-\sigma)}{u(s,t+1)} \leq \frac{4}{\beta^2}$$

for all large s and t. Therefore,

$$\frac{u(s-\tau,t-\sigma)}{u(s+1,t+1)} = \frac{u(s-\tau,t-\sigma)}{u(s+1,t-\sigma)}\frac{u(s+1,t-\sigma)}{u(s+1,t+1)} \le \Big(\frac{4}{\beta^2}\Big)^2$$

for all large s and t as required. □

Lemma 3.0.6 (Zhang et al., 1995, Lemma 2.7)**.** *Let $\sigma = 0$. Suppose that $\{u(i,j)\}$ is an eventually positive solution of* (3.19) *and there exists a positive number δ such that for all large m and n, $\sum_{i=m-\tau}^{m-1} p(i,n) \ge \delta$ holds. Then for all large s and t,*

$$\frac{u(s-\tau,t)}{u(s+1,t)} \le \frac{4}{\delta^2}. \tag{3.25}$$

Theorem 3.0.7 (Zhang et al., 1995, Theorem 3.4)**.** *Assume $\tau,\sigma \ge 1$, and*

$$\liminf_{m,n\to\infty} \sum_{i=m-\tau}^{m-1}\sum_{j=n-\sigma}^{n-1} p(i,j) > \frac{\tau+\sigma}{2}\left(\frac{\kappa}{\kappa+1}\right)^{\kappa+1}, \quad \textit{where} \quad \kappa := \frac{2\tau\sigma}{\tau+\sigma}. \tag{3.26}$$

Then, every solution of (3.19) *is oscillatory.*

Proof. Suppose to contrary that $\{u(m,n)\}$ is an eventually positive solution of (3.19). Since $\{u(m,n)\}$ is decreasing in m and n,

$$\begin{aligned} u(m+1,n)+u(m,n+1)-u(m,n) &= -p(m,n)u(m-\tau,n-\sigma) \\ &\le -p(m,n)u(m,n). \end{aligned}$$

Thus,

$$\begin{aligned} p(m,n) &\le 1 - \frac{u(m+1,n)+u(m,n+1)}{u(m,n)} \\ &\le 1 - \frac{[u(m+1,n)u(m,n+1)]^{\frac{1}{2}}}{u(m,n)}. \end{aligned}$$

In view of (3.26), we can choose a constant ψ such that

$$\gamma := \frac{\tau+\sigma}{2}\left(\frac{\kappa}{\kappa+1}\right)^{\kappa+1} \le \psi \le \frac{1}{\tau\sigma}\sum_{i=m-\tau}^{m-1}\sum_{j=n-\sigma}^{n-1} p(i,j)$$

for sufficiently large m and n. Thus,

$$\psi \leq \frac{1}{\tau\sigma} \sum_{i=m-\tau}^{m-1} \sum_{j=n-\sigma}^{n-1} \left(1 - \frac{[u(i+1,j)u(i,j+1)]^{\frac{1}{2}}}{u(i,j)}\right)$$
$$= 1 - \frac{1}{\tau\sigma} \sum_{i=m-\tau}^{m-1} \sum_{j=n-\sigma}^{n-1} \frac{[u(i+1,j)u(i,j+1)]^{\frac{1}{2}}}{u(i,j)}.$$

Note that by the well known inequality between the arithmetic and geometric means, we have

$$\sum_{i=m-\tau}^{m-1} \sum_{j=n-\sigma}^{n-1} \frac{[u(i+1,j)u(i,j+1)]^{\frac{1}{2}}}{u(i,j)}$$
$$\geq \sum_{i=m-\tau}^{m-1} \sigma \left(\prod_{i=m-\tau}^{m-1} \frac{[u(i+1,j)u(i,j+1)]^{1/2}}{u(i,j)}\right)^{\frac{1}{\sigma}}$$
$$= \sigma \sum_{i=m-\tau}^{m-1} \left(\frac{u(i,n)}{u(i,n-\sigma)} \prod_{i=m-\tau}^{m-1} \frac{u(i+1,j)}{u(i,j)}\right)^{\frac{1}{2\sigma}}$$
$$= \tau\sigma \prod_{i=m-\tau}^{m-1} \left(\frac{u(i,n)}{u(i,n-\sigma)}\right)^{\frac{1}{2\tau\sigma}} \prod_{i=m-\tau}^{m-1} \prod_{j=n-\sigma}^{n-1} \left(\frac{u(i+1,j)}{u(i,j)}\right)^{\frac{1}{2\tau\sigma}}$$
$$= \tau\sigma \prod_{i=m-\tau}^{m-1} \left(\frac{u(i,n)}{u(i,n-\sigma)}\right)^{\frac{1}{2\tau\sigma}} \prod_{i=m-\tau}^{m-1} \left(\frac{u(m,j)}{u(m-\tau,j)}\right)^{\frac{1}{2\tau\sigma}}$$
$$\geq \tau\sigma \left(\frac{u(m,n)}{u(m-\tau,n-\sigma)}\right)^{\frac{\tau+\sigma}{2\tau\sigma}}.$$

Thus,

$$1 - \psi \geq \left(\frac{u(m,n)}{u(m-\tau,n-\sigma)}\right)^{\frac{\tau+\sigma}{2\tau\sigma}} = \left(\frac{u(m,n)}{u(m-\tau,n-\sigma)}\right)^{\frac{1}{\kappa}}.$$

In particular, this implies $0 < \psi < 1$,

$$1 - \psi \leq \left(\frac{\kappa^\kappa}{(\kappa+1)^{\kappa+1}}\right)^{\frac{1}{\kappa}} = \psi^{-1/\kappa} = \left(\frac{\gamma}{\psi}\right)^{1/\kappa},$$

we thus obtain

$$\frac{u(m,n)}{u(m-\tau,n-\sigma)} \leq \frac{\gamma}{\psi},$$

say, for $m \geq M_1$ and $n \geq N_1$. If we apply the above equality to (3.19), we obtain

$$u(m+1,n)+u(m,n+1)-u(m,n) \leq -p(m,n)\Big(\frac{\gamma}{\psi}\Big)u(m,n), \quad m \geq M_1,\ n \geq N_1.$$

A similar procedure then leads to

$$\frac{u(m,n)}{u(m-\tau,n-\sigma)} \leq \Big(\frac{\gamma}{\psi}\Big)^2,$$

say, for $m \geq M_2 \geq M_1$ and $n \geq N_2 \geq N_1$. Inductively, we see that for any positive integer k, there are integers M_k and N_k such that

$$\frac{u(m,n)}{u(m-\tau,n-\sigma)} \leq \Big(\frac{\gamma}{\psi}\Big)^k,$$

for $m \geq M_k$ and $n \geq N_k$.
Thus, $\frac{u(m-\tau,n-\sigma)}{u(m,n)}$ diverges as m and n tend to infinity. On the other hand,

$$\begin{aligned}\frac{u(m-\tau,n-\sigma)}{u(m,n)} &= \frac{u(m-\tau,n-\sigma)}{u(m+1,n+1)}\frac{u(m+1,n+1)}{u(m,n)}\\ &\leq \frac{u(m-\tau,n-\sigma)}{u(m+1,n+1)}\frac{u(m+1,n)+u(m,n+1)}{u(m,n)}\\ &\leq \frac{u(m-\tau,n-\sigma)}{u(m+1,n+1)}[1-p(m,n)]\\ &\leq \frac{u(m-\tau,n-\sigma)}{u(m+1,n+1)}\end{aligned}$$

sufficiently large m and n. We may now apply Lemma 3.0.5 and deduce the fact that the last term in the above chain of inequalities is bounded. A contradiction is obtained. □

Theorem 3.0.8 (Zhang et al., 1995, Theorem 3.6)**.** *Assume* $\tau \geq 1$, $\sigma = 0$ *and*

$$\liminf_{m,n\to\infty} \sum_{i=m-\tau}^{m-1} p(i,n) > \left(\frac{\tau}{\tau+1}\right)^{\tau+1}. \tag{3.27}$$

Then, every solution of (3.19) *is oscillatory.*

Proof. Let $\{u(m,n)\}$ is an eventually positive solution of (3.19), then

$$\begin{aligned} u(m+1,n)-u(m,n) &\le u(m+1,n)+u(m,n+1)-u(m,n) \\ &\le -p(m,n)u(m-\tau,n) \le -p(m,n)u(m,n). \end{aligned}$$

In view of (3.27), we can choose a constant ψ such that

$$\begin{aligned} \gamma := \left(\frac{\tau}{\tau+1}\right)^{\tau+1} \le \psi &\le \frac{1}{\tau}\sum_{i=m-\tau}^{m-1} p(i,n) \\ &\le \frac{1}{\tau}\sum_{i=m-\tau}^{m-1}\left[1-\frac{u(i+1,n)}{u(i,n)}\right] \\ &\le 1-\frac{1}{\tau}\sum_{i=m-\tau}^{m-1}\frac{u(i+1,n)}{u(i,n)} \\ &\le 1-\left[\frac{u(m,n)}{u(m-\tau,n)}\right]^{\frac{1}{\tau}} \end{aligned}$$

for sufficiently large m and n. Thus, $0<\psi\le 1$ and

$$\left[\frac{u(m,n)}{u(m-\tau,n)}\right]^{\frac{1}{\tau}} \le 1-\psi \le \left[\frac{\tau^{\tau}}{(\tau+1)^{\tau+1}}\right]^{\frac{1}{\tau}} = \frac{\gamma}{\psi},$$

which implies

$$\frac{u(m,n)}{u(m-\tau,n)} = \frac{\gamma}{\psi} \quad \text{for all large } m \text{ and } n.$$

$\frac{u(s-\tau,t)}{u(s,t)}$ is unbounded as s and t tend to infinity like in the proof of Theorem 3.0.7. On the other hand,the same quantity $\frac{u(s-\tau,t)}{u(s,t)}$ is bounded by Lemma 3.0.6. This contradiction completes our proof. □

Later on, Zhang and Liu improved Theorem 3.0.7 and Theorem 3.0.8 with the following three results.

Let

$$\Lambda := \{\lambda > 0 :\ 1-\lambda p(m,n) > 0 \text{ for all } m,n \gg 0\}. \tag{3.28}$$

Theorem 3.0.9 (Zhang & Liu, 1997b, Theorem 2.1)**.** *Assume that*

$\limsup_{m,n\to\infty} p(m,n) > 0$ *and*

$$\limsup_{m,n\to\infty} \sup_{\lambda\in\Lambda} \left\{ \lambda \left(\prod_{i=m-\tau}^{m-1} \prod_{j=n-\sigma}^{n-1} [1-\lambda p(i,j)] \right)^{\frac{1}{S}} \right\} < 1, \tag{3.29}$$

where $S := \min\{\tau, \sigma\}$. *Then, every solution of* (3.19) *is oscillatory.*

Proof. Let $\{u(m,n)\}$ be an eventually positive solution of (3.19). Then, we have

$$u(m+1,n) + u(m,n+1) = -p(m,n)u(m-\tau,n-\sigma) \leq 0$$

and so $u(m+1,n) \leq u(m,n)$ and $u(m,n+1) \leq u(m,n)$. Then, $\{u(m,n)\}$ is decreasing in m and n. Using $u(m,n) \leq u(m-\tau,n-\sigma)$, we obtain

$$u(m+1,n) + u(m,n+1) - u(m,n) + p(m,n)u(m,n) \leq 0$$

or equivalently

$$u(m+1,n) + u(m,n+1) - [1-p(m,n)]u(m,n) \leq 0.$$

We assume that

$$\digamma(u) := \{\lambda > 0 : \ u(m+1,n)+u(m,n+1)-[1-\lambda p(m,n)]u(m,n) \leq 0, \ m,n \gg 0\}.$$

Note that if $\lambda \in \digamma(u)$, then since

$$0 < u(m+1,n) + u(m,n+1) \leq [1-p(m,n)]u(m,n),$$

we see that $\lambda \in \Lambda$. In other words, $\digamma(u)$ is subset of Λ. Then Λ is bounded, since $\limsup_{m,n\to\infty} p(m,n) > 0$ and $\digamma(u) \subset \Lambda$.

Clearly, $1 \in F(u)$ and so $F(u)$ is nonempty set. Let $\mu \in F(u)$, then

$$u(m+1,n) \le [1-\mu p(m,n)]u(m,n)$$

and

$$u(m,n+1) \le [1-\mu p(m,n)]u(m,n)$$

for all large m and n. Therefore,

$$\begin{aligned} u(m,n) &\le [1-\mu p(m-1,n)]u(m-1,n), \\ u(m-1,n) &\le [1-\mu p(m-2,n)]u(m-2,n), \\ &\vdots \\ u(m-\tau+1,n) &\le [1-\mu p(m-\tau,n)]u(m-\tau,n), \end{aligned} \tag{3.30}$$

so that

$$u(m,n) \le \prod_{i=m-\tau}^{m-1} [1-\mu p(i,n)]u(m-\tau,n) \tag{3.31}$$

and

$$u(m,n-j) \le \prod_{i=m-\tau}^{m-1} [1-\mu p(i,n-j)]u(m-\tau,n-j), \quad 0 \le j \le \sigma. \tag{3.32}$$

But then

$$\begin{aligned} u(m,n)^{\sigma} &\le u(m,n-1)u(m,n-2)\cdots u(m,n-\sigma) \\ &\le \left(\prod_{j=n-\sigma}^{n-1}\prod_{i=m-\tau}^{m-1} [1-\mu p(i,j)]\right)\big(u(m-\tau,n-\sigma)\big)^{\sigma}. \end{aligned} \tag{3.33}$$

Similarly, we may show by symmetric arguments that

$$u(m-i,n) \le \prod_{j=n-\sigma}^{n-1} [1-\mu p(m-i,j)]u(m-i,n-\sigma), \quad 0 \le i \le \tau, \tag{3.34}$$

and

$$u(m,n)^{\tau} \le \left(\prod_{i=m-\tau}^{m-1}\prod_{j=n-\sigma}^{n-1} [1-\mu p(i,j)]\right)\big(u(m-\tau,n-\sigma)\big)^{\tau}. \tag{3.35}$$

Combining (3.33) and (3.35), we see that

$$u(m,n) \le \left(\prod_{i=m-\tau}^{m-1} \prod_{j=n-\sigma}^{n-1} [1-\mu p(i,j)] \right)^{\frac{1}{S}} u(m-\tau, n-\sigma). \tag{3.36}$$

Substituting the above inequality (3.36) into (3.19), we obtain

$$\begin{aligned} &u(m+1,n)+u(m,n+1) \\ &-\left\{ 1-p(m,n)\left(\prod_{i=m-\tau}^{m-1} \prod_{j=n-\sigma}^{n-1} [1-\mu p(i,j)] \right)^{-\frac{1}{S}} \right\} u(m,n) \le 0 \end{aligned}$$

and hence

$$\begin{aligned} &u(m+1,n)+u(m,n+1) \\ &-\left\{ 1-p(m,n)\left(\sup_{m\ge M, n\ge N} \left(\prod_{i=m-\tau}^{m-1} \prod_{j=n-\sigma}^{n-1} [1-\mu p(i,j)] \right)^{\frac{1}{S}} \right)^{-1} \right\} u(m,n) \le 0. \end{aligned}$$

It follows that

$$\left(\sup_{m\ge M, n\ge N} \left\{ \left(\prod_{i=m-\tau}^{m-1} \prod_{j=n-\sigma}^{n-1} [1-\mu p(i,j)] \right)^{\frac{1}{S}} \right\} \right)^{-1} \in F(u).$$

On the other hand, the condition (3.29) implies the existence of a number $\beta \in (0,1)$ such that

$$\sup_{m\ge M, n\ge N} \left\{ \left(\prod_{i=m-\tau}^{m-1} \prod_{j=n-\sigma}^{n-1} [1-\mu p(i,j)] \right)^{\frac{1}{S}} \right\} \le \beta < 1.$$

Thus,

$$\left(\sup_{m\ge M, n\ge N} \left\{ \left(\prod_{i=m-\tau}^{m-1} \prod_{j=n-\sigma}^{n-1} [1-\mu p(i,j)] \right)^{\frac{1}{S}} \right\} \right)^{-1} \ge \frac{\mu}{\beta},$$

so that $\frac{\mu}{\beta} \in F(u)$. By induction, it is clear that $\frac{\mu}{\beta}^r \in F(u)$ for $r = 1, 2, \cdots$. But this implies that $F(u)$ is unbounded. Also, remember that $F(u)$ is a subset of Λ which is bounded by the assumption $\limsup_{m,n\to\infty} p(m,n) > 0$. This is a contradiction. The proof is complete. □

Remark 11. *An application of Theorem 3.0.9 to* (3.4) *yields the oscillation condition*

$$p\frac{(S+1)^{S+1}}{S^S} > 1.$$

Note that

$$\frac{(\tau+\sigma+1)^{\tau+\sigma+1}}{\tau^\tau \sigma^\sigma} > \left\{ \begin{array}{ll} \dfrac{(\sigma+1)^{\sigma+1}}{\sigma^\sigma}, & \tau \geq \sigma \\ \dfrac{(\tau+1)^{\tau+1}}{\tau^\tau}, & \sigma \geq \tau \end{array} \right\} = \frac{(S+1)^{S+1}}{S^S}.$$

Theorem 3.0.10 (Zhang & Liu, 1997b, Theorem 2.2)**.** *Assume that* $\tau, \sigma \geq 1$*,*

$$\limsup_{m,n\to\infty} p(m,n) > 0, \tag{3.37}$$

and

$$\limsup_{m,n\to\infty} \sup_{\lambda\in\Lambda} \left\{ \lambda \prod_{i=m-\tau}^{m-1} [1-\lambda p(i,n)] \prod_{j=n-\sigma}^{n-1} [1-\lambda p(m-\tau,j)] \right\} < 1. \tag{3.38}$$

Then, every solution of (3.19) *is oscillatory.*

Proof. As in the proof of Theorem 3.0.9, (3.31) and (3.34) hold so that

$$u(m,n) \leq \left(\prod_{i=m-\tau}^{m-1} [1-\mu p(i,n)] \prod_{j=n-\sigma}^{n-1} [1-\mu p(m-\tau,j)] \right) u(m-\tau,n-\sigma). \tag{3.39}$$

Substituting the above inequality (3.39) into (3.19), we obtain

$$\begin{aligned} & u(m+1,n) + u(m,n+1) \\ & - \left\{ 1 - p(m,n) \left(\prod_{i=m-\tau}^{m-1} [1-\mu p(i,n)] \prod_{j=n-\sigma}^{n-1} [1-\mu p(m-\tau,j)] \right)^{-1} \right\} u(m,n) \leq 0, \end{aligned}$$

and hence

$$\begin{aligned} & u(m+1,n) + u(m,n+1) \\ & - \left\{ 1 - p(m,n) \left(\sup_{m\geq M, n\geq N} \left\{ \prod_{i=m-\tau}^{m-1} [1-\mu p(i,n)] \prod_{j=n-\sigma}^{n-1} [1-\mu p(m-\tau,j) \right) \right\}^{-1} \right\} u(m,n) \leq 0. \end{aligned}$$

It follows that

$$\left(\sup_{m\geq M, n\geq N}\left(\prod_{i=m-\tau}^{m-1}[1-\mu p(i,n)]\prod_{j=n-\sigma}^{n-1}[1-\mu p(m-\tau,j)\right)\right)^{-1}\in \mathcal{F}(u) \tag{3.40}$$

On the other hand, the condition (3.38) implies the existence of a number $\beta \in (0,1)$ such that

$$\sup_{m\geq M, n\geq N}\left(\lambda\prod_{i=m-\tau}^{m-1}[1-\lambda p(i,n)]\prod_{j=n-\sigma}^{n-1}[1-\lambda p(m-\tau,j)]\right)\leq \beta < 1.$$

Thus,

$$\left(\sup_{m\geq M, n\geq N}\left(\prod_{i=m-\tau}^{m-1}[1-\mu p(i,n)]\prod_{j=n-\sigma}^{n-1}[1-\mu p(m-\tau,j)\right)\right)^{-1}\geq \frac{\mu}{\beta},$$

so that $\frac{\mu}{\beta}\in \mathcal{F}(u)$. By induction, it is clear that $\frac{\mu}{\beta}^r \in \mathcal{F}(u)$ for $r = 1, 2, \cdots$. But this implies that $\mathcal{F}(u)$ is unbounded. Also, remember that $\mathcal{F}(u)$ is a subset of Λ which is bounded by the assumption $\limsup_{m,n\to\infty} p(m,n) > 0$. This is a contradiction. The proof is complete. □

Theorem 3.0.11 (Zhang & Liu, 1997b, Theorem 2.4)**.** *Further, assume that* $\tau \geq 1$*,* (3.28)*, and*

$$\limsup_{m,n\to\infty}\sup_{\lambda\in\Lambda}\left\{\lambda\prod_{i=m-\tau}^{m-1}[1-\lambda p(i,n)]\right\}<1, \tag{3.41}$$

where Λ *is defined in* (3.28)*. Then, every solution of* (3.19) *is oscillatory.*

Proof. As in the proof of Theorem 3.0.9, substituting (3.41) into (3.19), we obtain

$$u(m+1,n)+u(m,n+1)-\left\{1-p(m,n)\left(\prod_{i=m-\tau}^{m-1}[1-\mu p(i,n)]\right)^{-1}\right\}u(m,n)\leq 0,$$

and hence

$$u(m+1,n)+u(m,n+1) - \left\{1-p(m,n)\left(\sup_{m\geq M, n\geq N}\left\{\prod_{i=m-\tau}^{m-1}[1-\mu p(i,n)]\right\}\right)^{-1}\right\}u(m,n)\leq 0.$$

It follows that

$$\left(\sup_{m\geq M, n\geq N}\left\{\prod_{i=m-\tau}^{m-1}[1-\mu p(i,n)]\right\}\right)^{-1}\in F(u). \tag{3.42}$$

On the other hand, the condition (3.41) implies the existence of a number $\beta \in (0,1)$ such that

$$\sup_{m\geq M, n\geq N}\left\{\lambda\prod_{i=m-\tau}^{m-1}[1-\lambda p(i,n)]\right\}\leq \beta < 1.$$

Thus,

$$\left(\sup_{m\geq M, n\geq N}\left\{\prod_{i=m-\tau}^{m-1}[1-\mu p(i,n)]\prod_{j=n-\sigma}^{n-1}[1-\mu p(m-\tau,j)\right)\right\}^{-1}\geq \frac{\mu}{\beta},$$

so that $\frac{\mu}{\beta}\in F(u)$. By induction, it is clear that $\frac{\mu}{\beta}^{r}\in F(u)$ for $r=1,2,\cdots$. But this implies that $F(u)$ is unbounded. Also, remember that $F(u)$ is a subset of Λ which is bounded by the assumption $\limsup_{m,n\to\infty}p(m,n)>0$. This is a contradiction. The proof is complete. □

CHAPTER FOUR
NEW RESULT

In this chapter, we prove our new theorems and give several corollaries.
The following lemma establishes equivalence between oscillation of solutions of equations of the forms

$$au(m+1,n)+bu(m,n+1)-cu(m,n)+p(m,n)u(m-\tau,n-\sigma)=0 \qquad (4.1)$$

for $m,n=0,1,\cdots$ and

$$u(m+1,n)+u(m,n+1)-u(m,n)+p(m,n)u(m-\tau,n-\sigma)=0 \qquad (4.2)$$

for $m,n=0,1,\cdots$.

Lemma 4.0.1. *Let $a,b,c\in\mathbb{R}^+$ and $\tau,\sigma\in\mathbb{Z}$. Then, the following statements are equivalent.*

(i) Every solution of (4.1) *is oscillatory.*

(ii) Every solution of

$$u(m+1,n)+u(m,n+1)-u(m,n)+\frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}}p(m,n)u(m-\tau,n-\sigma)=0 \quad (4.3)$$

for $m,n=0,1,\cdots$ is oscillatory.

Proof. The oscillation invariant substitution $u(m,n):=\frac{c^{m+n}}{a^m b^n}v(m,n)$ shows that oscillation and nonoscillation of (4.1) and (4.3) are equivalent. □

Remark 12. *It should be also noted that* (3.15) *admits a proper solution if and only if so does*

$$u(m+1,n)+u(m,n+1)-u(m,n)+\frac{pa^\tau b^\sigma}{c^{\tau+\sigma-1}}u(m-\tau,n-\sigma)=0, \quad m,n=0,1,\cdots.$$

Remark 13. *Lemma 4.0.1 allows us to apply oscillation/nonoscillation results (such as Theorem 3.0.7—Theorem 3.0.3.2) for* (4.2) *to* (4.1)*.*

Let $\gamma = \{(\alpha_i, \beta_i)\}$ be a path on the lattice $\mathbb{Z} \times \mathbb{Z}$ combining the points from (m, n) to $(m - \tau, n - \sigma)$ obtained by moving either left or downwards. Readily, any γ is obtained by moving τ-times left and σ-times downwards in certain orders, and consists of $(\tau + \sigma + 1)$ pairs, i.e., $\gamma = \{(\alpha_0, \beta_0), (\alpha_1, \beta_1), \cdots, (\alpha_{\tau+\sigma}, \beta_{\tau+\sigma})\}$ such that $(\alpha_0, \beta_0) = (m, n)$ and $(\alpha_{\tau+\sigma}, \beta_{\tau+\sigma}) = (m - \tau, n - \sigma)$. For instance, such a path is $\gamma = \{(m, n), (m - 1, n), \cdots, (m - \tau, n), (m - \tau, n - 1), \cdots, (m - \tau, n - \sigma)\}$. Mathematically, γ is a solution of the problem

$$\begin{cases} (\alpha_0, \beta_0) = (m, n) \\ (\alpha_i, \beta_i) = (\alpha_{i-1}, \beta_{i-1}) - (s_i, 1 - s_i), \quad i = 1, \cdots, \tau + \sigma \\ (\alpha_{\tau+\sigma}, \beta_{\tau+\sigma}) = (m - \tau, n - \sigma), \end{cases}$$

where $s_1, s_2, \cdots, s_{\tau+\sigma} \in \{0, 1\}$. Note that $\sum_{i=1}^{\tau+\sigma} s_i = \tau$.

Let Γ denote the set of all possible paths on $\mathbb{Z} \times \mathbb{Z}$ from (m, n) to $(m - \tau, n - \sigma)$ obtained by moving either left or downwards. Clearly, the number of paths in Γ is $\binom{\tau+\sigma}{\tau} = \frac{(\tau+\sigma)!}{\tau!\sigma!}$. Note that in the case where $\tau = 0$ or $\sigma = 0$, Γ consists of a single path

$$\gamma := \begin{cases} \{(m, n), (m - 1, n), \cdots, (m - \tau, n)\}, & \sigma = 0 \\ \{(m, n), (m, n - 1), \cdots, (m, n - \sigma)\}, & \tau = 0. \end{cases}$$

Now, we give a simple result, which generalizes (Zhang et al., 1995, Theorem 3.1).

Theorem 4.0.2. *Assume* $1 \notin \Lambda$*, where* Λ *is defined in* (3.28)*. Then, every solution of* (4.2) *is oscillatory.*

Proof. Thus, there exist increasing sequences of integers $\{\xi_k\}, \{\zeta_k\}$ such that $1 - p(\xi_k, \zeta_k) \leq 0$ for all k, i.e., $p(\xi_k, \zeta_k) \geq 1$ for all k. Now, suppose that $\{u(m, n)\}$ is an eventually positive solution of (4.2). Then, there is at least $m_1 \geq m_0$ and

$n_1 \geq n_0$ such that $u(m,n) > 0$ and $u(m-\tau, n-\sigma) > 0$ for $m = m_1, m_1+1, \cdots$ and $n = n_1, n_1+1, \cdots$. From the equation (4.2), we see that $u(m+1,n) + u(m,n+1) - u(m,n) \leq 0$ for $m = m_1, m_1+1, \cdots$ and $n = n_1, n_1+1, \cdots$, i.e., $u(m+1,n) < u(m,n)$ and $u(m+1,n) < u(m,n)$ for $m = m_1, m_1+1, \cdots$ and $n = n_1, n_1+1, \cdots$. Clearly, $\xi_k \geq m_1 + \tau$ and $\zeta_k \geq n_1 + \sigma$ for all $k \gg 0$. It follows from (4.2) that

$$\begin{aligned} 0 &> u(\xi_k+1, \zeta_k) + u(\xi_k, \zeta_k+1) - u(\xi_k, \zeta_k) + p(\xi_k, \zeta_k)u(\xi_k, \zeta_k) \\ &= u(\xi_k+1, \zeta_k) + u(\xi_k, \zeta_k+1) - [1 - p(\xi_k, \zeta_k)]u(\xi_k, \zeta_k) > 0 \end{aligned}$$

for all $k \gg 0$, which is a contradiction. □

As an immediate consequence of Theorem 4.0.2, we can give the following corollary.

Corollary 4.0.2.1. *Assume that there exist increasing sequences of integers* $\{\xi_k\}, \{\zeta_k\}$ *such that*

$$p(\xi_k, \zeta_k) \geq 1, \quad k = 1, 2, \cdots.$$

Then, every solution of (4.2) *is oscillatory.*

The following theorem is the main of this chapter.

Theorem 4.0.3. *Assume* $1 \in \Lambda$ *and*

$$\limsup_{m,n\to\infty} \sup_{\lambda\in\Lambda} \min_{\gamma=\{(\alpha_i,\beta_i)\}\in\Gamma} \left\{ \lambda \prod_{i=1}^{\tau+\sigma} [1 - \lambda p(\alpha_i, \beta_i)] \right\} < 1, \tag{4.4}$$

where γ *depends on both* m *and* n*. Then, all solutions of* (4.2) *oscillate.*

Before, we proceed with the proof of Theorem 4.0.3, we need to prove two lemmas.

Lemma 4.0.4. *If* (4.4) *holds, then* $\limsup_{m,n\to\infty} p(m,n) > 0$.

Proof. Assume the contrary that $\lim_{m,n\to\infty} p(m,n) = 0$. Then there exists $0 < \varepsilon \leq$

$\frac{(\tau+\sigma)^{\tau+\sigma}}{2(\tau+\sigma+1)^{\tau+\sigma+1}}$ such that $p(m,n) < \varepsilon$ for all $m,n \gg 0$. Thus, we estimate

$$\sup_{\lambda\in\Lambda}\left\{\lambda\prod_{i=1}^{\tau+\sigma}[1-\lambda p(\alpha_i,\beta_i)]\right\} \geq \sup_{\lambda\in\Lambda}\{\lambda(1-\varepsilon\lambda)^{\tau+\sigma}\} = \frac{(\tau+\sigma)^{\tau+\sigma}}{\varepsilon(\tau+\sigma+1)^{\tau+\sigma+1}} \geq 2$$

for all $\gamma = \{(\alpha_i,\beta_i)\} \in \Gamma$ and $m,n \gg 0$, which contradicts (4.4). □

Lemma 4.0.5. *If* $\limsup_{m,n\to\infty} p(m,n) > 0$ *holds, then* $\sup\{\Lambda\} < \infty$.

Proof. Assume the contrary that $\sup\{\Lambda\} = \infty$. There exist increasing sequences of integers $\{\xi_k\}$, $\{\zeta_k\}$ and $\delta > 0$ such that

$$p(\xi_k,\zeta_k) \geq \delta \quad \text{for all } k.$$

We have $\frac{2}{\delta} \in \Lambda$ since $\sup\{\Lambda\} = \infty$. Hence, it follows that

$$0 < 1 - \frac{2}{\delta}p(\xi_k,\zeta_k) \leq 1 - \frac{2}{\delta}\delta = -1 \quad \text{for all } k,$$

which is a contradiction. □

Now, we are in a position to prove Theorem 4.0.3.

Proof of Theorem 4.0.3. Assume the contrary that $\{u(m,n)\}$ is an eventually positive solution of (4.2). Then, there exists $m_1 \geq m_0$ and $n_1 \geq n_0$ such that $u(m,n) > 0$ and $u(m-\tau,n-\sigma) > 0$ for $m = m_1, m_1+1,\cdots$ and $n = n_1, n_1+1,\cdots$. From (4.2), we see that $u(m+1,n)+u(m,n+1)-u(m,n) \leq 0$ for $m = m_1, m_1+1,\cdots$ and $n = n_1, n_1+1,\cdots$, i.e., $u(m+1,n) < u(m,n)$ and $u(m+1,n) < u(m,n)$ for $m = m_1, m_1+1,\cdots$ and $n = n_1, n_1+1,\cdots$. Again from (4.2), we obtain

$$u(m+1,n)+u(m,n+1)-u(m,n)+p(m,n)u(m,n) \leq 0 \tag{4.5}$$

for $m = m_2, m_2+1,\cdots$ and $n = n_2, n_2+1,\cdots$, where $m_2 \geq m_1+\tau$ and $n_2 \geq n_1+\sigma$.

Define

$$\digamma(u) := \{\lambda > 0 : \ u(m+1,n)+u(m,n+1)-[1-\lambda p(m,n)]u(m,n) \leq 0, \ m,n \gg 0\}.$$

By (4.5), $1 \in \digamma(u)$, i.e., $\digamma(u) \neq \emptyset$. Readily, if $\lambda \in \digamma(u)$, then $\lambda \in \Lambda$, i.e., $\digamma(u) \subset \Lambda$. Further, for $\lambda \in \digamma(u)$, we have

$$\begin{cases} u(m+1,n) \leq [1-\lambda p(m,n)]u(m,n) \\ u(m,n+1) \leq [1-\lambda p(m,n)]u(m,n) \end{cases}$$

or equivalently

$$\begin{cases} u(m,n) \leq [1-\lambda p(m-1,n)]u(m-1,n) \\ u(m,n) \leq [1-\lambda p(m,n-1)]u(m,n-1) \end{cases} \tag{4.6}$$

for all $m,n \gg 0$. Letting $\gamma := \{(\alpha_0,\beta_0),(\alpha_1,\beta_1),\cdots,(\alpha_{\tau+\sigma},\beta_{\tau+\sigma})\} \in \Gamma$, (4.6) yields

$$\begin{aligned} u(m,n) &\leq [1-\lambda p(\alpha_1,\beta_1)]u(\alpha_1,\beta_1) \\ &\leq [1-\lambda p(\alpha_1,\beta_1)][1-\lambda p(\alpha_2,\beta_2)]u(\alpha_2,\beta_2) \\ &\vdots \\ &\leq \left(\prod_{i=1}^{\tau+\sigma}[1-\lambda p(\alpha_i,\beta_i)]\right)u(\alpha_{\tau+\sigma},\beta_{\tau+\sigma}) \\ &= \left(\prod_{i=1}^{\tau+\sigma}[1-\lambda p(\alpha_i,\beta_i)]\right)u(m-\tau,n-\sigma) \end{aligned} \tag{4.7}$$

for all $m,n \gg 0$. Substituting (4.7) into (4.2), we obtain

$$u(m+1,n)+u(m,n+1)-u(m,n)+\lambda p(m,n)\left(\prod_{i=1}^{\tau+\sigma}[1-\lambda p(\alpha_i,\beta_i)]\right)^{-1}u(m,n) \leq 0$$

for all $m,n \gg 0$ which yields

$$u(m+1,n)+u(m,n+1)-u(m,n)+\nu\lambda p(m,n)u(m,n) \leq 0 \quad \text{for all } m,n \gg 0$$

since

$$\sup_{\lambda\in\Lambda}\left\{\min_{\gamma=\{(\alpha_i,\beta_i)\}\in\Gamma}\left\{\lambda\prod_{i=1}^{\tau+\sigma}[1-\lambda p(\alpha_i,\beta_i)]\right\}\right\}\leq\frac{1}{\nu}<1\quad\text{for all } m,n\gg 0$$

showing that $\nu\lambda\in F(u)$. Repeating this procedure, we obtain $\nu^k\lambda\in F(u)$ for $k\in\mathbb{N}$, which yields $\sup\{F(u)\}=\infty$ since $\nu>1$. However, by Lemma 4.0.4 and Lemma 4.0.5, we have $\sup\{\Lambda\}<\infty$, which is a contradiction since $F(u)\subset\Lambda$. Therefore, the proof is completed. □

Theorem 4.0.6. *Assume* $\frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}}\notin\Lambda$, *where* Λ *is defined in* (3.28). *Then, every solution of* (4.1) *is oscillatory.*

Theorem 4.0.7. *Assume* $\frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}}\in\Lambda$ *and*

$$\limsup_{m,n\to\infty}\sup_{\lambda\in\Lambda}\min_{\gamma=\{(\alpha_i,\beta_i)\}\in\Gamma}\left\{\lambda\prod_{i=1}^{\tau+\sigma}\left[1-\lambda\frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}}p(\alpha_i,\beta_i)\right]\right\}<\frac{c^{\tau+\sigma-1}}{a^\tau b^\sigma},\qquad(4.8)$$

where γ *depends on both* m *and* n. *Then, every solution of* (4.1) *is oscillatory.*

Next, we list several corollaries extracted out from Theorem 4.0.7.

Letting $\gamma=\{(m,n),\cdots,(m-\tau,n),(m-\tau,n-1),\cdots,(m-\tau,n-\sigma)\}$, we obtain the following.

Corollary 4.0.7.1. *Assume* $\frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}}\in\Lambda$ *and*

$$\limsup_{m,n\to\infty}\sup_{\lambda\in\Lambda}\left\{\lambda\prod_{i=1}^{\tau}\left[1-\lambda\frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}}p(m-i,n)\right]\prod_{i=1}^{\sigma}\left[1-\lambda\frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}}p(m-\tau,n-i)\right]\right\}<\frac{c^{\tau+\sigma-1}}{a^\tau b^\sigma}.$$

Then, every solution of (4.1) *is oscillatory.*

Letting $\gamma=\{(m,n),\cdots,(m,n-\sigma),(m-1,n-\sigma),\cdots,(m-\tau,n-\sigma)\}$, we obtain the following.

Corollary 4.0.7.2. *Assume* $\frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}}\in\Lambda$ *and*

$$\limsup_{m,n\to\infty}\sup_{\lambda\in\Lambda}\left\{\lambda\prod_{i=1}^{\sigma}\left[1-\lambda\frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}}p(m,n-i)\right]\prod_{i=1}^{\tau}\left[1-\lambda\frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}}p(m-i,n-\sigma)\right]\right\}<\frac{c^{\tau+\sigma-1}}{a^\tau b^\sigma}.$$

Then, every solution of (4.1) *is oscillatory.*

Letting

$$\gamma = \begin{cases} \{(m,n),(m-1,n),(m-1,n-1),(m-2,n-1),\cdots,(m-\sigma,n-\sigma), \\ \qquad (m-\sigma-1,n-\sigma),\cdots,(m-\tau,n-\sigma)\}, \quad \tau \geq \sigma \\ \{(m,n),(m,n-1),(m-1,n-1),(m-1,n-2),\cdots,(m-\tau,n-\tau), \\ \qquad (m-\tau,n-\tau-1),\cdots,(m-\tau,n-\sigma)\}, \quad \sigma \geq \tau, \end{cases}$$

we obtain the following.

Corollary 4.0.7.3. *Assume* $\frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} \in \Lambda$ *and*

$$\begin{cases} \limsup_{m,n\to\infty} \sup_{\lambda\in\Lambda} \left\{ \lambda \prod_{i=1}^{\sigma} \left[1 - \lambda \frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} p(m-i,n-i+1)\right] \right. \\ \qquad \times \prod_{i=1}^{\sigma} \left[1 - \lambda \frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} p(m-i,n-i)\right] \\ \qquad \left. \times \prod_{i=\sigma+1}^{\tau} \left[1 - \lambda \frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} p(m-i,n-\sigma)\right] \right\} < \frac{c^{\tau+\sigma-1}}{a^\tau b^\sigma}, \quad \tau \geq \sigma \\ \limsup_{m,n\to\infty} \sup_{\lambda\in\Lambda} \left\{ \lambda \prod_{i=1}^{\tau} \left[1 - \lambda \frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} p(m-i+1,n-i)\right] \right. \\ \qquad \times \prod_{i=1}^{\tau} \left[1 - \lambda \frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} p(m-i,n-i)\right] \\ \qquad \left. \times \prod_{i=\tau+1}^{\sigma} \left[1 - \lambda \frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} p(m-\tau,n-i)\right] \right\} < \frac{c^{\tau+\sigma-1}}{a^\tau b^\sigma}, \quad \sigma \geq \tau. \end{cases}$$

Then, every solution of (4.1) *is oscillatory.*

Similar to Corollary 4.0.7.3, we can obtain the following.

Corollary 4.0.7.4. *Assume* $\frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} \in \Lambda$ *and*

$$\begin{cases} \limsup_{m,n\to\infty} \sup_{\lambda\in\Lambda} \left\{ \lambda \prod_{i=1}^{\tau-\sigma} \left[1 - \lambda \frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} p(m-i,n)\right] \right. \\ \qquad \times \prod_{i=\tau-\sigma+1}^{\tau} \left[1 - \lambda \frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} p(m-i,n-i+\tau-\sigma+1)\right] \\ \qquad \left. \times \prod_{i=\tau-\sigma+1}^{\tau} \left[1 - \lambda \frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} p(m-i,n-i+\tau-\sigma)\right] \right\} < \frac{c^{\tau+\sigma-1}}{a^\tau b^\sigma}, \quad \tau \geq \sigma \\ \limsup_{m,n\to\infty} \sup_{\lambda\in\Lambda} \left\{ \lambda \prod_{i=1}^{\sigma-\tau} \left[1 - \lambda \frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} p(m,n-i)\right] \right. \\ \qquad \times \prod_{i=\sigma-\tau+1}^{\sigma} \left[1 - \lambda \frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} p(m-i+\sigma-\tau+1,n-i)\right] \\ \qquad \left. \times \prod_{i=\sigma-\tau+1}^{\sigma} \left[1 - \lambda \frac{a^\tau b^\sigma}{c^{\tau+\sigma-1}} p(m-i+\sigma-\tau,n-i)\right] \right\} < \frac{c^{\tau+\sigma-1}}{a^\tau b^\sigma}, \quad \sigma \geq \tau. \end{cases}$$

Then, every solution of (4.1) *is oscillatory.*

Due to the inequality of arithmetic and geometric means, we obtain the following corollary from Theorem 4.0.3.

Corollary 4.0.7.5. *Assume*

$$\liminf_{m,n\to\infty} \max_{\gamma=\{(\alpha_i,\beta_i)\}\in\Gamma} \sum_{i=1}^{\tau+\sigma} p(\alpha_i,\beta_i) > \frac{c^{\tau+\sigma-1}}{a^\tau b^\sigma} \left(\frac{\tau+\sigma}{\tau+\sigma+1}\right)^{\tau+\sigma+1}, \tag{4.9}$$

where γ *depends on both* m *and* n*. Then, every solution of* (4.1) *is oscillatory.*

Combining Corollaries 4.0.7.1, 4.0.7.2 and 4.0.7.5, we obtain the following two results.

Corollary 4.0.7.6. *Assume*

$$\liminf_{m,n\to\infty} \left(\sum_{i=1}^{\tau} p(m-i,n) + \sum_{i=1}^{\sigma} p(m-\tau,n-i) \right) > \frac{c^{\tau+\sigma-1}}{a^\tau b^\sigma} \left(\frac{\tau+\sigma}{\tau+\sigma+1}\right)^{\tau+\sigma+1}. \tag{4.10}$$

Then, every solution of (4.1) *is oscillatory.*

Corollary 4.0.7.7. *Assume*

$$\liminf_{m,n\to\infty} \left(\sum_{i=1}^{\sigma} p(m,n-i) + \sum_{i=1}^{\tau} p(m-i,n-\sigma) \right) > \frac{c^{\tau+\sigma-1}}{a^\tau b^\sigma} \left(\frac{\tau+\sigma}{\tau+\sigma+1}\right)^{\tau+\sigma+1}. \tag{4.11}$$

Then, every solution of (4.1) *is oscillatory.*

In this part, we give a numerical example where all the previous known results fail to deduce any decision on the oscillation/nonoscillation of solutions but some of our new results succeed.

Example 7. Consider the partial difference equation

$$u(m+1,n)+u(m,n+1)-u(m,n)+p(m,n)u(m-3,n-2)=0 \text{ for } m,n=0,1,\cdots, \tag{4.12}$$

where

$$p(m,n):=\begin{cases}\frac{21}{64}, & m=0,3,6,\cdots,\ n=0,3,6,\cdots\\ \frac{1}{512}, & \text{otherwise.}\end{cases}$$

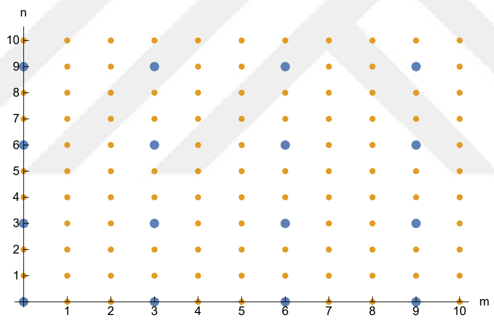


Figure 4.1 Plot of the coefficient sequence $\{p(m,n)\}$. Yellow: $\frac{1}{512}$, Blue: $\frac{21}{64}$

We see that $\Lambda=(0,\frac{64}{21})$. Note that

$$p(m,n)\geq\frac{1}{512}\quad\text{and}\quad\frac{1}{512}\leq\frac{3^3 2^2}{(3+2+1)^{3+2+1}}=\frac{1}{432}.$$

That is, every proper solution of the minimal autonomous equation

$$u(m+1,n)+u(m,n+1)-u(m,n)+\frac{1}{512}u(m-3,n-2)=0 \text{ for } m,n=0,1,\cdots$$

is nonoscillatory by Theorem 3.0.3.2. Further, we compute that

$$\begin{aligned}
&\sup_{0<\lambda<\frac{64}{21}} \left\{ \lambda \Big(\prod_{i=m-3}^{m-1} \prod_{j=n-2}^{n-1} [1-\lambda p(i,j)] \Big)^{\frac{1}{2}} \right\} \\
&= \left\{ \begin{array}{ll} \sup\limits_{0<\lambda<\frac{64}{21}} \big\{ \lambda (1-\frac{\lambda}{512})^3 \big\}, & m=0,3,6,\cdots \\ \sup\limits_{0<\lambda<\frac{64}{21}} \Big\{ \lambda (1-\frac{21\lambda}{64})^{\frac{1}{2}} (1-\frac{\lambda}{512})^{\frac{5}{2}} \Big\}, & \text{otherwise} \end{array} \right\} \\
&\geq \left\{ \begin{array}{ll} 2(1-\frac{2}{512})^3, & m=0,3,6,\cdots \\ 2(1-\frac{2\times 21}{64})^{\frac{1}{2}} (1-\frac{2}{512})^{\frac{5}{2}}, & \text{otherwise} \end{array} \right\} \\
&\approx \left\{ \begin{array}{ll} 1.97665, & m=0,3,6,\cdots \\ 1.16119, & \text{otherwise} \end{array} \right\} \geq 1.
\end{aligned}$$

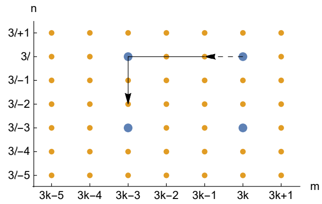
Hence, Theorem 3.0.9 does not apply for (4.12). For $k = 1, 2, 3$, we define

$$\begin{aligned}
\gamma &:= \{(\alpha_0, \beta_0), (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3), (\alpha_4, \beta_4), (\alpha_5, \beta_5)\} \\
&= \{(m,n), \cdots, (m-k,n), (m-k,n-1), (m-k,n-2), \cdots, (m-3,n-2)\}.
\end{aligned}$$

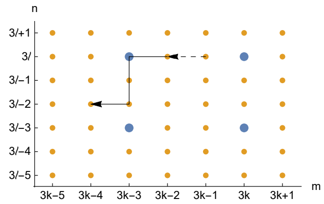
Thus, we compute

$$\sum_{i=1}^{5} p(\alpha_i, \beta_i) = 4 \times \frac{1}{512} + \frac{21}{64} = \underbrace{\frac{345}{1024}}_{\approx 0.337} > \left(\frac{3+2}{3+2+1} \right)^{3+2+1} = \underbrace{\frac{15625}{46656}}_{\approx 0.335}.$$

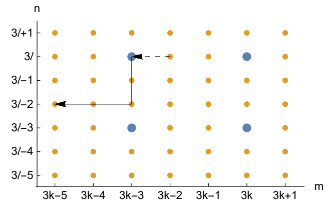
Due to Corollary 4.0.7.5, every solution of (4.12) is oscillatory.



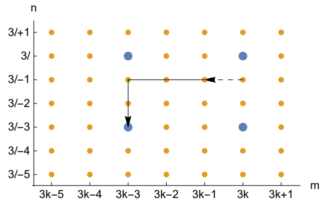
(a) $\gamma = \{(3k, 3\ell), (3k-1, 3\ell), (3k-2, 3\ell), (3k-3, 3\ell), (3k-3, 3\ell-1), (3k-3, 3\ell-2)\}$



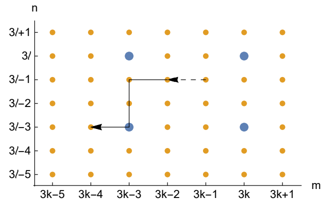
(b) $\gamma = \{(3k-1, 3\ell), (3k-2, 3\ell), (3k-3, 3\ell), (3k-3, 3\ell-1), (3k-3, 3\ell-2), (3k-4, 3\ell-2)\}$



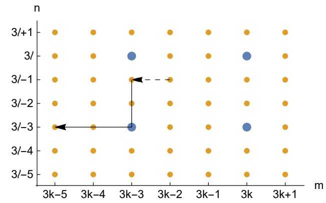
(c) $\gamma = \{(3k-2, 3\ell), (3k-3, 3\ell), (3k-3, 3\ell-1), (3k-3, 3\ell-2), (3k-4, 3\ell-2), (3k-5, 3\ell-2)\}$



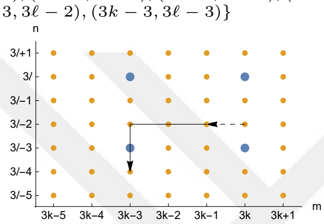
(d) $\gamma = \{(3k, 3\ell-1), (3k-1, 3\ell-1), (3k-2, 3\ell-1), (3k-3, 3\ell-1), (3k-3, 3\ell-2), (3k-3, 3\ell-3)\}$



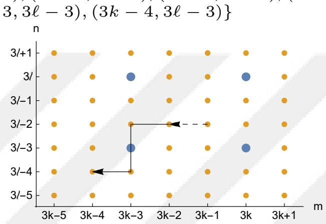
(e) $\gamma = \{(3k-1, 3\ell-1), (3k-2, 3\ell-1), (3k-3, 3\ell-1), (3k-3, 3\ell-2), (3k-3, 3\ell-3), (3k-4, 3\ell-3)\}$



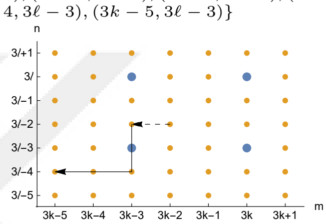
(f) $\gamma = \{(3k-2, 3\ell-1), (3k-3, 3\ell-1), (3k-3, 3\ell-2), (3k-3, 3\ell-3), (3k-4, 3\ell-3), (3k-5, 3\ell-3)\}$



(g) $\gamma = \{(3k, 3\ell-2), (3k-1, 3\ell-2), (3k-2, 3\ell-2), (3k-3, 3\ell-2), (3k-3, 3\ell-3), (3k-3, 3\ell-4)\}$



(h) $\gamma = \{(3k-1, 3\ell-2), (3k-2, 3\ell-2), (3k-3, 3\ell-2), (3k-3, 3\ell-3), (3k-3, 3\ell-4), (3k-4, 3\ell-4)\}$



(i) $\gamma = \{(3k-2, 3\ell-2), (3k-3, 3\ell-2), (3k-3, 3\ell-3), (3k-3, 3\ell-4), (3k-4, 3\ell-4), (3k-5, 3\ell-4)\}$

Figure 4.2 Paths for different (m, n) points

CHAPTER FIVE
CONCLUSION

In this section, we list some remarks on our main result Theorem 4.0.3 to conclude the thesis.

Remark 14. *Corollary 4.0.7.5 includes Theorem 3.0.8 with* $\sigma := 0$ *and* $\gamma := \{(m,n),(m-1,n),\cdots,(m-\tau,n)\}$.

Remark 15. *Theorem 4.0.3 improves Theorem 3.0.10 with* $\gamma := \{(m,n),(m-1,n),\cdots,(m-\tau,n),(m-\tau,n-1),\cdots,(m-\tau,n-\sigma)\}$ *by dropping the condition* (3.37).

Remark 16. *Theorem 4.0.3 improves Theorem 3.0.11 with* $\sigma := 0$ *and* $\gamma := \{(m,n),(m-1,n),\cdots,(m-\tau,n)\}$ *by dropping the condition* (3.37).

Remark 17. *Theorem 4.0.3 improves Theorem 3.0.9 by dropping the condition* (3.28). *For* $k = 1,2,\cdots,\tau$*, let* $\gamma_k := \{(\alpha_i^k, \beta_i^k)\}$ *be the path from* (m,n) *to* $(m-\tau,n-\sigma)$ *passing through the points* (m,n), $(m-k,n)$, $(m-k,n-\sigma)$ *and* $(m-\tau,n-\sigma)$*, and* $\gamma^* = \{(\alpha_i^*,\beta_i^*)\} \in \Gamma$ *be the path satisfying*

$$\min_{\gamma=\{(\alpha_i,\beta_i)\}\in\Gamma}\left\{\prod_{i=1}^{\tau+\sigma}[1-\lambda p(\alpha_i,\beta_i)]\right\} = \prod_{i=1}^{\tau+\sigma}[1-\lambda p(\alpha_i^*,\beta_i^*)].$$

Then, for $\lambda \in \Lambda$ *and* $m,n \gg 0$*, we estimate that*

$$\begin{aligned}&\underbrace{\prod_{i=m-\tau}^{m-1}\prod_{j=n-\sigma}^{n-1}[1-\lambda p(i,j)]}_{\tau\times\sigma\text{–terms}}\\ &\geq \prod_{k=1}^{\tau}\Bigg[\underbrace{\underbrace{\left(\prod_{i=m-\tau}^{m-(k+1)}[1-\lambda p(i,n-\sigma)]\right)}_{(\tau-k)\text{–terms}}\underbrace{\left(\prod_{j=n-\sigma}^{n-1}[1-\lambda p(m-k,j)]\right)}_{\sigma\text{–terms}}\underbrace{\left(\prod_{i=m-k}^{m-1}[1-\lambda p(i,n)]\right)}_{k\text{–terms}}}_{(\tau+\sigma)\text{–terms}}\Bigg]\\ &= \prod_{k=1}^{\tau}\prod_{j=1}^{\tau+\sigma}[1-\lambda p(\alpha_j^k,\beta_j^k)] \geq \prod_{k=1}^{\tau}\prod_{j=1}^{\tau+\sigma}[1-\lambda p(\alpha_j^*,\beta_j^*)] = \left(\prod_{j=1}^{\tau+\sigma}[1-\lambda p(\alpha_j^*,\beta_j^*)]\right)^{\tau},\end{aligned}$$

where the empty product is assumed to be 1*. Similarly, for* $\lambda \in \Lambda$ *and* $m,n \gg 0$*, we* *estimate that*

$$\prod_{i=m-\tau}^{m-1}\prod_{j=n-\sigma}^{n-1}[1-\lambda p(i,j)] \geq \left(\prod_{j=1}^{\tau+\sigma}[1-\lambda p(\alpha_j^*,\beta_j^*)]\right)^{\sigma}.$$

Thus, we obtain

$$\prod_{i=m-\tau}^{m-1}\prod_{j=n-\sigma}^{n-1}[1-\lambda p(i,j)] \geq \left(\prod_{j=1}^{\tau+\sigma}[1-\lambda p(\alpha_j^*,\beta_j^*)]\right)^{S}.$$

This shows that (4.4) *is better than* (3.29).

Remark 18. *By applying Theorem 3.0.3.2, Theorem 3.0.9 and Theorem 4.0.3 to* (3.15)*, we get the following conditions.*

- *Theorem 3.0.3.2:* $p > \frac{\tau^\tau\sigma^\sigma}{(\tau+\sigma+1)^{\tau+\sigma+1}}$.
- *Theorem 3.0.9:* $p > \frac{T^T}{(T+1)^{T+1}}$*, where* $T := \max\{\tau,\sigma\}$.
- *Theorem 4.0.3:* $p > \frac{(\tau+\sigma)^{\tau+\sigma}}{(\tau+\sigma+1)^{\tau+\sigma+1}}$.

Further, we obtain

$$\frac{\tau^\tau\sigma^\sigma}{(\tau+\sigma+1)^{\tau+\sigma+1}} < \frac{(\tau+\sigma)^\tau(\tau+\sigma)^\sigma}{(\tau+\sigma+1)^{\tau+\sigma+1}} < \begin{Bmatrix} \frac{\tau^\tau}{(\tau+1)^{\tau+1}}, & \tau\geq\sigma \\ \frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}}, & \tau\leq\sigma \end{Bmatrix}$$

or equivalently

$$\frac{\tau^\tau\sigma^\sigma}{(\tau+\sigma+1)^{\tau+\sigma+1}} < \frac{(\tau+\sigma)^{\tau+\sigma}}{(\tau+\sigma+1)^{\tau+\sigma+1}} < \frac{T^T}{(T+1)^{T+1}},$$

which shows that in the case of autonomous equations Theorem 3.0.3.2 is still better than Theorem 4.0.3.

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