DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS USING RESIDUAL METHOD

by
Volkan ÖĞER

May, 2016
İZMİR

NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS USING RESIDUAL METHOD

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by
Volkan ÖĞER

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İZMİR

Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS USING RESIDUAL METHOD**" completed by **VOLKAN ÖĞER** under supervision of **ASSIST. PROF. DR. MELTEM ADIYAMAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS USING RESIDUAL METHOD

ABSTRACT

In this thesis, Residual method is developed to approximate solutions of second order non-linear initial value problems with its singular case and non-linear boundary value problems. Application of the Residual method is based on the construction of the approximate solution using Bézier curves and determination of the unknown control points by minimizing the residual function. Advantage of this method is avoiding non-linear equations, as far as possible, while determining the control points. Error analysis of the proposed method is given and theoretical aspects compared with numerical results. Singular non-linear equations of Lane-Emden type and Troesch's problem are used to illustrate the high accuracy and implementation of the method. Numerical experiments confirm the effectiveness of the suggested approach.

Keywords: Bézier curves, Bernstein polynomials, continuous approximation, Lane-Emden equations, singular non-linear initial value problems, Troesch's problem, unstable boundary value problems.

DİFERANSİYEL DENKLEMLERİN KALINTI METODU KULLANIMI İLE SAYISAL ÇÖZÜMLERİ

ÖZ

Bu tezde, tekil olma durumlarıyla birlikte ikinci mertebeden doğrusal olmayan başlangıç değer problemlerinin ve doğrusal olmayan sınır değer problemlerinin çözümlerine yaklaşmak için kalıntı metodu geliştirildi. Kalıntı metodunun uygulanışı, Bézier eğrileri kullanılarak yaklaşım fonksiyonunun oluşturulması ve kalıntı fonksiyonunu minimize edecek kontrol noktalarının bulunmasına dayanır. Bu metodun avantajı, kontrol noktalarının bulunması sırasında doğrusal olmayan denklemlerden olabildiğince kaçınmasıdır. Metodun hata analizi yapılmış ve teorik sonuçlar sayısal sonuçlarla kıyaslanmıştır. Metodun uygulanışı ve doğruluğu, doğrusal olmayan tekil Lane-Emden tipi denklemler ve Troesch problemi ile gösterilmiştir. Sayısal deneyimler önerilen tekniğin ne kadar etkili olduğunu desteklemektedir.

Anahtar kelimeler: Bézier eğrileri, Bernstein polinomları, sürekli yaklaşım, Lane-Emden denklemleri, doğrusal olmayan tekil başlangıç değer problemleri, Troesch problemi, stabil olmayan sınır değer problemleri

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CHAPTER ONE
INTRODUCTION

In the fields of applied mathematics, physics and engineering second order initial and boundary value problems are often encountered. Most of these problems are non-linear or have singularity at any point in their domains. In this case, computation of analytic solution of problems have some difficulties. Because of this, large number of methods, such as finite difference schemes, finite element methods and Tau methods have been developed to provide accurate numerical solutions. Most of them reduce initial value problems to a non-linear system of equations. Consequently another numerical method has to be used to find approximate solution of the non-linear system of equations and the convergence of the method has to be guaranteed.

In this thesis, we consider second order non-linear initial value problems, singular non-linear initial value problems such as Lane-Emden type equations and non-linear boundary value problems such as Troesch's problem. We develop Residual method with nth degree Bézier curves to find approximate solutions of investigated problems.

First, we construct the method for second order non-linear initial value problems

$$y'' = F(x, y, y'), \quad x \in [a, b], \tag{1.1}$$

with initial conditions

$$y(a) = \alpha, \ y'(a) = \beta, \tag{1.2}$$

where $F \in C^{n-2}([a, b]) \times C(D_1) \times C(D_2)$, D_1 and D_2 are closed intervals in $\mathbb{R}$, a, b, α and β are finite constants, n is positive constant. We divide the interval $[a, b]$ into subintervals. On each subinterval, we write an initial value problem whose initial conditions are calculated from the approximate solution which obtained on previous subinterval. For the approximate solutions of these initial value problems, we construct Bézier curves with unknown control points. Our goal is to determine the control points in order to minimize the residual function. Finally, we get a

continuously differentiable piecewise approximate solution on $[a, b]$. This method provides the control of error bound by decreasing the step size and by increasing the degree of approximation function as well. Unlike the other methods, Residual method reduces the non-linear initial value problems to linear systems of equations, which give lower triangular systems with non-zero diagonals, by using the properties of Bernstein polynomials. So, most of the non-linear problems can be solved linearly using suggested technique. This creates the significant advantage of our method.

After that, we apply Residual method to Lane-Emden type equations

$$y'' + \frac{a}{x}y' + f(x, y) = g(x), \quad 0 < x \leq b, \tag{1.3}$$

with the initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{1.4}$$

where the prescribed function $g(x)$, which represents the source term, is sufficiently smooth; $f \in C^{n-1}([a, b] \times D_1)$, D_1 is a closed interval in $\mathbb{R}$ and a, b, α and β are finite constants, n is positive constant.

Lane-Emden type equations are frequently encountered in a various physical phenomena such as reactant concentration in a chemical reactor, boundary layer theory, control and optimization theory, and flow networks in biology, some areas of astrophysics, the thermal behaviour of a spherical cloud of gas, isothermal gas spheres, and the theory of thermionic currents. But the solutions of (1.3) and (1.4) neither exists, nor are so practical except for some special cases. So, improving both analytical and numerical schemes for finding the solution becomes very important. So far many methods have been developed to provide accurate numerical solutions as in Bengochea (2014), Marzban et al. (2008), Parand et al. (2010), Turkyilmazoglu (2013), Wang et al. (2014), Wazwaz (2001, 2014).

Application of Residual method to Lane-Emden type equations involves difficulty in the first subinterval because of the singularity at the initial point. Thence, we have non-linear system of equations only for the first interval while computing the

unknown control points. After finding the approximate solution on the first subinterval, the approximate solutions on the rest of subintervals are found by the same technique obtained for second order non-linear initial value problems.

Residual Method is mainly developed for initial value problems. So, by making small modifications, we adapt Residual Method to boundary value problems as

$$y''(x) = F(x, y(x), y'(x)), \tag{1.5}$$

with the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta, \tag{1.6}$$

where $F \in C^{n-2}([a,b] \times D_1 \times D_2)$, D_1 and D_2 are closed intervals in $\mathbb{R}$ and a, b, α and β are finite constants. To approximate the initial slope, without dividing the interval $[a,b]$, we apply Residual method to corresponding initial value problem with a parameter s, $s = y'(a)$, that has the same solution with the boundary value problem. After that, again we apply Residual method to corresponding initial value problem and so we have the approximate solution of the boundary value problem (1.5) and (1.6). We illustrate our technique for boundary value problems with highly sensitive Troesch's problem

$$\begin{aligned} &y'' = \lambda \sinh(\lambda y), \\ &y(0) = 0,\ y(1) = 1, \end{aligned} \tag{1.7}$$

where λ is a positive constant.

Troesch's problem arises in the investigation of the confinement of a plasma column by radiation pressure. It was initially introduced by Troesch (1960, 1976) and Weibel (1958). In Roberts & Shipman (1976), the closed form solution of Troesch's problem is given as

$$y(x) = \frac{2}{\lambda} \sinh^{-1}\left[\frac{y'(0)}{2} \mathrm{sc}(\lambda x|m)\right],$$

where $\mathrm{sc}(n|m)$ is Jacobian elliptic function, $m = 1 - \frac{1}{4}(y'(0))^2$ and satisfies the

transcendental equation

$$\text{sc}(\lambda|m)(1-m)^{\frac{1}{2}} = \sinh(\frac{\lambda}{2}).$$

Before approximating Troesch's problem, we define a new dependent variable

$$u(x) = \tanh\left(\frac{\lambda y(x)}{4}\right)$$

or equivalently

$$y(x) = \frac{4}{\lambda}\tanh^{-1}(u(x)) \tag{1.8}$$

as in Chang (2010b). After the above transformation, Troesch's problem becomes

$$\begin{aligned} &(1-u)^2u'' + 2u(u')^2 = \lambda^2 u(1+u^2), \\ &u(0) = 0,\ u(1) = \tanh\left(\tfrac{\lambda}{4}\right). \end{aligned} \tag{1.9}$$

Boundary value problem (1.9) does not contain any hyperbolic non-linear term, so it is not stiff as the original Troesch's problem.

It has been interested by so many researchers for the last 50 years and they have seen that significant difficulties appear for large values of the sensitivity parameter λ, especially in the boundary layer near $x = 1$. Thence, significant amount of numerical methods have been applied to approximate Troesch's problem such as the shooting method applied by Jones (1973), Kubiček & Hlaváček (1975), Troesch (1976), Chang (2010a), non iterative method given in Chiou & Na (1975), the invariant embedded method as in Scott (1974), a combined superposition and quasi linearization procedure given in Miele et al. (1974), Vemuri & Raefsky (1979), the inverse shooting method as in Snyman (1979), the modified homotopy perturbation technique applied by Feng et al. (2007), the finite-element method and discontinuous Galerkin methods as in Khuri & Sayfy (2011), Temimi (2012) and the sinc-Galerkin method as in Zarebnia & Sajjadian (2012). They provide accurate approximate solutions for various values of λ, but they cannot provide accurate solutions of Troesch's problem for larger $\lambda \geq 20$. Approximate solutions and their graphs of Troesch's problem are given in Temimi & Kurkcu (2014)

for large values of $\lambda = 30,\ 40,\ 50$ and 60 in the literature for the first time. But most recent approximate results of Troesch's problem and their graphs for larger values of $\lambda = 70,\ 80$ are given in this thesis.

The organization of this thesis is as follows.

In Chapter 2, Residual method is developed for second order non-linear initial value problems. Further error analysis and numerical results of the method are given. The application of the Residual method to singular non-linear equations of Lane-Emden type is discussed in Chapter 3. Error analysis and numerical results by comparing the results of other methods are also given. In Chapter 4, we present application of Residual method to boundary value problems with the error analysis. We illustrate the technique with Troesch's problem. In the conclusion, we summarize the study and present our suggestions regarding future works. Definition and properties of Bézier curves are given in Appendix A.1. Required theorems for boundary value problems are given in Appendix A.2.

CHAPTER TWO
RESIDUAL METHOD FOR SECOND ORDER NON-LINEAR INITIAL VALUE PROBLEMS

2.1 Construction of the Residual Method

In this chapter, we introduce Residual Method for second order initial value problems as

$$y''(x) = F(x, y(x), y'(x)) \tag{2.1}$$

with the initial conditions

$$y(a) = \alpha \quad y'(a) = \beta, \tag{2.2}$$

where $F \in C^{n-2}([a, b] \times D_1 \times D_2)$, D_1 and D_2 are closed intervals in $\mathbb{R}$ and α, a, b , β are finite constants.

Our method based on approximating to solution of (2.1) and (2.2) via composite Bézier curves over $[a, b]$. So, we divide the interval $[a, b]$ into equally spaced subintervals $[a_{i-1}, a_i]$, where

$$a_i = a + ih, \quad i = 0, 1, ..., N,$$

$h = (b - a)/N$, N denotes number of intervals. We define the initial value problem (2.1) and (2.2) piecewisely as

$$y_i''(x) = F(x, y_i(x), y_i'(x)), \quad x \in S_i, \tag{2.3}$$

where $S_i = [a_{i-1}, a_i]$ for $i = 1, \ldots, N$ and

$$\begin{gathered} y_1(a_0) = \alpha, \quad y_1'(a_0) = \beta, \\ y_i(a_{i-1}) = y_{i-1}(a_{i-1}), \quad y_i'(a_{i-1}) = y_{i-1}'(a_{i-1}) \end{gathered} \tag{2.4}$$

for $i = 2, \ldots, N$.

Our strategy is to use a Bézier spline curve to approximate to the solution $y_i(x)$ over the interval S_i. Let nth degree Bézier curve be

$$u_i(x) = \sum_{j=0}^{n} c_j^i B_j^n(x; S_i), \quad x \in S_i, \tag{2.5}$$

where

$$B_j^n(x; S_i) = \binom{n}{j} \frac{1}{h^n} (x - a_{i-1})^j (a_i - x)^{n-j}$$

are the Bernstein polynomials over the interval S_i and c_j^i are unknown control points. So, we have $n + 1$ unknown control points for $u_i(x)$.

The initial conditions (2.4) should be applied to the approximate solution, that is

$$u_1(a_0) = \alpha, \qquad u_1'(a_0) = \beta. \tag{2.6}$$

To have continuously differentiable approximate solution, $u_i(x)$ must satisfy the following conditions

$$u_i(a_{i-1}) = u_{i-1}(a_{i-1}), \qquad u_i'(a_{i-1}) = u_{i-1}'(a_{i-1}), \tag{2.7}$$

for $i = 2, \ldots, N$. So, we have piecewise

$$u(x) = \begin{cases} u_1(x), & \text{if } x \in S_1, \\ u_2(x), & \text{if } x \in S_2, \\ \vdots & \\ u_N(x), & \text{if } x \in S_N, \end{cases} \tag{2.8}$$

which is continuously differentiable.

Applying the end point property (A.2) and the end point derivative property (A.5) of the Bézier curves to Equations (2.6), we get first two control points of $u_1(x)$, such

that

$$\begin{aligned} c_0^1 &= \alpha, \\ c_1^1 &= \beta\frac{h}{n} + \alpha, \end{aligned} \tag{2.9}$$

and for C^1 continuity of composite Bézier curves, using (A.5) and (A.6), Equations (2.7) give

$$\begin{aligned} c_0^i &= c_n^{i-1}, \\ c_1^i &= 2c_n^{i-1} - c_{n-1}^{i-1}, \end{aligned} \tag{2.10}$$

for $i = 2, \ldots, N$. After that, we have $n-1$ unknown control points for each subinterval S_i.

Substituting (2.5) into the differential equation (2.3) for $i = 1, \ldots, N$, we have the piecewise residual function

$$R(x) = \begin{cases} R_1(x), & \text{if } x \in S_1, \\ R_2(x), & \text{if } x \in S_2, \\ \vdots & \\ R_N(x), & \text{if } x \in S_N, \end{cases}$$

where

$$R_i(x) = u_i''(x) - F(x, u_i(x), u_i'(x)), \qquad x \in S_i. \tag{2.11}$$

The goal is to determine such unknown control points c_j^i that the sufficiently differentiable residual function $R_i(x)$ becomes minimum in S_i. For minimizing residual function $R_i(x)$, we force the first $n-1$ terms in Taylor's expansion of $R_i(x)$ at $x = a_{i-1}$ be zero, that is

$$\begin{aligned} R_i(a_{i-1}) &= 0, \\ R_i'(a_{i-1}) &= 0, \\ R_i''(a_{i-1}) &= 0, \\ &\vdots \\ R_i^{(n-2)}(a_{i-1}) &= 0. \end{aligned} \tag{2.12}$$

Since

$$R_i(a_{i-1}) = u_i''(a_{i-1}) - F(a_{i-1}, u_i(a_{i-1}), u_i'(a_{i-1})),$$

using the end point derivative property of the Bézier curves (Equation (A.5)), Equations (2.9) and (2.10), we have the following linear equation

$$\frac{n(n-1)}{h^2}(c_2^i - 2c_1^i + c_0^i) - F\left(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\right) = 0. \tag{2.13}$$

Likewise, the rest of equations (2.12)

$$R_i^{(k)}(a_{i-1}) = u_i^{(k+2)}(a_{i-1}) - F^{(k)}\left(a_{i-1}, u_i(a_{i-1}), u_i'(a_{i-1})\right) = 0, \quad k = 1, \ldots, n-2,$$

where $F^{(k)}(x, y(x), z(x)) = \dfrac{\partial^k}{\partial x^k} F(x, y(x), z(x))$ becomes

$$\frac{n(n-1)\ldots(n-k-1)}{h^{k+2}}\Delta^{k+2}c_0^i - F^{(k)}\left(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\right) = 0, \tag{2.14}$$

where

$$\Delta^{k+2}c_0^i = \sum_{j=0}^{k+2} \binom{k+2}{j} (-1)^{k+2-j}\, c_j^i.$$

Using above equations, we have the following lower triangular system

$$\begin{bmatrix} 1 & 0 & \ldots & \ldots & 0 \\ & & & & \\ -3 & 1 & 0 & \ldots & 0 \\ 6 & -4 & 1 & 0 & \vdots \\ \vdots & & & \ddots & 0 \\ (-1)^{n-2}\binom{n}{n-2} & (-1)^{n-3}\binom{n}{n-3} & \ldots & (-1)\binom{n}{1} & 1 \end{bmatrix} \begin{bmatrix} c_2^i \\ c_3^i \\ c_4^i \\ \\ \vdots \\ c_n^i \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \\ \vdots \\ g_{n-1} \end{bmatrix},$$

where

$$g_j = (-1)^{j-1}c_0^i + (-1)^{j-2}(j+1)c_1^i - \frac{(n-j-1)}{n!}h^{j+1}F^{(n-2)}\left(a_{i-1}, \alpha, \frac{n}{h}(c_1^i - c_0^i)\right).$$

We solve $c_2^i, c_3^i, \ldots, c_n^i$ from above lower triangular system. So, we obtain the approximate solutions $u_i(x)$ such that $R_i(x)$ will be minimum in S_i, for $i = 1, \ldots, N$.

2.2 Error Analysis

Lemma 2.2.1. *Residual functions $R_i(x)$ are order of h^{n-1} for $i = 1, \ldots, N$.*

Proof. Writing Taylor's expansion of $R_i(x)$ at $x = a_{i-1}$, we get

$$R_i(x) = R_i(a_{i-1}) + R_i'(a_{i-1})(x - a_{i-1}) + \cdots + R_i^{(n-2)}(a_{i-1})\frac{(x - a_{i-1})^{n-2}}{(n-2)!} + R_i^{(n-1)}(\zeta_i)\frac{(x - a_{i-1})^{n-1}}{(n-1)!}$$

and using (2.12), we have

$$R_i(x) = R_i^{(n-1)}(\zeta_i)\frac{(x - a_{i-1})^{n-1}}{(n-1)!}$$

where $a_{i-1} < \zeta_i < x < a_i$. Since $|x - a_{i-1}| \leq h$,

$$|R_i(x)| \leq M_i\, h^{n-1}, \tag{2.15}$$

where

$$M_i = \frac{1}{(n-1)!} \max_{x \in [a_{i-1}, a_i]} |R_i^{(n-1)}(x)|$$

which means that $R_i(x) = O(h^{n-1})$. □

Lemma 2.2.2. *Let $\tilde{u}_i(x)$ be the nth degree auxiliary approximate solution of piecewise initial value problem* (2.3) *with initial conditions* (2.4) *as shown in Figure 2.1. Then, we have*

$$y_i^{(l)}(a_{i-1}) = \tilde{u}_i^{(l)}(a_{i-1}), \quad l = 0, 1, \ldots, n, \tag{2.16}$$

where $y_i(x)$ is the corresponding exact solution.

Proof. Since $\tilde{u}_i(x)$ is approximate solution of piecewise initial value problem (2.3) with initial conditions (2.4), i.e.

$$\tilde{u}_i(a_{i-1}) = y_{i-1}(a_{i-1}) \quad \text{and} \quad \tilde{u}_i'(a_{i-1}) = y_{i-1}'(a_{i-1}), \tag{2.17}$$

and $y_i(x)$ is the corresponding exact solution, then

$$\tilde{u}_i(a_{i-1}) = y_i(a_{i-1}) \qquad \text{and} \qquad \tilde{u}_i'(a_{i-1}) = y_i'(a_{i-1}). \tag{2.18}$$

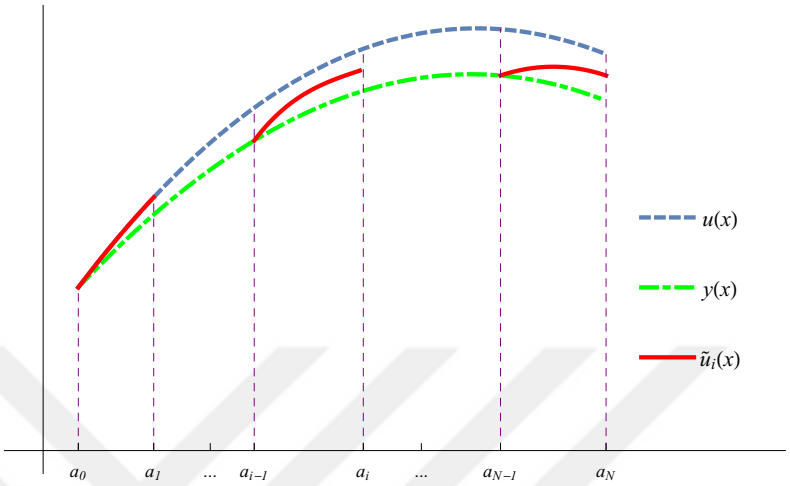


Figure 2.1 An illustration of auxiliary approximate solutions $\tilde{u}_i(x)$, approximate solution $u(x)$ and exact solution $y(x)$ of non-linear initial value problem (2.1) on $[a_0, a_N]$.

Define the residual function and its derivatives as

$$\tilde{R}_i(x) = \tilde{u}_i''(x) - F(x, \tilde{u}_i(x), \tilde{u}_i'(x)), \tag{2.19}$$

$$\tilde{R}_i^{(k)}(x) = \tilde{u}_i^{(k+2)}(x) - F^{(k)}(x, \tilde{u}_i(x), \tilde{u}_i'(x)), \tag{2.20}$$

where

$$F^{(k)}(x, y(x), z(x)) = \frac{\partial^k}{\partial x^k} F(x, y(x), z(x))$$

for $k = 1, \ldots, n-2$ and $i = 1, 2, \ldots, N$. We find the control points of $\tilde{u}_i(x)$ using

$$\tilde{R}_i^{(k)}(a_{i-1}) = 0, \tag{2.21}$$

for $k = 0, 1, \ldots, n-2$ as in (2.12). (2.20) and (2.21) yields

$$\tilde{u}_i^{(k+2)}(a_{i-1}) - F^{(k)}(a_{i-1}, \tilde{u}_i(a_{i-1}), \tilde{u}_i'(a_{i-1})) = 0.$$

Using (2.18), we have

$$\tilde{u}_i^{(k+2)}(a_{i-1}) - F^{(k)}(a_{i-1}, y_i(a_{i-1}), y_i'(a_{i-1})) = 0.$$

Since $y_i^{(k+2)}(a_{i-1}) = F^{(k)}(a_{i-1}, y_i(a_{i-1}), y_i'(a_{i-1}))$, we get

$$\tilde{u}_i^{(k+2)}(a_{i-1}) - y_i^{(k+2)}(a_{i-1}) = 0.$$

Hence

$$\tilde{u}_i^{(k+2)}(a_{i-1}) = y_i^{(k+2)}(a_{i-1}), \qquad \text{for } k = 0, 1, \ldots, n-2. \tag{2.22}$$

Therefore, (2.18) and (2.22) yield,

$$\tilde{u}_i^{(l)}(a_{i-1}) = y_i^{(l)}(a_{i-1}) \qquad \text{for } l = 0, 1, \ldots, n.$$

□

Lemma 2.2.3. *Let $\tilde{u}_i(x)$ be the nth degree auxiliary approximate solution and $y_i(x)$ be the exact solution of piecewise initial value problem* (2.3) *with initial conditions* (2.4)*, then*

$$|y_i(x) - \tilde{u}_i(x)| \leq Kh^{n+1} \tag{2.23}$$

and

$$|y_i'(x) - \tilde{u}_i'(x)| \leq (n+1)Kh^n, \tag{2.24}$$

where $x \in S_i$ and

$$K = \frac{1}{(n+1)!} \max_{x \in [a,b]} |y^{(n+1)}(x)|. \tag{2.25}$$

Proof. Writing the Taylor's expansion of the difference $y_i(x) - \tilde{u}_i(x)$ about $x = a_{i-1}$, we have

$$
\begin{aligned}
y_i(x) - \tilde{u}_i(x) =& \Big(y_i(a_{i-1}) - \tilde{u}_i(a_{i-1})\Big) \\
&+(x-a_{i-1})\Big(y_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})\Big) \\
&+\frac{(x-a_{i-1})^2}{2!}\Big(y_i''(a_{i-1}) - \tilde{u}_i''(a_{i-1})\Big) + \cdots \\
&+\frac{(x-a_{i-1})^n}{n!}\Big(y_i^{(n)}(a_{i-1}) - \tilde{u}_i^{(n)}(a_{i-1})\Big) \\
&+\frac{(x-a_{i-1})^{n+1}}{(n+1)!}\Big(y_i^{(n+1)}(\xi_{i-1}) - 0\Big),
\end{aligned}
$$

where $a_{i-1} < \xi_{i-1} < x \leq a_i$. And using (2.16), we get

$$
y_i(x) - \tilde{u}_i(x) = \frac{(x-a_{i-1})^{n+1}}{(n+1)!}\Big(y_i^{(n+1)}(\xi_{i-1})\Big),
$$

which gives (2.23).

Similarly, expanding the Taylor series of the difference $y_i'(x) - \tilde{u}_i'(x)$ at $x = a_{i-1}$ and using (2.16), we obtain the required bound in (2.24), where K is constant defined in (2.25). □

Lemma 2.2.4. *Let $\tilde{u}_i(x)$ be the nth degree auxiliary approximate solution of piecewise initial value problem* (2.3) *with initial conditions* (2.4) *and $u_i(x)$ be the approximate solution of* (2.3) *with the initial conditions* (2.6) *or* (2.7)*, then*

$$
\begin{aligned}
|u_i(x) - \tilde{u}_i(x)| \leq & |u_{i-1}(a_{i-1}) - y_{i-1}(a_{i-1})| \left(1 + \left(\frac{h^2}{2!} + \cdots + \frac{h^n}{n!}\right) C_1\right) \\
& + |u_{i-1}'(a_{i-1}) - y_{i-1}'(a_{i-1})| \left(h + \left(\frac{h^2}{2!} + \cdots + \frac{h^n}{n!}\right) C_2\right), \quad (2.26)
\end{aligned}
$$

where $x \in S_i$, and

$$
C_1 = \max_{k=0,1,\ldots,n-2} \left\{ \max_{(x,v,z)\in\Omega} |F_v^{(k)}(x,v,z)| \right\} \quad (2.27)
$$

and

$$
C_2 = \max_{k=0,1,\ldots,n-2} \left\{ \max_{(x,v,z)\in\Omega} |F_z^{(k)}(x,v,z)| \right\}, \quad (2.28)
$$

for $\Omega = \{(x,v,z)|\ x \in [a,b],\ v \in D_1,\ z \in D_2\}$.

Proof. Write Taylor's expansion of $u_i(x) - \tilde{u}_i(x)$ at $x = a_{i-1}$

$$\begin{aligned} u_i(x) - \tilde{u}_i(x) =& u_i(a_{i-1}) - \tilde{u}_i(a_{i-1}) \\ &+ (x - a_{i-1})(u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})) + \frac{(x - a_{i-1})^2}{2!}(u_i''(a_{i-1}) - \tilde{u}_i''(a_{i-1})) \\ &+ \cdots + \frac{(x - a_{i-1})^n}{n!}(u_i^{(n)}(a_{i-1}) - \tilde{u}_i^{(n)}(a_{i-1})), \end{aligned}$$

using (2.12), (2.19), (2.20) and (2.21), we have

$$\begin{aligned} u_i(x) - \tilde{u}_i(x) =& u_i(a_{i-1}) - \tilde{u}_i(a_{i-1}) \\ &+ (x - a_{i-1})(u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})) \\ &+ \frac{(x - a_{i-1})^2}{2!}\Big(F(a_{i-1}, u_i(a_{i-1}), u_i'(a_{i-1})) \\ &\qquad - F(a_{i-1}, \tilde{u}_i(a_{i-1}), \tilde{u}_i'(a_{i-1}))\Big) + \cdots \\ &+ \frac{(x - a_{i-1})^n}{n!}\Big(F^{(n-2)}(a_{i-1}, u_i(a_{i-1}), u_i'(a_{i-1})) \\ &\qquad - F^{(n-2)}(a_{i-1}, \tilde{u}_i(a_{i-1}), \tilde{u}_i'(a_{i-1}))\Big). \end{aligned}$$

Adding and subtracting the terms $F^{(j)}(a_{i-1}, u_i(a_{i-1}), \tilde{u}_i'(a_{i-1}))$, $j = 0, 1, \ldots, n-2$ and using Mean Value Theorem, we get

$$\begin{aligned} u_i(x) - \tilde{u}_i(x) =& u_i(a_{i-1}) - \tilde{u}_i(a_{i-1}) + (x - a_{i-1})(u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})) \\ &+ \frac{(x - a_{i-1})^2}{2!}\Big(F_z(a_{i-1}, u_i(a_{i-1}), \xi_i^{(0)})(u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})) \\ &\qquad + F_v(a_{i-1}, \eta_i^{(0)}, \tilde{u}_i'(a_{i-1}))(u_i(a_{i-1}) - \tilde{u}_i(a_{i-1}))\Big) + \cdots \\ &+ \frac{(x - a_{i-1})^n}{n!}\Big(F_z^{(n-2)}(a_{i-1}, u_i(a_{i-1}), \xi_i^{(n-2)})(u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})) \\ &\qquad + F_v^{(n-2)}(a_{i-1}, \eta_i^{(n-2)}, \tilde{u}_i'(a_{i-1}))(u_i(a_{i-1}) - \tilde{u}_i(a_{i-1}))\Big), \end{aligned}$$

where $\eta_i^{(j)}$ are between $u_i(a_{i-1})$ and $\tilde{u}_i(a_{i-1})$, and $\xi_i^{(j)}$ are between $u_i'(a_{i-1})$ and $\tilde{u}_i'(a_{i-1})$, $j = 0, \ldots, n-2$. Factoring out $u_i(a_{i-1}) - \tilde{u}_i(a_{i-1})$ and $u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})$, we have

$$u_i(x)-\tilde{u}_i(x) = (u_i(a_{i-1}) - \tilde{u}_i(a_{i-1})) \left(1 + \frac{(x-a_{i-1})^2}{2!} F_v(a_{i-1}, \eta_i^{(0)}, \tilde{u}_i'(a_{i-1})) + \cdots \right.$$
$$\left. + \frac{(x-a_{i-1})^n}{n!} F_v^{(n-2)}(a_{i-1}, \eta_i^{(n-2)}, \tilde{u}_i'(a_{i-1}))\right)$$
$$+(u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1}))\left((x-a_{i-1}) + \frac{(x-a_{i-1})^2}{2!} F_z(a_{i-1}, u_i(a_{i-1}), \xi_i^{(0)}) + \cdots \right.$$
$$\left. + \frac{(x-a_{i-1})^n}{n!} F_z^{(n-2)}(a_{i-1}, u_i(a_{i-1}), \xi_i^{(n-2)})\right).$$

Taking absolute value of both sides and using $|x - a_{i-1}| < h$,

$$|u_i(x) - \tilde{u}_i(x)| \leq |u_i(a_{i-1}) - \tilde{u}_i(a_{i-1})| \left(1 + \frac{h^2}{2!} |F_v(a_{i-1}, \eta_i^{(0)}, \tilde{u}_i'(a_{i-1}))| + \cdots \right.$$
$$\left. + \frac{h^n}{n!} |F_v^{(n-2)}(a_{i-1}, \eta_i^{(n-2)}, \tilde{u}_i'(a_{i-1}))|\right)$$
$$+|u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})| \left(h + \frac{h^2}{2!} |F_z(a_{i-1}, u_i(a_{i-1}), \xi_i^{(0)})| + \cdots \right.$$
$$\left. \frac{h^n}{n!} |F_z^{(n-2)}(a_{i-1}, u_i(a_{i-1}), \xi_i^{(n-2)})|\right).$$

Hence,

$$|u_i(x) - \tilde{u}_i(x)| \leq |u_i(a_{i-1}) - \tilde{u}_i(a_{i-1})| \left(1 + \left(\frac{h^2}{2!} + \cdots + \frac{h^n}{n!}\right) C_1\right)$$
$$+ |u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})| \left(h + \left(\frac{h^2}{2!} + \cdots + \frac{h^n}{n!}\right) C_2\right),$$

where constant C_1 and C_2 are defined in (2.27) and (2.28). Using initial conditions (2.4) and (2.7), we obtain

$$|u_i(x) - \tilde{u}_i(x)| \leq |u_{i-1}(a_{i-1}) - y_{i-1}(a_{i-1})| \left(1 + \left(\frac{h^2}{2!} + \cdots + \frac{h^n}{n!}\right) C_1\right)$$
$$+ |u_{i-1}'(a_{i-1}) - y_{i-1}'(a_{i-1})| \left(h + \left(\frac{h^2}{2!} + \cdots + \frac{h^n}{n!}\right) C_2\right).$$

□

Lemma 2.2.5. *Let $y_i(x)$ be the exact solution of* (2.3) *with the initial conditions* (2.4)*, $\tilde{u}_i(x)$ be the nth degree auxiliary approximate solution of piecewise initial value problem* (2.3) *with initial conditions* (2.4) *and $u_i(x)$ be the nth degree approximate solution of* (2.3) *with the initial conditions* (2.6) *or* (2.7)*, then*

$$\begin{aligned}|y_i'(a_i) - u_i'(a_i)| \leq & |y_{i-1}(a_{i-1}) - u_{i-1}(a_{i-1})| \left(h + \frac{h^2}{2!} + \cdots + \frac{h^{n-1}}{(n-1)!}\right) C_1 \\ & + |y_{i-1}'(a_{i-1}) - u_{i-1}'(a_{i-1})| \left(1 + \left(h + \frac{h^2}{2!} + \cdots + \frac{h^{n-1}}{(n-1)!}\right) C_2\right) \\ & + (n+1)Kh^n, \end{aligned} \tag{2.29}$$

where $C_1,\ C_2$ are given as in (2.27) *and* (2.28) *and K is given as in Equation* (2.25)*.*

Proof. Write Taylor's expansion of $u_i'(x) - \tilde{u}_i'(x)$ at $x = a_{i-1}$, we get

$$\begin{aligned}u_i'(x) - \tilde{u}_i'(x) = & u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1}) + (x - a_{i-1})(u_i''(a_{i-1}) - \tilde{u}_i''(a_{i-1})) + \cdots \\ & + \frac{(x - a_{i-1})^{n-1}}{(n-1)!}(u_i^{(n)}(a_{i-1}) - \tilde{u}_i^{(n)}(a_{i-1})).\end{aligned}$$

(2.12), (2.19), (2.20) and (2.21) yield

$$\begin{aligned}u_i'(x) - \tilde{u}_i'(x) = & u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1}) \\ & + (x - a_{i-1})\Big(F(a_{i-1}, u_i(a_{i-1}), u_i'(a_{i-1})) \\ & \qquad - F(a_{i-1}, \tilde{u}_i(a_{i-1}), \tilde{u}_i'(a_{i-1}))\Big) + \cdots \\ & + \frac{(x - a_{i-1})^{n-1}}{(n-1)!}\Big(F^{(n-2)}(a_{i-1}, u_i(a_{i-1}), u_i'(a_{i-1})) \\ & \qquad - F^{(n-2)}(a_{i-1}, \tilde{u}_i(a_{i-1}), \tilde{u}_i'(a_{i-1}))\Big).\end{aligned}$$

Adding and subtracting the terms $F^{(j)}(a_{i-1}, u_i(a_{i-1}), \tilde{u}_i'(a_{i-1}))$, $j = 0, 1, \ldots, n-2$ and using Mean Value Theorem, we get

$$
\begin{aligned}
u_i'(x) - \tilde{u}_i'(x) =& u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1}) \\
&+ (x - a_{i-1})\Big(F_z(a_{i-1}, u_i(a_{i-1}), \kappa_i^{(0)})(u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})) \\
&\qquad + F_v(a_{i-1}, \tau_i^{(0)}, \tilde{u}_i'(a_{i-1}))(u_i(a_{i-1}) - \tilde{u}_i(a_{i-1}))\Big) + \cdots \\
&+ \frac{(x - a_{i-1})^{n-1}}{(n-1)!}\Big(F_z^{(n-2)}(a_{i-1}, u_i(a_{i-1}), \kappa_i^{(n-2)})(u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})) \\
&\qquad + F_v^{(n-2)}(a_{i-1}, \tau_i^{(n-2)}, \tilde{u}_i'(a_{i-1}))(u_i(a_{i-1}) - \tilde{u}_i(a_{i-1}))\Big),
\end{aligned}
$$

where $\tau_i^{(j)}$ are between $u_i(a_{i-1})$ and $\tilde{u}_i(a_{i-1})$, and $\kappa_i^{(j)}$ are between $u_i'(a_{i-1})$ and $\tilde{u}_i'(a_{i-1})$, $j = 0, \ldots, n-2$. After factoring out, we get

$$
\begin{aligned}
u_i'(x) - \tilde{u}_i'(x) =& (u_i(a_{i-1}) - \tilde{u}_i(a_{i-1}))\Big((x - a_{i-1})F_v(a_{i-1}, \tau_i^{(0)}, \tilde{u}_i'(a_{i-1})) + \cdots \\
&\qquad + \frac{(x - a_{i-1})^{n-1}}{(n-1)!}F_v^{(n-2)}(a_{i-1}, \tau_i^{(n-2)}, \tilde{u}_i'(a_{i-1}))\Big) \\
&+ (u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1}))\Big(1 + (x - a_{i-1})F_z(a_{i-1}, u_i(a_{i-1}), \kappa_i^{(0)}) + \cdots \\
&\qquad + \frac{(x - a_{i-1})^{n-1}}{(n-1)!}F_z^{(n-2)}(a_{i-1}, u_i(a_{i-1}), \kappa_i^{(n-2)})\Big).
\end{aligned}
$$

Taking absolute value of both sides and using $|x - a_{i-1}| < h$, we have

$$
\begin{aligned}
|u_i'(x) - \tilde{u}_i'(x)| \leq& |u_i(a_{i-1}) - \tilde{u}_i(a_{i-1})|\Big(h|F_v(a_{i-1}, \tau_i^{(0)}, \tilde{u}_i'(a_{i-1}))| + \cdots \\
&\qquad + \frac{h^{n-1}}{(n-1)!}|F_v^{(n-2)}(a_{i-1}, \tau_i^{(n-2)}, \tilde{u}_i'(a_{i-1}))|\Big) \\
&+ |u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})|\Big(1 + h|F_z(a_{i-1}, u_i(a_{i-1}), \kappa_i^{(0)})| + \cdots \\
&\qquad + \frac{h^{n-1}}{(n-1)!}|F_z^{(n-2)}(a_{i-1}, u_i(a_{i-1}), \kappa_i^{(n-2)})|\Big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|u_i'(x) - \tilde{u}_i'(x)| \leq& |u_i(a_{i-1}) - \tilde{u}_i(a_{i-1})|\left(h + \frac{h^2}{2!} + \cdots + \frac{h^{n-1}}{(n-1)!}\right)C_1 \\
&+ |u_i'(a_{i-1}) - \tilde{u}_i'(a_{i-1})|\left(1 + \left(h + \frac{h^2}{2!} + \cdots + \frac{h^{n-1}}{(n-1)!}\right)C_2\right),
\end{aligned}
$$

where constant C_1 and C_2 are defined in (2.27) and (2.28). Using initial conditions

(2.4) and (2.7), we obtain

$$|u_i'(a_i) - \tilde{u}_i'(a_i)| \leq |u_{i-1}(a_{i-1}) - y_{i-1}(a_{i-1})| \left(h + \frac{h^2}{2!} + \cdots + \frac{h^{n-1}}{(n-1)!}\right) C_1 + |u_{i-1}'(a_{i-1}) - y_{i-1}'(a_{i-1})| \left(1 + \left(h + \frac{h^2}{2!} + \cdots + \frac{h^{n-1}}{(n-1)!}\right) C_2\right). \tag{2.30}$$

Since

$$|y_i'(a_i) - u_i'(a_i)| \leq |y_i'(a_i) - \tilde{u}_i'(a_i)| + |\tilde{u}_i'(a_i) - u_i'(a_i)|,$$

by (2.24) and (2.30), we have

$$\begin{aligned} |y_i'(a_i) - u_i'(a_i)| \leq & |y_{i-1}(a_{i-1}) - u_{i-1}(a_{i-1})| \left(h + \frac{h^2}{2!} + \cdots + \frac{h^{n-1}}{(n-1)!}\right) C_1 \\ & + |y_{i-1}'(a_{i-1}) - u_{i-1}'(a_{i-1})| \left(1 + \left(h + \frac{h^2}{2!} + \cdots + \frac{h^{n-1}}{(n-1)!}\right) C_2\right) \\ & + (n+1) K h^n. \end{aligned}$$

□

Theorem 2.2.6. *Let $y(x)$ be the exact solution of the second order non-linear initial value problem* (2.1) *and $u(x)$ be the corresponding nth degree approximate function* (2.8)*. Then, we have the inequality*

$$|y(x) - u(x)| \leq M h^{n-1}, \qquad x \in [a, b], \tag{2.31}$$

where $M = K(b-a)^2(n+1)$ and K is defined in (2.25)*.*

Proof. We will use mathematical induction to prove the following inequalities which will be useful while proving the theorem. First, we must show that for any $j = 2, 3, \ldots, N$

$$|y_j(x) - u_j(x)| \leq K h^{n+1} \mu_j + O(h^{n+2}) \tag{2.32}$$

and

$$|y_j'(a_j) - u_j'(a_j)| \leq j(n+1) K h^n + O(h^{n+1}), \tag{2.33}$$

where

$$\mu_j = 1 + \mu_{j-1} + (j-1)(n+1), \qquad \mu_1 = 1.$$

Using Inequality (2.23) and Inequality (2.26), we have

$$\begin{aligned}
|y_2(x) - u_2(x)| \leq & |y_2(x) - \tilde{u}_2(x)| + |\tilde{u}_2(x) - u_2(x)| \\
\leq & Kh^{n+1} + |\tilde{u}_2(x) - u_2(x)| \\
\leq & Kh^{n+1} + |u_1(a_1) - y_1(a_1)|(1 + O(h^2)) \\
& + |u_1'(a_1) - y_1'(a_1)|(h + O(h^2)),
\end{aligned}$$

where K as in (2.25). Since $u_1(x) = \tilde{u}_1(x)$ in S_1, last expression can be written as

$$\begin{aligned}
|y_2(x) - u_2(x)| \leq & Kh^{n+1} + |\tilde{u}_1(a_1) - y_1(a_1)|(1 + O(h^2)) \\
& + |\tilde{u}_1'(a_1) - y_1'(a_1)|(h + O(h^2)).
\end{aligned}$$

Using (2.23) and (2.24), we get

$$\begin{aligned}
|y_2(x) - u_2(x)| \leq & Kh^{n+1} + Kh^{n+1}(1 + O(h^2)) \\
& + (n+1)Kh^n(h + O(h^2)) \\
\leq & Kh^{n+1}\Big(1 + 1 + (n+1)\Big) + O(h^{n+2}).
\end{aligned} \tag{2.34}$$

(2.29) yields

$$\begin{aligned}
|y_2'(a_2) - u_2'(a_2)| \leq & |y_1(a_1) - u_1(a_1)|O(h) + |y_1'(a_1) - u_1'(a_1)|(1 + O(h)) \\
& + (n+1)Kh^n,
\end{aligned}$$

where K is defined as in (2.25). Since $u_1(x) = \tilde{u}_1(x)$, using (2.23) and (2.24), we obtain

$$\begin{aligned}
|y_2'(a_2) - u_2'(a_2)| \leq & Kh^{n+1}O(h) + (n+1)Kh^n(1 + O(h)) + (n+1)Kh^n \\
\leq & 2(n+1)Kh^n + O(h^{n+1}),
\end{aligned}$$

which means that Inequalities (2.32) and (2.33) are viable for $j = 2$.

Assume that inequalities (2.32) and (2.33) are true for $j-1$, i.e.

$$|y_{j-1}(x) - u_{j-1}(x)| \leq Kh^{n+1}\mu_{j-1} + O(h^{n+2})$$

and

$$|y'_{j-1}(a_{j-1}) - u'_{j-1}(a_{j-1})| \leq (j-1)(n+1)Kh^n + O(h^{n+1}),$$

where

$$\mu_j = 1 + \mu_{j-1} + (j-1)(n+1), \qquad \mu_1 = 1.$$

Now we will prove that they are also true for j. To prove (2.32), we use

$$|y_j(x) - u_j(x)| \leq |y_j(x) - \tilde{u}_j(x)| + |u_j(x) - \tilde{u}_j(x)|, \quad \text{for } j = 3, \ldots, N.$$

Using (2.23) and (2.26), we get

$$\begin{aligned} |y_j(x) - u_j(x)| \leq & Kh^{n+1} + |y_{j-1}(a_{j-1}) - u_{j-1}(a_{j-1})|(1 + O(h^2)) \\ & + |y'_{j-1}(a_{j-1}) - u'_{j-1}(a_{j-1})|(h + O(h^2)) \end{aligned} \tag{2.35}$$

and using assumptions, we have

$$\begin{aligned} |y_j(x) - u_j(x)| \leq & Kh^{n+1} + \Big(Kh^{n+1}\mu_{j-1} + O(h^{n+2})\Big)(1 + O(h^2)) \\ & + \Big((j-1)(n+1)Kh^n + O(h^{n+1})\Big)(h + O(h^2)). \end{aligned}$$

Rearranging rigth side of above inequality gives

$$\begin{aligned} |y_j(x) - u_j(x)| \leq & Kh^{n+1} + Kh^{n+1}\mu_{j-1} + (j-1)(n+1)Kh^{n+1} + O(h^{n+2}) \\ \leq & Kh^{n+1}\Big(1 + \mu_{j-1} + (j-1)(n+1)\Big) + O(h^{n+2}) \\ \leq & Kh^{n+1}\mu_j + O(h^{n+2}). \end{aligned}$$

Using inequality (2.29) and assumptions, we have

$$\begin{aligned}
|y_j'(a_j) - u_j'(a_j)| \leq & |y_{j-1}(a_{j-1}) - u_{j-1}(a_{j-1})|O(h) \\
& + |y_{j-1}'(a_{j-1}) - u_{j-1}'(a_{j-1})|(1 + O(h)) + (n+1)Kh^n \\
\leq & \Big(Kh^{n+1}\mu_{j-1} + O(h^{n+2})\Big)O(h) + ((j-1)(n+1)Kh^n \\
& + O(h^{n+1}))(1 + O(h)) + (n+1)Kh^n \\
\leq & j(n+1)Kh^n + O(h^{n+1}).
\end{aligned}$$

We have proved that, for $j = 2, 3, \ldots, N$

$$|y_j(x) - u_j(x)| \leq Kh^{n+1}\mu_j + O(h^{n+2})$$

and

$$|y_j'(a_j) - u_j'(a_j)| \leq j(n+1)Kh^n + O(h^{n+1}),$$

where

$$\mu_j = 1 + \mu_{j-1} + (j-1)(n+1), \qquad \mu_1 = 1.$$

We have first order linear difference equation on μ_j, and its solution can be found as

$$\mu_j = \Big(\frac{1-n}{2}\Big)j + \Big(\frac{1+n}{2}\Big)j^2.$$

Hence, for sufficiently small h, we get

$$\begin{aligned}
|y_N(x) - u_N(x)| \leq & Kh^{n+1}\mu_N + O(h^{n+2}) \\
\leq & Kh^{n+1}\Big(\Big(\frac{1-n}{2}\Big)N + \Big(\frac{1+n}{2}\Big)N^2\Big) + O(h^{n+2}) \\
\leq & Kh^{n+1}N^2\Big(\Big(\frac{1-n}{2}\Big)\frac{1}{N} + \Big(\frac{1+n}{2}\Big)\Big) + O(h^{n+2}) \\
\leq & Kh^{n+1}\Big(\frac{b-a}{h}\Big)^2\Big(\Big(\frac{1-n}{2}\Big)\frac{h}{b-a} + \Big(\frac{1+n}{2}\Big)\Big) + O(h^{n+2}) \\
\leq & Kh^{n-1}(b-a)^2\Big(\frac{1+n}{2}\Big) + O(h^n) \\
\leq & Kh^{n-1}(b-a)^2(n+1) \\
\leq & Mh^{n-1},
\end{aligned}$$

where $M = K(b-a)^2(n+1)$. Hence we have showed that

$$|y_N(x) - u_N(x)| = O(h^{n-1}).$$

□

2.3 Algorithm of Method

The algorithm of the method can be written as

S 1 Set $N, h = 1/N$ and n.

S 2 Compute $a_i = a + ih$ for $i = 0, 1, 2, \ldots, N$.

S 3 Define the functions

$$f_k(x, y, z) := F^{(k)}(x, y, z), \quad k = 0, 1, \ldots, n-2.$$

S 4 Set $c_0^1 = \alpha$, $c_1^1 = \beta\frac{h}{n} + \alpha$.

S 5 For $i = 1$ to $i = N$ do the followings

- Solve c_k^i, $k = 2, 3, \ldots, n$ from following linear equations

$$\frac{n(n-1)\ldots(n-k+1)}{h^k}\Delta^k c_0^i - f_{k-2}\left(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\right) = 0,$$

  where Δ is difference operator.

- Increase i.
- Compute $c_0^i = c_n^{i-1}$ and $c_1^i = 2c_n^{i-1} - c_{n-1}^{i-1}$.

S 6 Using computed control points c_j^i for $j = 0, 1, \ldots, n$ and $i = 1, 2, \ldots, N$, write

the piecewise approximate solution as

$$u(x) = \begin{cases} u_1(x), & x \in [a_0, a_1], \\ u_2(x), & x \in [a_1, a_2], \\ \quad \vdots & \\ u_N(x), & x \in [a_{N-1}, a_N], \end{cases}$$

where

$$u_i(x) = \sum_{j=0}^{n} c_j^i B_j^n(x; S_i),$$

and

$$B_j^n(x; S_i) = \binom{n}{j} \frac{1}{h^n} (x - a_{i-1})^j (a_i - x)^{n-j},$$

$S_i = [a_{i-1}, a_i]$.

2.4 Numerical Examples

In this section, we have solved the following three second order non-linear initial value problems to validate the proposed method.

Define error function $E_N(x)$ and maximum error e_N as

$$E_N(x) = |u(x; N) - y(x)|, \qquad x \in [a, b],$$

and

$$e_N = \max_{x \in [a,b]} E_N(x),$$

respectively, where $y(x)$ is the exact solution of (2.1) and (2.2), $u(x; N)$ is the corresponding approximate solution obtained by N subintervals.

In Table 2.2, Table 2.4 and Table 2.6, the observed orders are computed using

$$\text{ord}(N) = \frac{\log(\frac{e_N}{e_{2N}})}{\log 2}, \tag{2.36}$$

and we see that observed orders are well confirm the theoretical results.

Example 2.4.1. *Consider the problem*

$$y'' = (y')^2 - 2yy', \qquad x \in [0, 1],$$

$$y(0) = 0, \qquad y'(0) = 2,$$

with the exact solution $y(x) = e^{2x} - 1$. *Table 2.1 gives maximum errors and Table 2.2 gives observed orders for* $n = 3, 4, 5$ *and* $N = 20, 40, 80, 160$. *Graphs of error functions are shown in Figure 2.2.*

Table 2.1 Comparison of the maximum errors of Example 2.4.1, for $N = 20, 40, 80, 160$ and $n = 3, 4, 5$.

N	$n = 3$	$n = 4$	$n = 5$
20	$1.38(-1)$	$4.11(-3)$	$8.41(-5)$
40	$4.26(-2)$	$5.67(-4)$	$5.69(-6)$
80	$1.16(-2)$	$7.39(-5)$	$3.69(-7)$
160	$2.99(-3)$	$9.41(-6)$	$2.32(-8)$

Table 2.2 Comparison observed orders of Example 2.4.1, for $N = 20, 40, 80, 160$ and $n = 3, 4, 5$.

N	$n = 3$	$n = 4$	$n = 5$
20/40	1.69	2.86	3.88
40/80	1.88	2.94	3.94
80/160	1.95	2.97	3.99

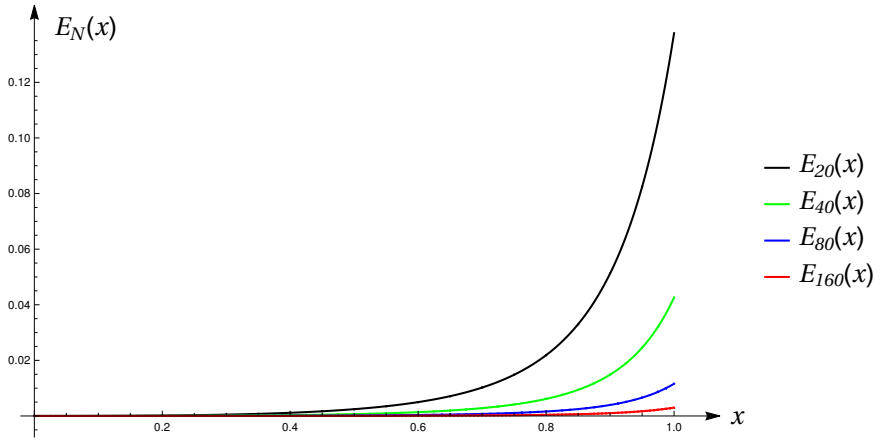


Figure 2.2 Graphs of error functions $E_{20}(x)$, $E_{40}(x)$, $E_{80}(x)$ and $E_{160}(x)$ of Example 2.4.1 obtained using 3rd degree approximate function.

Example 2.4.2. *Consider the problem*

$$y'' = (y')^2 - y - 3\sin(2x) - 4\cos^2(2x), \qquad x \in [0, 1],$$

$$y(0) = 0, \qquad y'(0) = 2,$$

with the exact solution $y(x) = \sin(2x)$. *Graphs of error functions are shown in Figure 2.3. Table 2.3 gives maximum errors and Table 2.4 gives observed orders for* $n = 3,\ 4,\ 5$ *and* $N = 20,\ 40,\ 80,\ 160$.

Table 2.3 Comparison of the maximum errors of Example 2.4.2, for $N = 20,\ 40,\ 80,\ 160$ and $n = 3,\ 4,\ 5$.

N	$n = 3$	$n = 4$	$n = 5$
20	$2.45(-3)$	$1.32(-4)$	$1.19(-6)$
40	$6.53(-4)$	$1.72(-5)$	$8.05(-8)$
80	$1.69(-4)$	$2.20(-6)$	$5.24(-9)$
160	$4.29(-5)$	$2.78(-7)$	$3.53(-10)$

Table 2.4 Comparison of observed orders of Example 2.4.2, for $N = 20,\ 40,\ 80,\ 160$ and $n = 3,\ 4,\ 5$.

N	$n = 3$	$n = 4$	$n = 5$
20/40	1.91	2.94	3.89
40/80	1.95	2.97	3.94
80/160	1.98	2.99	3.89

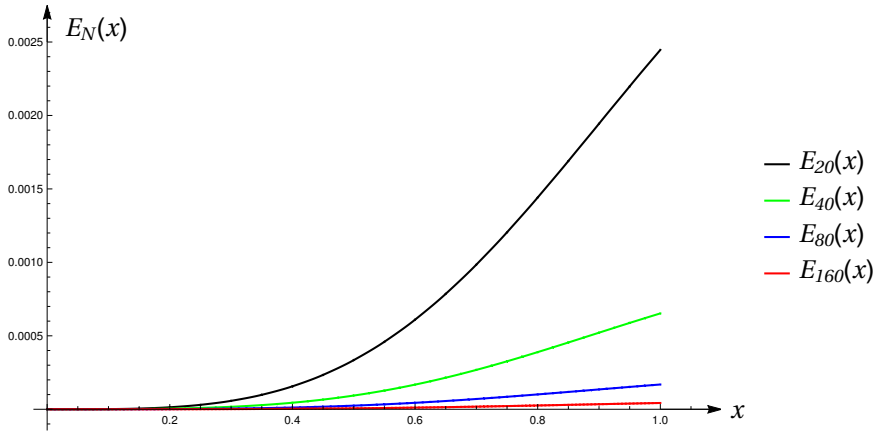


Figure 2.3 Graphs of error functions $E_{20}(x)$, $E_{40}(x)$, $E_{80}(x)$ and $E_{160}(x)$ of Example 2.4.2 obtained using 3rd degree approximate function.

Example 2.4.3. *Consider the problem*

$$y'' = -(y')^2 - e^y + 1 + x, \qquad x \in [0, 1],$$

$$y(0) = 0, \qquad y'(0) = 1,$$

with the exact solution $y(x) = \ln(x + 1)$. *Graphs of error functions are shown in Figure 2.4. Table 2.5 gives maximum errors and Table 2.6 observed orders for* $n = 3, 4, 5$ *and* $N = 20, 40, 80, 160$.

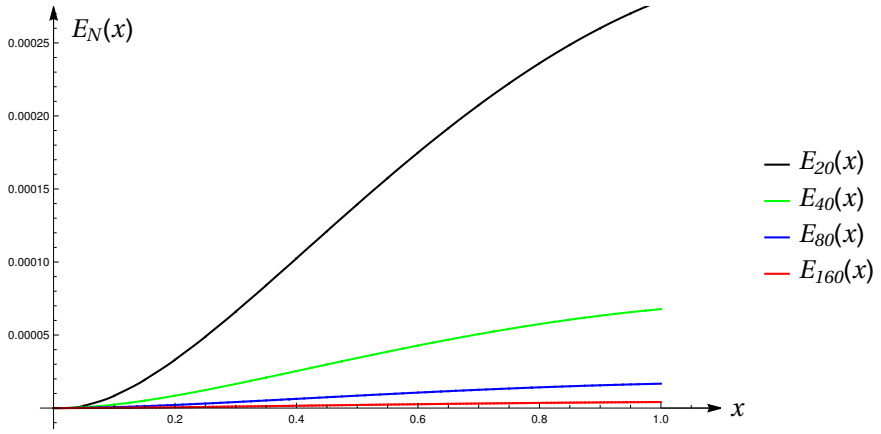


Figure 2.4 Graphs of error functions $E_{20}(x)$, $E_{40}(x)$, $E_{80}(x)$ and $E_{160}(x)$ of Example 2.4.3 obtained using 3rd degree approximate function.

Table 2.5 Comparison of the maximum errors of Example 2.4.3, for $N = 20, 40, 80, 160$ and $n = 3, 4, 5$.

N	$n = 3$	$n = 4$	$n = 5$
20	$2.79(-4)$	$1.17(-5)$	$5.04(-7)$
40	$6.78(-5)$	$1.41(-6)$	$3.00(-8)$
80	$1.67(-5)$	$1.73(-7)$	$1.83(-9)$
160	$4.14(-6)$	$2.14(-8)$	$1.16(-10)$

Table 2.6 Comparison of observed orders of Example 2.4.3, for $N = 20, 40, 80, 160$ and $n = 3, 4, 5$.

N	$n = 3$	$n = 4$	$n = 5$
20/40	2.04	3.05	4.07
40/80	2.02	3.03	4.04
80/160	2.01	3.01	3.98

CHAPTER THREE
RESIDUAL METHOD FOR SINGULAR NON-LINEAR EQUATIONS OF LANE-EMDEN TYPE

3.1 Lane-Emden Equation

We consider the following singular non-linear Lane-Emden equation

$$\begin{aligned} y'' + \frac{a}{x}y' + f(x,y) = g(x), \quad 0 < x \leq b, \\ y(0) = \alpha, \quad y'(0) = \beta, \end{aligned} \tag{3.1}$$

where the prescribed function $g(x)$, which represents the source term, is sufficiently smooth; $f \in C^{n-1}([a,b] \times D_1)$, D_1 is a closed interval in $\mathbb{R}$ and a, b, α, β are finite constants, n is positive constant.

3.2 Application of Residual Method to Lane-Emden Equation

First, we define the initial value problem on each subinterval $S_i = [a_{i-1}, a_i]$, where $a_i = a_0 + ih$, $i = 0, 1, \ldots, N$, $a_0 = 0$, $h = 1/N$, N denotes number of subintervals, as in previous Chapter, such that,

$$y_i''(x) = F(x, y_i(x), y_i'(x)), \quad x \in S_i, \tag{3.2}$$

for $1 \leq i \leq N$, where

$$F(x, y_i(x), y_i'(x)) = -\frac{a}{x}y_i'(x) - f(x, y_i(x)) + g(x),$$

and

$$\begin{aligned} y_1(0) = \alpha, \quad y_1'(0) = \beta, \\ y_i(a_{i-1}) = y_{i-1}(a_{i-1}), \quad y_i'(a_{i-1}) = y_{i-1}'(a_{i-1}), \end{aligned} \tag{3.3}$$

for $2 \leq i \leq N$.

We have to change our technique only for the first interval, since the residual function

$$R_1(x) = u_1''(x) + \frac{a}{x}u_1'(x) + f(x, u_1(x)) - g(x), \quad x \in S_1$$

and its derivatives are not defined at $a_0 = 0$. So, we choose a point $\eta \in (0, a_1]$ to force the residual function and its derivatives be 0, i.e.,

$$R_1^{(k)}(\eta) = 0, \quad \text{for} \quad k = 0, \ldots, n-2. \tag{3.4}$$

In this case, we have some non-linear equations to find the unknown control points for the first interval, since we cannot use the end point derivative property of the Bézier curves.(Equations (A.5) and (A.6)) But this situation does not affect the order of convergence (this will be proved in Section 3.3). We apply the Newton's method to solve the non-linear equations approximately and for the initial values, we use the initial conditions (3.1).

After finding the unknown control points on the interval S_1, we continue our procedure as described in Chapter 2. First, we find c_0^i and c_1^i from continuity of approximate function, then to find the unknown control points, we force the residual function and its $n-2$ derivatives be 0 at $x = a_{i-1}$ on S_i, $2 \leq i \leq N$, i.e.,

$$R_i^{(k)}(a_{i-1}) = 0, \tag{3.5}$$

for $k = 0, \ldots, n-2$, where

$$R_i(x) = u_i''(x) + \frac{a}{x}u_i'(x) + f(x, u_i(x)) - g(x). \tag{3.6}$$

In this case, we have $n-1$ linear equations to find rest of the unknown control points as well, but for only the first interval S_1, we have $n-1$ non-linear equations because of the singularity of the residual function at a_0.

3.3 Error Analysis

Lemma 3.3.1. *Let $\tilde{u}_1(x)$ be the auxiliary approximate solution of piecewise initial value problem* (3.2) *with initial conditions* (3.3)*. Then, we have*

$$|y_1(x) - u_1(x)| \leq KLh^{n+1} + O(h^{n+2}), \quad \forall x \in S_1 \tag{3.7}$$

$$|y_1'(x) - u_1'(x)| \leq K\bar{L}h^{n} + O(h^{n+1}), \quad \forall x \in S_1, \tag{3.8}$$

where $y_1(x)$ is the corresponding exact solution,

$$K = \frac{1}{(n+1)!} \max_{x\in(a,b]} |y^{(n+1)}(x)|, \qquad L = 2^{n+1} - n - 2 \quad \textit{and} \quad \bar{L} = (n+1)(2^n - 1).$$

Proof. This lemma can be proved using (3.5) and Taylor's expansion of $u_1(x) - y_1(x)$ and $u_1'(x) - y_1'(x)$ at $x = a_0$, and $u_1^{(k)}(x) - y_1^{(k)}(x)$ at $x = \eta$. □

The following corollary can be proved in same manner of proving Theorem 2.2.6, using the inequalities (3.7) and (3.8).

Corollary 3.3.2. *Let $y(x)$ be the exact solution of the Lane-Emden type equation* (3.1) *and $u(x)$ be the corresponding nth degree approximate function. Then, we have*

$$|y(x) - u(x)| \leq Mh^{n-1}, \qquad x \in [0, b],$$

where M is a constant, which does not depend on h.

3.4 Algorithm of Method

The algorithm of the modified method can be written as

S 1 Set $N, h = 1/N$ and n.

S 2 Compute $a_i = a + ih$ for $i = 0, 1, 2, \ldots, N$.

S 3 Define

$$F(x,y,z) := -\frac{\tilde{a}}{x}z - f(x,y) + g(x)$$

$$u_i(x) := \sum_{j=0}^{n} c_j^i B_j^n(x;S_i)$$

$$B_j^n(x;S_i) := \binom{n}{j}\frac{1}{h^n}(x-a_{i-1})^j(a_i-x)^{n-j}$$

S 4 Define the functions

$$f_k(x,y,z) := \frac{\partial^k}{\partial x^k}F(x,y(x),z(x)), \quad k=0,1,\ldots,n-2.$$

S 5 Set $c_0^1 = \alpha$, $c_1^1 = \beta\frac{h}{n} + \alpha$.

S 6 Set η from $(a_0, h]$

- Solve c_k^1, $k = 2,3,\ldots,n$ from following non-linear equations

$$\frac{n(n-1)\ldots(n-k+1)}{h^k}\Delta^k c_0^1 - f_{k-2}\left(\eta, u_1(\eta), u_1'(\eta)\right) = 0,$$

  where Δ is difference operator.

*S*7 For $i = 2$ to $i = N$ do the followings

- Compute $c_0^i = c_n^{i-1}$ and $c_1^i = 2c_n^{i-1} - c_{n-1}^{i-1}$.
- Solve c_k^i, $k = 2,3,\ldots,n$ from following linear equations

$$\frac{n(n-1)\ldots(n-k+1)}{h^k}\Delta^k c_0^i - f_{k-2}\left(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\right) = 0,$$

  where Δ is difference operator.

- Increase i.

S 8 Using computed control points c_j^i for $j = 0, 1, \ldots, n$ and $i = 1, 2, \ldots, N$, write the piecewise approximate solution as

$$u(x) = \begin{cases} u_1(x), & x \in [a_0, a_1], \\ u_2(x), & x \in [a_1, a_2], \\ \quad \vdots & \\ u_N(x), & x \in [a_{N-1}, a_N]. \end{cases}$$

3.5 Numerical Examples

In this section, we investigate the following numerical examples to illustrate the applicability of the proposed method. The tables and figures demonstrate the power of the current study.

Example 3.5.1. *Consider the Lane-Emden equation in Wang et al. (2014)*

$$\begin{gathered} y'' + \frac{2}{x}y' + y^5 = 0, \quad \textit{for} \quad 0 < x \leq 1, \\ y(0) = 1, \qquad y'(0) = 0 \end{gathered} \tag{3.9}$$

with the exact solution $y(x) = (1 + x^2/3)^{-1/2}$.

Table 3.1 gives a comparison of numerical results with the results of the reproducing kernel method (RKM) given in Wang et al. (2014) and we observe that the accuracy obtained is high enough. In Table 3.2, we give the observed orders obtained using (2.36). It can be seen that observed orders are well to confirm the theoretical results. Figure 3.1 shows the error functions for $N = 50$, 100, 200.

Table 3.1 Comparison of the errors of the present method with RKM errors.

x	Exact Values	RKM ($N = 50$) (in Wang[1])	Presented Method $n = 3$ $N = 50$	$n = 4$ $N = 50$	$n = 5$ $N = 50$	$n = 5$ $N = 200$
0.	1.	0.	0.	0.	0.	0.
0.1	0.998337	5.115(-7)	2.243(-7)	1.762(-10)	3.194(-11)	1.893(-13)
0.2	0.993399	7.322(-7)	6.302(-7)	1.633(-9)	8.934(-11)	6.780(-13)
0.3	0.985329	7.218(-7)	1.144(-6)	5.405(-9)	1.596(-10)	8.862(-13)
0.4	0.974355	2.963(-7)	1.711(-6)	1.199(-8)	2.250(-10)	1.556(-12)
0.5	0.960769	1.077(-6)	2.248(-6)	2.127(-8)	2.660(-10)	1.960(-12)
0.6	0.944911	8.175(-7)	2.675(-6)	3.259(-8)	2.682(-10)	1.169(-12)
0.7	0.927146	1.459(-6)	2.927(-6)	4.496(-8)	2.253(-10)	3.251(-13)
0.8	0.907841	1.701(-6)	2.960(-6)	5.721(-8)	1.390(-10)	5.829(-13)
0.9	0.887357	2.491(-6)	2.757(-6)	6.828(-8)	1.770(-11)	6.768(-13)
1.	0.866025	3.596(-6)	2.324(-6)	7.732(-8)	1.265(-10)	9.200(-13)

Table 3.2 Observed orders of Example 3.5.1 with $n = 3, 4, 5$ and $N = 25, 50, 100, 200$.

N	$n = 3$	$n = 4$	$n = 5$
25/50	2.281	3.028	4.339
50/100	2.119	3.017	4.176
100/200	2.049	3.010	3.278

[1]Wang et al. (2014)

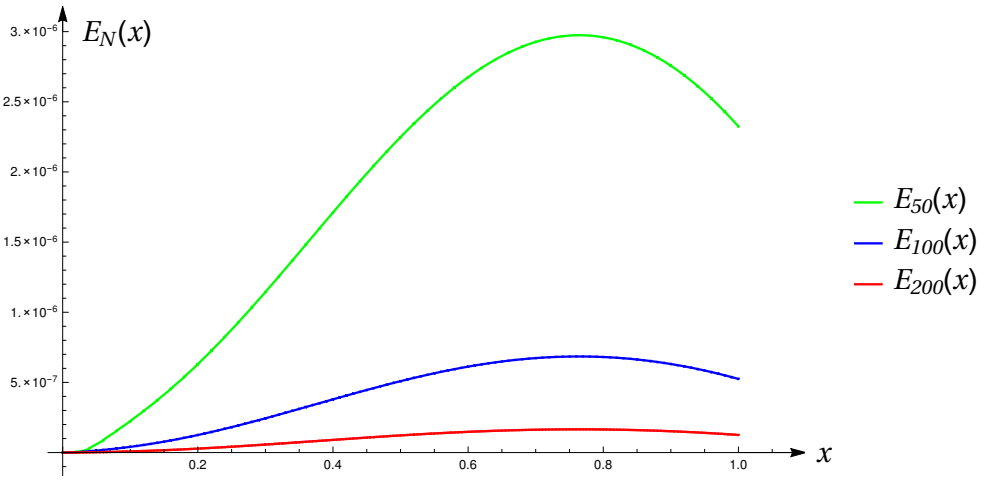


Figure 3.1 Graphs of Error functions of Example 3.5.1 obtained using 3rd degree approximate function with number of intervals $N = 50, 100, 200$.

Example 3.5.2. *Consider the Lane-Emden equation in Parand et al. (2010) and Wang et al. (2014)*

$$\begin{aligned} y'' + \frac{2}{x}y' + 4(2e^y + e^{y/2}) = 0, \quad \textit{for } x \geq 0 \\ y(0) = 0, \qquad y'(0) = 0, \end{aligned} \tag{3.10}$$

with the exact solution $y(x) = -2\ln(1 + x^2)$.

Table 3.4 gives a comparison of numerical results of Residual method with the results of Hermite functions collocation method (HFC) and RKM given in Parand et al. (2010) and Wang et al. (2014), respectively. In Table 3.3, observed orders obtained using (2.36) are given.

Table 3.3 Observed orders of Example 3.5.2 with $n = 3, 4, 5$ and $N = 25, 50, 100, 200$.

N	$n = 3$	$n = 4$	$n = 5$
25/50	1.900	3.000	2.264
50/100	2.093	3.033	3.903
100/200	2.061	3.028	4.022

Table 3.4 Comparison of the errors of the present method with RKM and HFC errors.

x	Exact Values	RKM[2] ($N = 50$)	HFC[3]	Presented Method $N = 50$, $n = 5$
0	0.	0	0	0.
0.01	-0.00019999	6.666(-13)	2.931(-6)	1.886(-13)
0.1	-0.0199007	3.829(-8)	3.939(-6)	2.224(-7)
0.5	-0.446287	1.026(-7)	3.018(-6)	4.755(-6)
1	-1.38629	4.361(-6)	9.314(-7)	2.632(-6)
2	-3.21888	4.175(-6)	4.999(-7)	5.099(-6)
3	-4.60517	1.860(-7)	8.104(-7)	2.257(-6)
4	-5.66643	3.312(-6)	7.692(-7)	1.967(-7)

Example 3.5.3. *Consider the Lane-Emden equation in Parand et al. (2010)*

$$\begin{aligned} y'' + \frac{2}{x}y' + \sin y = 0, \quad \textit{for } x \geq 0, \\ y(0) = 1, \qquad y'(0) = 0, \end{aligned} \tag{3.11}$$

with the asymptotic solution given in Wazwaz (2001)

$$\begin{aligned} y(x) \simeq \; & 1 - \frac{1}{6}k_1 x^2 + \frac{1}{120}k_1 k_2 x^4 \\ & + k_1\left(\tfrac{1}{3024}k_1^2 - \tfrac{1}{5040}k_2^2\right)x^6 + k_1 k_2\left(-\frac{113}{3265920}k_1^2 + \frac{1}{362880}k_2^2\right)x^8 \\ & + k_1\left(\frac{1781}{898128000}k_1^2 k_2^2 - \frac{1}{399168000}k_2^4 - \frac{19}{23950080}k_1^4\right)x^{10}, \end{aligned}$$

where $k_1 = \sin(1)$ *and* $k_2 = \cos(1)$.

Table 3.5 gives some values of the asymptotic solution given in Wazwaz (2001) and comparison of the errors of the presented method with errors of HFC given in Parand et al. (2010).

[2]Wang et al. (2014)
[3]Parand et al. (2010)

Table 3.5 Asymptotic solution and comparison of the errors of the present method with HFC errors.

x	Wazwaz[4]	Parand[5]	Presented Method
0	1.	0	0.
0.1	0.998598	7.207(-6)	8.427(-9)
0.2	0.994396	9.997(-6)	8.446(-9)
0.5	0.965178	1.038(-5)	8.538(-9)
1	0.863681	7.027(-6)	2.267(-8)
1.5	0.705042	1.049(-5)	3.309(-6)
2	0.506372	9.672(-5)	9.156(-6)

In Table 3.6, we give orders using

$$\text{ord}(N/2N) = \frac{\log(\frac{\max|u(x;N)-u(x;2N)|}{\max|u(x;2N)-u(x;4N)|})}{\log(2)},$$

where $u(x;N),\ u(x;2N),\ u(x;4N)$ are approximate solutions of $y(x)$ obtained by $N,\ 2N,\ 4N$ subintervals, respectively. From Table 3.2, Table 3.3 and Table 3.6, it is concluded that observed orders are equal to $n-1$ as proved in Corollary 3.3.2.

Table 3.6 Observed orders of Example 3.5.3 with $n = 3, 4, 5$ and $N = 25, 50, 100$.

N	$n=3$	$n=4$	$n=5$
25/50	2.345	2.975	4.423
50/100	2.163	2.998	4.010

[4]Wazwaz (2001)
[5]Parand et al. (2010)

CHAPTER FOUR
RESIDUAL METHOD FOR BOUNDARY VALUE PROBLEMS

Residual Method, described in Chapter 2, is mainly developed for initial value problems. In this chapter, we adapt Residual Method to boundary value problems as

$$y''(x) = F(x, y(x), y'(x)), \tag{4.1}$$

with the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta, \tag{4.2}$$

where $F \in C^{n-2}([a, b] \times D_1 \times D_2)$, D_1 and D_2 are closed intervals in $\mathbb{R}$ and a, b, α, β are finite constants.

4.1 Application of Residual Method to Boundary Value Problems

We define the following initial value problem

$$y''(x) = F(x, y, y'), \quad x \in [a, b], \tag{4.3}$$

$$y(a) = \alpha, \qquad y'(a) = s, \tag{4.4}$$

where s is the root of equation

$$y(b; s) - \beta = 0. \tag{4.5}$$

So, solution of the initial value problem (4.3) and (4.4) is also the solution of the boundary value problem (4.1) and (4.2). To find approximate solution of (4.5), we apply residual method to (4.3) and (4.4) without dividing the interval $[a, b]$, with high degree polynomial and we denote the approximate solution as $u(x; s)$.

An approximation of (4.5) can be obtained by solving the following non-linear equation

$$u(b; s) - \beta = 0. \tag{4.6}$$

Approximate root, s^*, of (4.6) can be estimated by using Newton's method with starting value α. Hence, we have the following initial value problem

$$\begin{aligned} &y''(x) = F(x, y, y'), \qquad x \in [a, b], \\ &y(a) = \alpha, \qquad y'(a) = s^*. \end{aligned} \tag{4.7}$$

Now, we can apply Residual Method, which is described in Chapter 2, to (4.7). In next section, we prove that using accurate approximate value for initial slope does not change order of convergence.

4.2 Error Analysis

Lemma 4.2.1. *Let the function $F(x, v, z)$ in* (4.1) *satisfy*

$$0 < \frac{\partial F}{\partial v} < L_1$$

for some positive constant L_1. Then

$$|s - y'(a)| \leq C\, h^n,$$

where C is a positive constant, y is the exact solution of the boundary value problem (4.1) *and* (4.2)*, s is the approximate root of* (4.6) *and h is the step size of residual method.*

Proof. This Lemma can be proved using Theorem A.2.1, Lemma A.2.2, Theorem A.2.3 and Theorem A.2.4 given in Keller (1968). □

Lemma 4.2.2. *Let $y_1(x)$ be the exact solution of* (4.3) *and* (4.4) *and $u_1(x)$ be the approximate solution of* (4.7) *on $x \in [a, h]$ with $|y_1'(a) - s| \leq \tilde{K}h^n$, where s is the approximate root of* (4.6) *and $\tilde{K}$ is a positive constant. Then,*

$$|y_1(x) - u_1(x)| \leq \bar{K}h^{n+1} + O(h^{n+2}) \tag{4.8}$$

and

$$|y_1'(x) - u_1'(x)| \leq (n+1)\bar{K}h^n + O(h^{n+1}), \tag{4.9}$$

where $\bar{K} = \tilde{K} + K$, $K = \dfrac{1}{(n+1)!} \max\limits_{x\in[a,b]} |y^{(n+1)}(x)|$.

Proof. Let us write Taylor's expansion of $|u_1(x) - y_1(x)|$ at $x = a$,

$$\begin{aligned} u_1(x) - y_1(x) =& (u_1(a) - y_1(a)) + (x-a)(u_1'(a) - y_1'(a)) \\ &+ \frac{(x-a)^2}{2}(u_1''(a) - y_1''(a)) + \cdots + \frac{(x-a)^{n+1}}{(n+1)!}(u_1^{(n+1)}(\xi) - y_1^{(n+1)}(\xi)), \end{aligned}$$

where ξ is between x and a. Taking absolute value of both sides and using $|x-a| < h$, we get

$$\begin{aligned} |u_1(x) - y_1(x)| \leq& |u_1(a) - y_1(a)| + h|u_1'(a) - y_1'(a)| \\ &+ \frac{h^2}{2}|u_1''(a) - y_1''(a)| + \cdots + \frac{h^{n+1}}{(n+1)!}|y_1^{(n+1)}(\xi)|. \end{aligned}$$

Since $u_1(x)$ is approximate solution of (4.7) in S_1, then $u_1(a) = y_1(a) = \alpha$ and $u_1'(a) = s$. Arranging the right hand side, we have

$$|u_1(x) - y_1(x)| \leq h|s - y_1'(a)| + \frac{h^2}{2}|u_1''(a) - y_1''(a)| + \cdots + \frac{h^{n+1}}{(n+1)!}|y_1^{(n+1)}(\xi)|.$$

Using Equations (2.3), (2.11) and (2.12), we have

$$\begin{aligned}|u_1(x) - y_1(x)| \leq & h|s - y_1'(a)| + \frac{h^2}{2}|F(a, u_1(a), u_1'(a)) - F(a, y_1(a), y_1'(a))| \\ & + \frac{h^3}{3!}|F'(a, u_1(a), u_1'(a)) - F'(a, y_1(a), y_1'(a))| + \cdots \\ & + \frac{h^n}{n!}|F^{(n-2)}(a, u_1(a), u_1'(a)) - F^{(n-2)}(a, y_1(a), y_1'(a))| \\ & + \frac{h^{n+1}}{(n+1)!}|y_1^{(n+1)}(\xi)|.\end{aligned}$$

Mean value theorem yields

$$\begin{aligned}|u_1(x) - y_1(x)| \leq & |s - y_1'(a)|h + \frac{h^2}{2}|F_z(a, \alpha, \xi^{(1)})|\ |s - y_1'(a)| \\ & + \frac{h^3}{3!}|F_z'(a, \alpha, \xi^{(2)})|\ |s - y_1'(a)| + \cdots \\ & + \frac{h^n}{n!}|F_z^{(n-2)}(a, \alpha, \xi^{(n-1)})|\ |s - y_1'(a)| \\ & + \frac{h^{n+1}}{(n+1)!}|y_1^{(n+1)}(\xi)|,\end{aligned}$$

where $\xi^{(i)},\ i = 1, 2, \ldots, n-1$, are between s and $y_1'(a)$. Rearranging the rigth hand side, we have

$$\begin{aligned}|u_1(x) - y_1(x)| \leq & |s - y_1'(a)|h \left(1 + \frac{h}{2}|F_z(a, \alpha, \xi^{(1)})| + \frac{h^2}{3!}|F_z'(a, \alpha, \xi^{(2)})| + \cdots \right. \\ & \left. + \frac{h^{n-1}}{n!}|F_z^{(n-2)}(a, \alpha, \xi^{(n-1)})|\right) + \frac{h^{n+1}}{(n+1)!}|y_1^{(n+1)}(\xi)|,\end{aligned}$$

and using $|y_1'(a) - s| \leq \tilde{K}h^n$, we get

$$\begin{aligned}|u_1(x) - y_1(x)| \leq & \tilde{K}h^n h \left(1 + \frac{h}{2}|F_z(a, \alpha, \xi^{(1)})| + \frac{h^2}{3!}|F_z'(a, \alpha, \xi^{(2)})| \right. \\ & \left. \cdots + \frac{h^{n-1}}{n!}|F_z^{(n-2)}(a, \alpha, \xi^{(n-1)})|\right) + Kh^{n+1} \\ \leq & h^{n+1}(\tilde{K} + K) + h^{n+2}\left(\frac{1}{2}|F_z(a, \alpha, \xi^{(1)})| + \frac{h}{3!}|F_z'(a, \alpha, \xi^{(2)})| \right. \\ & \left. \cdots + \frac{h^{n-2}}{n!}|F_z^{(n-2)}(a, \alpha, \xi^{(n-1)})|\right) \\ \leq & \bar{K}h^{n+1} + h^{n+2}C_2\left\{\frac{1}{2} + \frac{h}{3!} + \cdots + \frac{h^{n-2}}{n!}\right\},\end{aligned}$$

where $\bar{K} = K + \tilde{K}$ and C_2 is constant as defined in Lemma 2.2.4. Therefore,

$$|u_1(x) - y_1(x)| \leq \bar{K}h^{n+1} + O(h^{n+2}).$$

By the similar way, it can be shown that

$$|y_1'(x) - u_1'(x)| \leq (n+1)\bar{K}h^n + O(h^{n+1}).$$

□

Comparing (4.8) and (4.9) with (2.23) and (2.24), we can say obviously that accurate approximation to the first derivative at $x = a$ using Residual Method does not change the order of convergence of our method.

Theorem 4.2.3. *Let $y(x)$ be the exact solution of the second order non-linear boundary value problem* (4.1) *and* (4.2)*, and $u(x)$ be the nth degree approximate solution of* (4.7)*. Then, we have the inequality*

$$|y(x) - u(x)| \leq \bar{M}h^{n-1}, \qquad x \in [a, b], \tag{4.10}$$

where $\bar{M} = \bar{K}(b-a)^2(n+1)$ and $\bar{K}$ is defined in Lemma 4.2.2.

Proof. Proof of this theorem can be done likewise in Theorem 2.2.6, using Lemma 2.2.2, Lemma 2.2.3 and Lemma 2.2.4 together with 4.2.2. □

4.3 Algorithm of Method

The algorithm of the modified method can be written as

S 1 Set a, b and $\tilde{n}$.

S 2 Define $F(x, y, z)$ and

$$\tilde{u}(x) := \sum_{j=0}^{\tilde{n}} \tilde{c}_j B_j^{\tilde{n}}(x; [a, b])$$

$$B_j^{\tilde{n}}(x; [a, b]) := \binom{\tilde{n}}{j} \frac{1}{(b-a)^{\tilde{n}}} (x-a)^j (b-x)^{\tilde{n}-j}$$

S 3 Define the functions

$$f_k(x, y, z) := \frac{\partial^k}{\partial x^k} F(x, y(x), z(x)), \quad k = 0, 1, \ldots, \tilde{n} - 2.$$

S 4 Set $\tilde{c}_0 = \alpha$, $\tilde{c}_1 = s\dfrac{b-a}{\tilde{n}} + \alpha$.

S 5 Solve $\tilde{c}_k(s)$, $k = 2, 3, \ldots, \tilde{n}$ from following linear equations

$$\frac{\tilde{n}(\tilde{n}-1)\ldots(\tilde{n}-k+1)}{(b-a)^k} \Delta^k \tilde{c}_0 - f_{k-2}\left(\eta, \tilde{c}_0, s\frac{b-a}{\tilde{n}}\right) = 0,$$

where Δ is difference operator.

S 6 Find approximate root, s^*, from non-linear equation

$$\tilde{c}_{\tilde{n}}(s) - \beta = 0.$$

S 7 Set $N, h = 1/N$, and n.

S 8 Compute $a_i = a + ih$ for $i = 0, 1, 2, \ldots, N$.

S 9 Define

$$u_i(x) := \sum_{j=0}^{n} c_j^i B_j^n(x; S_i)$$

$$B_j^n(x; S_i) := \binom{n}{j} \frac{1}{h^n} (x - a_{i-1})^j (a_i - x)^{n-j}$$

S 10 Set $c_0^1 = \alpha$, $c_1^1 = s^*\dfrac{h}{n} + \alpha$.

S 11 For $i = 1$ to $i = N$ do the followings

– Solve c_k^i, $k = 2, 3, \ldots, n$ from following linear equations

$$\frac{n(n-1)\ldots(n-k+1)}{h^k}\Delta^k c_0^i - f_{k-2}\left(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\right) = 0,$$

where Δ is difference operator.

– Compute $c_0^i = c_n^{i-1}$ and $c_1^i = 2c_n^{i-1} - c_{n-1}^{i-1}$.

– Increase i.

S 12 Using computed control points c_j^i for $j = 0, 1, \ldots, n$ and $i = 1, 2, \ldots, N$, write the piecewise approximate solution as

$$u(x) = \begin{cases} u_1(x), & x \in [a_0, a_1], \\ u_2(x), & x \in [a_1, a_2], \\ \quad \vdots & \\ u_N(x), & x \in [a_{N-1}, a_N]. \end{cases}$$

4.4 Numerical Examples

In this section, we apply Residual method to modified Troesch's problem (1.9), after that we use back transformation to find the approximate solution of the original Troesch's problem. We compare numerical results with theoretical aspects and the numerical results obtained using other methods. After that approximate solutions for $\lambda = 70$ and 80 will be given in literature for the *first time*.

Approximate value of the initial slope for $\lambda = 10, \ldots, 80$ are given in Table 4.1.

Table 4.1 Initial slope $y'(0)$ of Troesch's problem for $\lambda = 10, \ldots, 80$.

	Exact Value Scott[1]	Presented Method	Temimi[2]	Chang[3]
$\lambda = 10$	3.58337770 (-04)	3.583377846(-04)	3.583377846(-04)	3.58337785 (-04)
$\lambda = 20$	-	1.6487731(-08)	1.6487731(-08)	-
$\lambda = 30$	-	7.4860937(-13)	7.4860937(-13)	-
$\lambda = 40$	-	3.3986833(-17)	3.3986833(-17)	-
$\lambda = 50$	-	1.5429998(-21)	1.5429998(-21)	-
$\lambda = 60$	-	7.0052162(-26)	7.0052086(-26)	-
$\lambda = 70$	-	3.1814705(-30)	-	-
$\lambda = 80$	-	1.4657416(-34)	-	-

In the following tables, we give the observed orders of the Residual method obtained using (2.36) for $\lambda = 10$ and $\lambda = 50$.

Table 4.2 Observed orders of Residual Method for $n = 3,\ 4,\ 5,\ 6$ for $\lambda = 10$.

N	$n = 3$	$n = 4$	$n = 5$	$n = 6$
40/80	1.65372	2.42298	3.86542	3.5544
80/160	1.81684	2.6417	3.9322	4.1192
160/320	1.91445	2.82207	3.94779	4.57789
320/640	1.95775	2.90413	3.70984	4.57768

Table 4.3 Observed orders of Residual Method for $n = 3,\ 4,\ 5,\ 6$ for $\lambda = 50$.

N	$n = 3$	$n = 4$	$n = 5$	$n = 6$
200/400	1.51762	2.72791	3.71636	4.73639
400/800	1.80989	2.87723	3.86019	4.86846
800/1600	1.92195	2.94087	3.93075	4.93471
1600/3200	1.96569	2.97076	3.96534	4.96579

[1]Scott (1974)
[2]Temimi & Kurkcu (2014)
[3]Chang (2010b)

Table 4.4 contains exact values of Troesch's problem for $\lambda = 10$ given in Scott (1974), approximate values of it using Residual method proposed in this research, asymptotic approximation given in Temimi & Kurkcu (2014) and results of variational iteration method given in Chang (2010b).

Table 4.4 Comparison of exact and approximation solutions of Troesch's problem for $\lambda = 10$.

x	Exact Value Scott (1974)	Residual Method	Reference Temimi & Kurkcu (2014)	Reference Chang (2010b)
0.000	0	0	0	0
0.100	4.21118367 (-05)	4.211189927 (-05)	4.211189(-05)	4.2111895 (-05)
0.200	1.29963923 (-04)	1.299641158 (-04)	1.299641(-04)	1.2996410 (-04)
0.300	3.58977885 (-04)	3.589784013 (-04)	3.589784(-04)	3.5897837 (-04)
0.400	9.77901422 (-04)	9.779027717 (-04)	9.779027(-04)	9.7790270 (-04)
0.500	2.65901717 (-03)	2.659020490 (-03)	2.659020(-03)	2.6590203 (-03)
0.600	7.22892469 (-03)	7.228931212 (-03)	7.228931(-03)	7.2289309 (-03)
0.700	1.96640602 (-02)	1.966406309 (-02)	1.966406(-02)	1.9664062 (-02)
0.800	5.37303295 (-02)	5.373032934 (-02)	5.373032(-02)	5.3730328 (-02)
0.900	1.52114078 (-01)	1.521140763 (-01)	1.521140(-01)	1.5211407 (-01)
0.925	2.02001685 (-01)	2.020016837 (-01)	-	2.0200168 (-01)
0.950	2.76267734 (-01)	2.762677337 (-01)	2.762677(-01)	2.7626773 (-01)
0.970	3.72264333 (-01)	3.722643326 (-01)	3.722643(-01)	3.7226433 (-01)
0.980	4.48233038 (-01)	4.482330385 (-01)	4.482330(-01)	4.4823303 (-01)
0.990	5.74076498 (-01)	5.740764995 (-01)	5.740764(-01)	5.7407649 (-01)
0.995	6.90114941 (-01)	6.901149443 (-01)	-	6.9011494 (-01)
0.997	7.65769726 (-01)	7.657697278 (-01)	-	7.6576972 (-01)
0.998	8.18032828 (-01)	8.180328294 (-01)	-	8.1803282 (-01)
0.999	8.88993117 (-01)	8.889931171 (-01)	-	8.8899311 (-01)
1.000	1.0	1	1	9.9999999 (-01)

Errors of the obtained results are given in the following table.

Table 4.5 Errors of Troesch's problem for $\lambda = 10$.

x	Exact Value[4]	Presented Method	Temimi[5]	Chang[6]
0.000	0	0	0	0
0.100	4.21118367 (-05)	6.2475(-11)	6.2473(-11)	5.82157(-11)
0.200	1.29963923 (-04)	1.91991(-10)	1.91971(-10)	1.79479(-10)
0.300	3.58977885 (-04)	5.15829(-10)	5.15752(-10)	4.85475(-10)
0.400	9.77901422 (-04)	1.34905(-09)	1.34909(-09)	1.28167(-09)
0.500	2.65901717 (-03)	3.31212(-09)	3.31194(-09)	3.17111(-09)
0.600	7.22892469 (-03)	6.51713(-09)	6.51679(-09)	6.23612(-09)
0.700	1.96640602 (-02)	2.83862(-09)	2.83335(-09)	2.31252(-09)
0.800	5.37303295 (-02)	2.40573(-10)	2.3567(-10)	1.12171(-09)
0.900	1.52114078 (-01)	2.40133(-09)	2.3863(-09)	3.5678(-09)
0.925	2.02001685 (-01)	1.74066(-09)	-	2.9082(-09)
0.950	2.76267734 (-01)	1.10607(-09)	1.1042(-09)	2.2155(-09)
0.970	3.72264333 (-01)	3.69479(-10)	3.645(-10)	1.3629(-09)
0.980	4.48233038 (-01)	7.60166(-11)	2.83999(-11)	9.629(-10)
0.990	5.74076498 (-01)	1.33713(-09)	1.5507(-09)	6.658(-10)
0.995	6.90114941 (-01)	2.66173(-09)	-	2.2828(-09)
0.997	7.65769726 (-01)	1.70616(-09)	-	1.6102(-09)
0.998	8.18032828 (-01)	1.19488(-09)	-	1.3604(-09)
0.999	8.88993117 (-01)	6.42548(-11)	-	5.789(-10)
1.000	1.0	4.44089(-16)	0	2.99982(-13)

In Table 4.6, comparison of the approximate solutions obtained using Residual method and asymptotic approximation given in Temimi & Kurkcu (2014) for $\lambda = 50$ and $\lambda = 60$ are presented. Finally, graphs of the approximate solutions of Troesch's problem for $\lambda = 10, 40, 70$ and 80 are given in Figure 4.1.

[4]Scott (1974)
[5]Temimi & Kurkcu (2014)
[6]Chang (2010b)

Table 4.6 Approximate solutions of Troesch's problem for $\lambda = 50$ for $\lambda = 60$.

x	$\lambda = 50$		$\lambda = 60$	
	Presented Method	Temimi[7]	Presented Method	Temimi[7]
0.000	0	0	0	0
0.100	2.28991(-21)	2.289910897(-21)	2.35507(-25)	2.355071244(-25)
0.200	3.39868(-19)	3.398683397(-19)	9.5011 (-23)	9.501093884(-23)
0.300	5.04409(-17)	5.044093407(-17)	3.83302(-20)	3.833014842(-20)
0.400	7.48609(-15)	7.486098374(-15)	1.54635(-17)	1.546348553(-17)
0.500	1.11103(-12)	1.111035509(-12)	6.23842(-15)	6.238415312(-15)
0.600	1.64892(-10)	1.648922897(-10)	2.51676(-12)	2.516756362(-12)
0.700	2.44721(-08)	2.447218563(-08)	1.01533(-09)	1.015331982(-09)
0.800	3.63198(-06)	3.631994383(-06)	4.09615(-07)	4.096141568(-07)
0.900	5.39042(-04)	5.390439175(-04)	1.65251(-04)	1.652504835(-04)
0.950	6.58158(-03)	6.581608721(-03)	3.32189(-03)	3.321884417(-03)
0.970	1.81558(-02)	1.815582947(-02)	1.1122 (-02)	1.112197262(-02)
0.980	3.08773(-02)	3.087747331(-02)	2.07222(-02)	2.072216283(-02)
0.990	5.62729(-02)	5.627316454(-02)	4.1112 (-02)	4.111194396(-02)
1.000	1.0	1	0	1

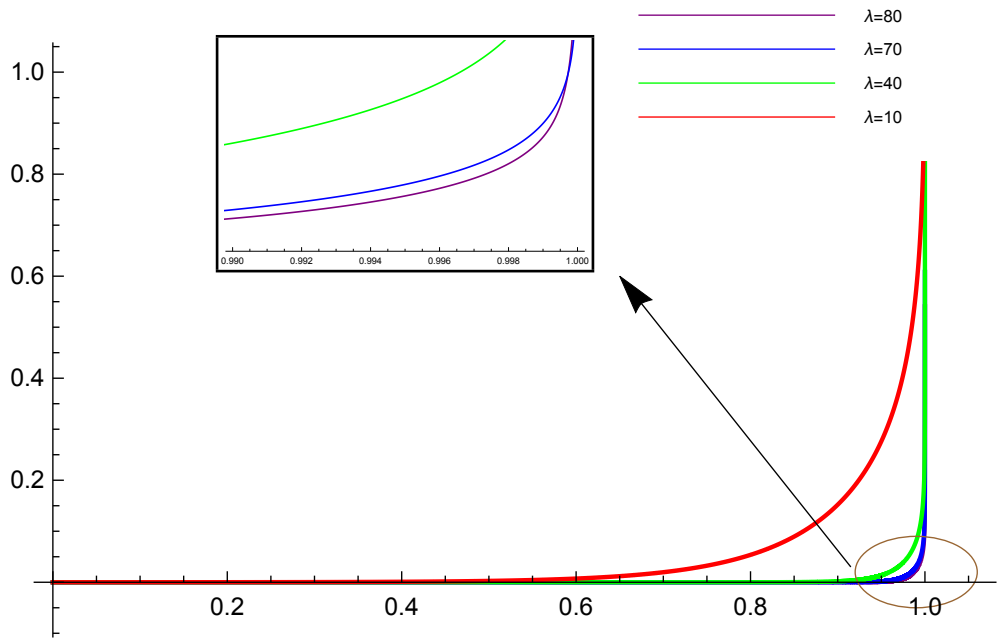


Figure 4.1 Graphs of the approximate solutions of Troesch's problem for $\lambda = 10$, 40, 70 and 80.

[7]Temimi & Kurkcu (2014)

In Temimi and Kurkcu's research Temimi & Kurkcu (2014), it is pointed that approximate solutions for $\lambda > 60$ couldn't be obtained because of the accuracy problem. But by using Residual Method, we obtain them for $\lambda = 70$ and 80, and present the approximate values in the literature for the first time in Table 4.7.

Table 4.7 Approximate solutions of Troesch's problem for $\lambda = 70$ and $\lambda = 80$.

x	$\lambda = 70$	$\lambda = 80$
0.000	0	0
0.100	2.49207(-29)	2.73082(-33)
0.200	2.73289(-26)	8.14046(-30)
0.300	2.99698(-23)	2.42664(-26)
0.400	3.28659(-20)	7.23371(-23)
0.500	3.60418(-17)	2.15634(-19)
0.600	3.95247(-14)	6.42795(-16)
0.700	4.3344 (-11)	1.9165 (-12)
0.800	4.75325(-08)	5.71195(-09)
0.900	5.21258(-05)	1.70271(-05)
0.925	2.99966(-04)	1.25714(-04)
0.950	1.72669(-03)	9.29754(-04)
0.970	7.03529(-03)	4.61766(-03)
0.980	1.4393 (-02)	1.03949(-02)
0.990	3.11425(-02)	2.46208(-02)
0.995	5.01167(-02)	4.14993(-02)
0.997	6.45461(-02)	5.47281(-02)
0.998	7.60966(-02)	6.56525(-02)
0.999	9.59375(-02)	8.56815(-02)
1.000	1.0	1

CHAPTER FIVE
CONCLUSION

We constructed Residual method to approximate solution of the second order non-linear differential equations, especially having singularity, by using Bézier curves. Applying Residual method, we reduced the non-linear problems to linear systems of equations. But if problem has a singularity, while applying proposed method, we solved non-linear equations only for the first interval. We also modified Residual method for boundary value problems. In this case, again we solved a non-linear equation to find initial slope. We obtained theoretical results and compared them with numerical results. It is seen that the numerical results coincide with theoretical aspects. We applied presented method to singular Lane-Emden type equations and highly sensitive Troesch's problem. Approximate solutions of Troesch's problem for $\lambda = 70$ and $\lambda = 80$ are obtained in literature for the first time. This method can be extended to chaotic system of equations, singularly perturbed boundary value problems. Residual method can also be modified for partial differential equations to find numerical solutions of various problems.

REFERENCES

Bengochea, G. (2014). Algebraic approach to the Lane-Emden equation. *Applied Mathematics and Computation*, *232*, 424-430.

Chang, S.-H. (2010a). Numerical solution of Troesch's problem by simple shooting method. *Applied Mathematics and Computation*, *216*(11), 3303-3306.

Chang, S.-H. (2010b). A variational iteration method for solving Troesch's problem. *Journal of Computational and Applied Mathematics*, *234*(10), 3043-3047.

Chiou, J., & Na, T. (1975). On the solution of Troesch's nonlinear two-point boundary value problem using an initial value method. *Journal of Computational Physics*, *19*(3), 311-316.

Feng, X., Mei, L., & He, G. (2007). An efficient algorithm for solving Troesch's problem. *Applied Mathematics and Computation*, *189*(1), 500-507.

Jones, D. (1973). Solution of Troesch's, and other, two point boundary value problems by shooting techniques. *Journal of Computational Physics*, *12*(3), 429-434.

Keller, H. B. (1968). *Numerical methods for two-point boundary-value problems*. Waltham: Blaisdell Publishing Company.

Khuri, S., & Sayfy, A. (2011). Troesch's problem: A B-spline collocation approach. *Mathematical and Computer Modelling*, *54*(9–10), 1907-1918.

Kubiček, M., & Hlaváček, V. (1975). Solution of Troesch's two-point boundary value problem by shooting technique. *Journal of Computational Physics*, *17*(1), 95-101.

Marzban, H., Tabrizidooz, H., & Razzaghi, M. (2008). Hybrid functions for nonlinear initial-value problems with applications to Lane-Emden type equations. *Physics Letters A*, *372*(37), 5883-5886.

Miele, A., Aggarwal, A., & Tietze, J. (1974). Solution of two-point boundary-value problems with Jacobian matrix characterized by large positive eigenvalues. *Journal of Computational Physics*, *15*(2), 117-133.

Parand, K., Dehghan, M., Rezaei, A., & Ghaderi, S. (2010). An approximation algorithm for the solution of the nonlinear Lane-Emden type equations arising in astrophysics using Hermite functions collocation method. *Computer Physics Communications*, *181*(6), 1096-1108.

Roberts, S., & Shipman, J. (1976). On the closed form solution of Troesch's problem. *Journal of Computational Physics*, *21*(3), 291-304.

Scott, M. (1974). On the conversion of boundary-value problems into stable initial-value problems via several invariant imbedding algorithms. In Aziz, A. K. (Ed.), *Numerical solutions of boundary value problems for ordinary differential equations*, (89-148). New York: Academic Press.

Snyman, J. (1979). Continuous and discontinuous numerical solutions to the Troesch's problem. *Journal of Computational and Applied Mathematics*, *5*(3), 171-175.

Temimi, H. (2012). A discontinuous Galerkin finite element method for solving the Troesch's problem. *Applied Mathematics and Computation*, *219*(2), 521-529.

Temimi, H., & Kurkcu, H. (2014). An accurate asymptotic approximation and precise numerical solution of highly sensitive Troesch's problem. *Applied Mathematics and Computation*, *235*, 253-260.

Troesch, B. (1976). A simple approach to a sensitive two-point boundary value problem. *Journal of Computational Physics*, *21*(3), 279-290.

Troesch, B. A. (1960). Intrinsic difficulties in the numerical solution of a boundary value problem. *Space Technologies Labrotories, Technical Notes*, *TN-142*.

Turkyilmazoglu, M. (2013). Effective computation of exact and analytic approximate solutions to singular nonlinear equations of Lane-Emden-Fowler type. *Applied Mathematical Modelling*, *37*(14), 7539-7548.

Vemuri, V., & Raefsky, A. (1979). On a method of solving sensitive boundary value problems. *Journal of the Franklin Institute*, *307*(4), 217-243.

Wang, Y., Yu, H., & Tan, F. (2014). Using an effective numerical method for solving a class of Lane-Emden equations. *Abstract and Applied Analysis*, *2014*.

Wazwaz, A.-M. (2001). A new algorithm for solving differential equations of Lane-Emden type. *Applied Mathematics and Computation*, *118*(2), 287-310.

Wazwaz, A.-M. (2014). The variational iteration method for solving the Volterra integro-differential forms of the Lane–Emden equations of the first and the second kind. *Journal of Mathematical Chemistry*, *52*, 613-626.

Weibel, E. S. (1958). Confinement of a plasma column by radiation pressure. In Landshoff, R. K. M. (Ed.), *The Plasma in a Magnetic Field: A Symposium on Magneto Hydrodynamics*, (60-67). Stanford: Stanford University.

Zarebnia, M., & Sajjadian, M. (2012). The sinc–Galerkin method for solving Troesch's problem. *Mathematical and Computer Modelling*, *56*(9–10), 218-228.

Zheng, J., Sederberg, T. W., & Johnson, R. W. (2004). Least squares methods for solving differential equations using Bézier control points. *Applied Numerical Mathematics*, *48*(2), 237-252.

APPENDICES

A.1: Bézier Curves

In this section, we describe Bézier Curves as described in Zheng et al. (2004). Any nth degree polynomial $u(t)$ can be represented as

$$u(t) = \sum_{j=0}^{n} c_j B_j^n(t; [a, b]), \tag{A.1}$$

where

$$B_j^n(t; [a, b]) = \binom{n}{j} \left(\frac{t-a}{b-a}\right)^j \left(\frac{b-t}{b-a}\right)^{n-j}$$

are Bernstein Polynomials over the interval $[a, b]$. The representation (A.1) is called Bernstein-Bézier form (or Bézier form for brevity). If $u(t)$ is a vector-valued polynomial, $u(t)$ is a parametric Bézier curve and the Bézier coefficients c_j are the control points. If $u(t)$ is a scalar-valued polynomial, we call the function $y = u(t)$ an explicit Bézier curve given by $(t, u(t))$. The control points of an explicit curve are $(j/n, c_j)$, where the x-coordinates j/n are called the Greville abscissa. In this thesis, we also call c_j the Bézier control points for convenience.

Bézier curves have many nice properties involving the control points. For example, end point interpolation says first and last control points are the end points of Bézier curve. In the explicit case,

$$u(a) = c_0 \tag{A.2}$$

and

$$u(b) = c_n. \tag{A.3}$$

In addition, many operations on the Bézier curves can be performed using control points. For example, the derivatives of the Bézier curve (A.1) can be written

$$\frac{d^k}{dt^k} u(t) = \frac{1}{(b-a)^k} \frac{n!}{(n-k)!} \sum_{j=0}^{n-k} \Delta^k c_j \; B_j^{n-k}(t; [a, b]) \tag{A.4}$$

for $k = 0, \ldots, n$, where Δ is the difference operator such that $\Delta c_j = c_{j+1} - c_j$. In particular, derivatives at the end points can be written simply

$$\frac{d^k}{dt^k} u(t)\Big|_{t=a} = \frac{1}{(b-a)^k} \frac{n!}{(n-k)!} \Delta^k c_0, \tag{A.5}$$

$$\frac{d^k}{dt^k} u(t)\Big|_{t=b} = \frac{1}{(b-a)^k} \frac{n!}{(n-k)!} \Delta^k c_{n-k}. \tag{A.6}$$

These formulas can be used to set constraints on Bézier segments to make them connect with C^k continuity and to form a Bézier spline. For example, the curve which consists of the following two Bézier segments

$$u_1(t) = \sum_{j=0}^{n} c_j^1 B_j^n(t; [a, b])$$

and

$$u_2(t) = \sum_{j=0}^{n} c_j^2 B_j^n(t; [b, c])$$

satisfies C^1 continuity if control points of them satisfy the following equations

$$c_0^2 = c_n^1 \tag{A.7}$$

and

$$c_1^2 - c_0^2 = c_n^1 - c_{n-1}^1 \tag{A.8}$$

as seeing in Figure A.1.

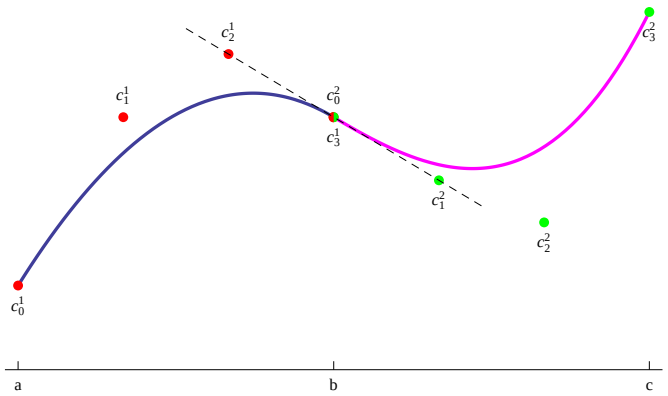


Figure A.1 C^1 continuity of two Bézier segments.

A.2: Some Required Theorems for Boundary Value Problems

In this part, we give some theorems about accuracy of approximation techniques for boundary value problems. We also give the existence and uniqueness theorem for boundary value problems. Proofs of the given theorems can be found in Keller (1968).

Consider second order differential equations, which may be non-linear, of the form

$$y'' = F(x, y, y'), \qquad a \leq x \leq b, \tag{A.9}$$

subject to the general two-point boundary conditions

$$\begin{aligned} a_0 y(a) - a_1 y'(a) &= \alpha, \qquad |a_0| + |a_1| \neq 0, \\ b_0 y(b) + b_1 y'(b) &= \beta, \qquad |b_0| + |b_1| \neq 0. \end{aligned} \tag{A.10}$$

Theorem A.2.1 (Existence and Uniqueness)**.** *Let the function* $F(x, y, z)$ *in Equation* (A.9) *be continuous on the infinite strip*

$$R : a \leq x \leq b, \qquad y^2 + z^2 < \infty$$

and satisfy there a uniform Lipschitz condition in y *and* z*. And let* $F(x, y, z)$ *have continuous derivatives on* R *which satisfy, for some positive constant* M*,*

$$\frac{\partial f}{\partial y} > 0, \qquad \left|\frac{\partial f}{\partial z}\right| \leq M.$$

Let the coefficients in (A.10) *satisfy*

$$a_0 a_1 \geq 0, \qquad b_0 b_1 \geq 0, \qquad |a_0| + |b_0| \neq 0.$$

Then the boundary value problem (A.9) *and* (A.10) *has a unique solution.*

Consider second order differential equation

$$y'' = F(x, y, y'), \qquad a \leq x \leq b, \tag{A.11}$$

and the boundary conditions

$$\begin{aligned} a_0 y(a) - a_1 y'(a) &= \alpha, \quad a_1 \geq 0, \\ b_0 y(b) + b_1 y'(b) &= \beta, \quad b_1 \geq 0; \quad a_0 + b_0 > 0. \end{aligned} \tag{A.12}$$

And let $F(x, y, z)$, in Equation (A.11), satisfy the hypothesis of Theorem A.2.1 to assure existence and uniqueness of the solution.

Corresponding initial value problem with the boundary value problem (A.11) and (A.12) can be written as

$$u'' = F(x, u, u'), \qquad a \leq x \leq b, \tag{A.13}$$

$$u(a) = a_1 s - c_1 \alpha, \qquad u'(a) = a_0 s - c_0 \alpha, \tag{A.14}$$

where c_0 and c_1 are any constant such that

$$a_1 c_0 - a_0 c_1 = 1. \tag{A.15}$$

Solution of this problem, denoted by $u(x; s)$, will be a solution of the boundary value problem (A.11) and (A.12) if and only if s is a root of

$$\phi(s) := b_0 u(b; s) + b_1 u'(b; s) - \beta = 0. \tag{A.16}$$

In Keller (1968), it is shown that $\phi(s)$ has a unique solution by Theorem A.2.1.

First order system of initial value problem of (A.13) and (A.14) can be written as

$$\begin{aligned} u' &= v, & u(a) &= a_1 s - c_1 \alpha, \\ v' &= F(x, u, v) & v(a) &= a_0 s - c_0 \alpha. \end{aligned} \tag{A.17}$$

If the solution of this system is written as

$$u = u(x; s), \qquad v = v(x; s),$$

the function $\phi(s)$ as defined in (A.16) becomes

$$\phi(s) = b_0 u(b; s) + b_1 v(b; s) - \beta.$$

The net

$$x_j = a + jh, \qquad j = 0, 1, \ldots, J, \quad h = \frac{b-a}{J}$$

is used to obtain numerical solution of (A.17) which is denoted by

$$U_j(s), \qquad V_j(s), \qquad 0 \le j \le J \tag{A.18}$$

for each value of s.

If the numerical method, which is used to get $U_j(s)$ and $V_j(s)$, has order p and is stable, and solutions of (A.17) are sufficiently smooth, then

$$|U_j(s) - u(x_j; s)| = O(h^p), \tag{A.19}$$

$$|V_j(s) - u'(x_j; s)| = O(h^p), \qquad 0 \le j \le J. \tag{A.20}$$

Using these numerical solutions $U_j(s), V_j(s)$, we define the function

$$\Phi(s) := b_0 U_J(s) + b_1 V_J(s) - \beta, \tag{A.21}$$

it follows that

$$|\Phi(s) - \phi(s)| = O(h^p).$$

Thus, in general we cannot hope to evaluate $\phi(s)$ exactly but rather we get an $O(h^p)$ approximation to it.

If root of Equation (A.16), s^* is known, the solution of boundary value problem

(A.11) and (A.12) can be approximated at least $O(h^p)$.

Lemma A.2.2. *Let $u(x;s)$ and $v(x;s)$ denote the solutions of the initial value problem* (A.17)*, where $F(x,y,z)$ satisfies the conditions of Theorem A.2.1. Let $\{U_j(s)\}$ and $\{V_j(s)\}$ denote the numerical solution of* (A.17) *using a stable, pth-order-accurate scheme. Then if the solutions of* (A.17)*, $u(x;s)$ and $v(x;s)$, are sufficiently smooth, we have for any s and t*

$$|U_j(t) - u(x_j;s)| = O(h^p) + O(|s-t|), \tag{A.22}$$

$$|V_j(t) - u'(x_j;s)| = O(h^p) + O(|s-t|), \qquad 0 \le j \le J. \tag{A.23}$$

To approximate the root of (A.16), we write the equation in the form

$$s = g(s) := s - m\phi(s), \qquad m \neq 0,$$

and consider the functional iteration scheme: s_0 =arbitrary,

$$s_{i+1} = g(s_i), \qquad i = 0, 1, 2, \ldots. \tag{A.24}$$

Theorem A.2.3. *Let $F(x,y,y')$ in Equation* (A.11) *satisfy the hypothesis of Theorem A.2.1, and in addition for some positive constant N,*

$$\frac{\partial f}{\partial y} \le N.$$

Then the function $\phi(s)$ in (A.16) *has a continuous derivative, $\phi'(s)$, which satisfies*

$$0 < \gamma \le \phi'(s) \le \Gamma,$$

where

$$\gamma := b_0 \left[a_1 + a_0 \left(\frac{1 - e^{-M(b-a)}}{M} \right) \right] + b_1 a_0 e^{-M(b-a)},$$

$$\Gamma := \frac{e^{M(b-a)/2}}{2\sigma}\Big\{\Big[\Big(\sigma - \frac{M}{2}\Big)a_1 + a_0\Big]\Big[\Big(\sigma + \frac{M}{2}\Big)b_1 + b_0\Big]e^{\sigma(b-a)} - \Big[\Big(\sigma + \frac{M}{2}\Big)a_1 - a_0\Big]\Big[\Big(\sigma - \frac{M}{2}\Big)b_1 - b_0\Big]e^{-\sigma(b-a)}\Big\},$$

$$\sigma := \sqrt{\Big(\frac{M}{2}\Big)^2 + N}$$

for any s_0 *and fixed* m *in*

$$0 < m < 2/\Gamma$$

the iterates in Equations (A.24) *converge to the root* s^* *of Equation* (A.16). *In particular, with the choice* $m = 2/(\gamma + \Gamma)$*, the iterates satisfy*

$$|s_i - s^*| \leq \left(\frac{1 - (\gamma/\Gamma)}{1 + (\gamma/\Gamma)}\right)^i \frac{|\phi(s_0)|}{\gamma}; \qquad i = 1, 2, \ldots.$$

Theorem A.2.4. *Let* $F(x, y, y')$ *in Equation* (A.11) *satisfy the hypothesis of Theorem A.2.3 and the numerical solutions* (A.18), (A.19) *and* (A.20) *of the initial value problem* (A.17) *be computed as in Lemma A.2.2. Let the sequence* $\{s_i\}$ *be defined by*

$$s_{i+1} = s_i - m\Phi(s_i), \qquad i = 0, 1, 2, \ldots,$$

where $\Phi(s)$ *as defined in Equation* (A.21). *Then for any* m *in* $0 < m < 2/\Gamma$ *and for some positive* $\lambda = \lambda(m) < 1$*; we have*

$$|U_j(s_i) - y(x_j)| \leq O(h^p) + O(\lambda^i),$$
$$|V_j(s_i) - y'(x_j)| \leq O(h^p) + O(\lambda^i),$$

for $0 \leq j \leq J$, $i = 0, 1, 2, \ldots$, *where* $y(x)$ *is the solution of the boundary value problem* (A.9) *and* (A.10).

It should be noted that the converge factor λ, in the above theorem, is independent

of the net spacing h. Thus, for each h one can find an integer $i = i(h, p, \lambda)$ such that

$$\lambda^i \leq h^p,$$

and so, in principle, the total error can be made $O(h^p)$.

Newton's method may have arbitrarily small convergence factors, λ. Thus, we indicate the proper way in which to employ this method in the present problem. The iterations $\{s_i\}$ are now defined by, s_0 =arbitrary,

$$s_{i+1} = s_i - \frac{\phi(s_i)}{\phi'(s_i)}, \qquad i = 0, 1, \ldots.$$

We have previously indicated how $\phi(s)$ in Equation (A.16) is to be evaluated, or rather approximated, by $\Phi(s)$, in terms of a numerical solution of the initial value problem (A.17).