DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

LIE GROUP APPLICATIONS ON ROBOTICS

by
Soner ERKUŞ

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İZMİR

LIE GROUP APPLICATIONS ON ROBOTICS

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Soner ERKUŞ

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Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"LIE GROUP APPLICATIONS ON ROBOTICS"** completed by **SONER ERKUŞ** under supervision of **ASSOC.PROF.DR. İLHAN KARAKILIÇ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

.....

Assoc.Prof.Dr. İlhan KARAKILIÇ

Supervisor

.....

Assist. Prof. Dr. Ahmet Z. ÖZÇELİK

Thesis Committee Member

.....

Assoc. Prof. Hasan KARABIYIK

Thesis Committee Member

.....

Assoc.Prof.Dr. Mesut Şahin

Examining Committee Member

.....

Assoc.Prof.Dr. Gözde Yazgı Tütüncü

Examining Committee Member

.....

Prof. Dr. Emine İlknur CÖCEN

Director

Graduate School of Natural and Applied Sciences

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LIE GROUP APPLICATIONS ON ROBOTICS

ABSTRACT

In this thesis, the exponential map and the Cayley map, from the Lie algebra $\mathfrak{se}(2)$ of the planar motion group $SE(2)$, to the group itself are investigated. The comparison between these maps on $SE(2)$ is given by means of the Rodrigues vector and as an application a three joint mechanism is discussed. The canonical coordinate system and instantaneous invariants for planar motion group $SE(2)$ are introduced. The relation of instantanous invariants and the velocity, accelaration matrices is given.

Keywords:Kinematics, Lie group, planar motion group, exponential map, Cayley map, Rodrigues vector, instantaneous invariants.

ROBOT MEKANİZMALARINDA LİE GRUP UYGULAMALARI

ÖZ

Bu tezde üssel fonksiyon ve Cayley fonksiyonu düzlemsel Lie grubu $SE(2)$'den Lie cebiri temsili olan $\mathfrak{se}(2)$'e birer fonksiyon olarak incelenmiştir. Bu fonksiyonların $SE(2)$'deki karşılaştırmaları sonucu Rodrigues vektörü anlam bulmuştur ve bu karşılaştırmalar kullanılarak üç kollu mekanizma uygulama olarak incelenmiştir. Kanonik koordinat ve anlık değişmezler düzlemsel hareket grubu $SE(2)$'de tanımlanmıştır. Hareketin hız ve ivme matrisleri ile anlık değişmezler arasındaki bağlantı verilmiştir.

Anahtar kelimeler:Kinematik, Lie grup, düzlemsel hareket grubu, üssel fonksiyon, Cayley fonksiyonu, Rodrigues vektör, anlık değişmezler.

CONTENTS

Page

THESIS EXAMINATION RESULT FORM ii
ACKNOWLEDGEMENTS iii
ABSTRACT iv
ÖZ v
LIST OF FIGURES viii

CHAPTER ONE – INTRODUCTION 1

CHAPTER TWO – LIE GROUPS 3

2.1 Smooth Manifold 3
2.1.1 Coordinate Charts 3
2.2 Lie Group 8
2.2.1 Matrix Lie Groups 9
2.2.2 Group Actions and Products 13
2.2.3 The Proper (Special) Euclidean Group 14

CHAPTER THREE – LIE ALGEBRA AND THE EXPONENTIAL MAP 17

3.1 Derivation and Vector Fields 17
3.2 The Lie algebra associated with a Lie group 21
3.2.1 Examples of the Lie algebra 22
3.3 The Exponential Map 23
3.3.1 Examples of the Exponential Map 24

CHAPTER FOUR – KINEMATIC COMPARISON OF EXPONENTIAL AND CAYLEY MAPS IN THE PLANAR MOTION GROUP 26

4.1 The Matrix Lie Group in the Plane Kinematics 26
4.2 The Exponential and The Cayley Maps 27

4.3 Comparison Between The Exponential and The Cayley Maps 31
4.3.1 Application (A Three-Link Mechanism) 34

CHAPTER FIVE – CONTINUOUS DISPLACEMENTS WITH EXPONENTIAL MAPPING IN THE PLANAR MOTION GROUP 38

5.1 Active and Passive Transformation 38
5.2 Instantaneous Geometric Invariants in the Planar Motion Group 44
5.3 Time Dependent Motion in the Planar Motion Group 48

CHAPTER SIX – CONCLUSION 50

REFERENCES 51

LIST OF FIGURES

Page

Figure 2.1 Smooth function 5
Figure 2.2 Unit circle 9
Figure 4.1 The instantaneous rotation axis for a rigid body and the rhombuses drawn by $\vec{r_1}$, $\vec{r_2}$ and $\vec{r_1'}$, $\vec{r_2'}$. 33
Figure 4.2 A three-link planar manipulator 35
Figure 4.3 The length of Rodrigues vector and the location of joint positions with $-\frac{\pi}{2} \leq \theta_2 \leq \frac{\pi}{4}$ (dark) and $-\frac{\pi}{2} \leq \theta_3 \leq \frac{\pi}{4}$ (light). 37
Figure 5.1 Change of Coordinate Frames 39
Figure 5.2 Change of Coordinate Frames 44
Figure 5.3 Inflection Circle 47
Figure 5.4 Euler-Savary Equation 47

CHAPTER ONE
INTRODUCTION

Kinematics, the common subject of mathematics and physics, date back to old times. It deals with the motion of a point or a rigid body regardless of mass. On the other hand Lie group theory, a modern study, states an analogue of Galois theory for differential equations. However there is a large number of studies which connects these popular classic and modern topics.

Lie group applications to robotics are discussed in several studies. Selig's lecture notes Selig (2004) contains the basic knowledge from Chasles's theorem, lower Reuleaux's pairs to the Rodrigues formula. For the usage of Lie groups to the large rotations, it is worth to see Paraskevopoulos & Natsiavas (2013). The relation between the derivatives in Lie algebra and classical centrode curves of a rigid body motion is established in Selig (2006) and point-plane constraints are studied in Selig (2011). The Lie group theory applications in computer science is constituted in Selig (2005), Gallier (2011). One can examine the configuration manifolds for mobile robots Ouerfelli et al. (2000), Murray et al. (1994), Ebetiuc & Haraldw (2005), for the extensive usage of this context. Cayley transformation is well known in $SO(3)$, but it is generalized to $SE(3)$ by Selig. He expressed the comparison between the Cayley map and the exponential map in $SE(3)$ Selig (2007).

One important problem in kinematics is to keep track of rigid bodies. It is well known that any rigid transformation can be composed from a rotation, a translation and a reflection. Physical machines cannot perform reflections and so we should really speak of proper rigid transformations, excluding the reflections. The proper rigid transformations can be represented by using 3×3 matrices. These can be written in partitioned form as

$$S = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}.$$

where R is a rotation matrix and t is translation vector in Euclidean space $\mathbb{E}^2$.

In the view of Selig's work Selig (2007) we investigated similar decisions in the planar motion group $SE(2)$ and original result, equation (4.17), is derived by the use of Rodrigues vector Müller (1962), Duffy (1991), Bottema & Roth (1990). As an example, the length of Rodrigues vector and the position of end effector in an application of three joint mechanism is graphically signified.

In chapter two, some basic definitions to Lie group and matrix Lie group are given. In chapter three Lie algebra and exponential map are stated. In chapter four, kinematic comparison of exponential and Cayley maps in the planar motion group are discussed, and originally the relation between the Rodrigues vector, the rotation angle and eigenvalue of the skew-symmetric matrix of the Cayley map is obtained and also a three link mechanism is given as an application. Instanteneous invariants in spatial kinematics are studied in canonical coordinate systems. Veldkamp (1967), Roth (2015). In chapter five a brief introduction to instantaneous invariants in plane kinematics by use of matrix Lie groups are given.

CHAPTER TWO
LIE GROUPS

The goal of this chapter is to define Lie groups. Before doing this, we review standard concepts from the differential geometry of smooth manifolds and outlines the notation we adopt in this thesis.

2.1 Smooth Manifold

2.1.1 Coordinate Charts

Definition 2.1.1 *Let M be a topological space and $U \subseteq M$ an open set. Let $V \subseteq \mathbb{R}^n$ be open. A homeomorphism $\varphi : U \to V$, $\varphi(u) = (x_1(u), x_2(u), ..., x_n(u))$ is called* ***coordinate system*** *on U, and the functions $x_1, x_2, ...x_n$ the coordinate functions. The pair (U, φ) is called* ***coordinate chart*** *on M. The inverse map φ^{-1} is a parametrization of U.*

Definition 2.1.2 *An $n-$dimensional* ***atlas*** *on M is a collection of coordinate charts $\{U_\alpha, \varphi_\alpha\}_{\alpha \in I}$ such that*

1. *M is covered by the $\{U_\alpha\}_{\alpha \in I}$.*
2. *For each map $\alpha, \beta \in I$, $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$.*
3. *The transition map $\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \hookrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is C^∞ with C^∞ inverse.*

Definition 2.1.3 *A second countable, Hausdorff topological space M is an* ***n-dimensional topological manifold*** *if it admits an atlas $\{U_\alpha, \varphi_\alpha\}_{\alpha \in I}$, $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, $n \in \mathbb{N}$. It is a smooth manifold if all transition maps are C^∞ diffeomorphisms, that is, all partial derivatives exist and are continuous.*

Example 2.1 *1. The Euclidean space $\mathbb{R}^n$ is an n-dimensional manifold, covered by only one chart $U = \mathbb{R}^n$ with $\varphi : U \to \mathbb{R}^n$ the identity map*

2. *The n-sphere $S^n = \{y = (y_1, y_2, ...y_{n+1}) \in \mathbb{R}^{n+1} : y_1^2 + y_2^2 + ... + y_{n+1}^2 = 1\}$ in $\mathbb{R}^{n+1}$ is a manifold of dimension n. It can be covered by two charts $U_+ = \{y \in S^n : y_{n+1} > -1\}$ with $\varphi_+ : U_+ \to \mathbb{R}^n$ by $\varphi_+(y) = \left(\frac{y_1}{1+y_{n+1}}, \frac{y_2}{1+y_{n+1}}, ..., \frac{y_n}{1+y_{n+1}}\right)$ and $U_- = \{y \in S^n : y_{n+1} < 1\}$ with $\varphi_- : U_- \to \mathbb{R}^n$ by $\varphi_-(y) = \left(\frac{y_1}{1-y_{n+1}}, \frac{y_2}{1-y_{n+1}}, ..., \frac{y_n}{1-y_{n+1}}\right)$. The maps φ_+ and φ_- are called stereographic projections.*

3. *Any open subset U of a smooth manifold M is itself a smooth manifold. The charts of U are the intersections of U with the charts of M.*

4. *If M and N are smooth manifolds, then the Cartesian product $M \times N$ is also a smooth manifold of dimension equal to the sum of the dimensions of M and N.*

By using charts we can define differentiability for functions between smooth manifolds.

Definition 2.1.4 *Let M and N be two smooth manifolds and let $f : M \to N$ be a continuous function. Then f is called* ***smooth*** *(or differentiable) if and only if all charts $\varphi : U \subset M \to \mathbb{R}^k$ and $\psi : V \subset N \to \mathbb{R}^m$, the map*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \mathbb{R}^m \tag{2.1}$$

is a smooth (or differentiable) function between Euclidean spaces.

Example 2.2 *If $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^m$ are submanifolds and $F : \mathbb{R}^k \to \mathbb{R}^m$ is smooth with $F(M) \subset N$ then $F : M \to N$ is smooth.*

Example 2.3 *The general linear group which is defined by*

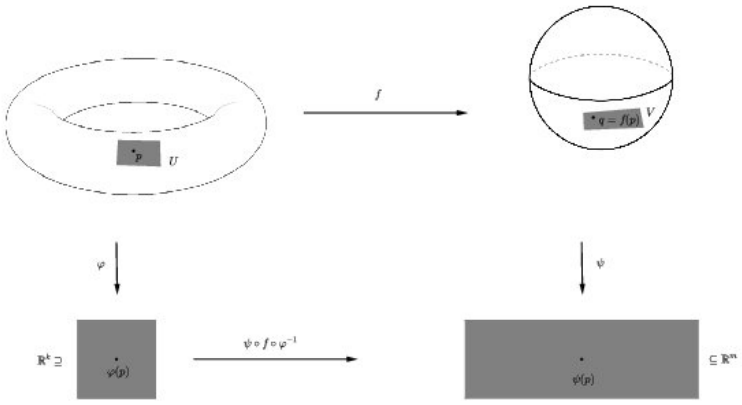


Figure 2.1 Smooth function

$GL(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} | detA \neq 0\}$ *is open in* $\mathbb{R}^{n\times n}$ *and* $\mu : GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \to GL(n,\mathbb{R}); (A,B) \mapsto A.B$ *and* $\iota : GL(n,\mathbb{R}) \to GL(n,\mathbb{R}); A \mapsto A^{-1}$ *are smooth (or differentiable).*

Definition 2.1.5 *A* ***diffeomorphism*** $f : M \to N$ *is a smooth function that has an inverse which is also smooth.*

Now we will proceed tangent vectors and vector fields. Let $\mathcal{F}(M)$ be the set of all smooth real-valued functions on a manifold M.

Definition 2.1.6 *Let* p *be a point of a manifold* M*. A* ***tangent vector*** *to* M *at* p *is a real-valued function* $\vartheta : \mathcal{F}(M) \to \mathbb{R}$ *that satisfies:*

1. $\vartheta(af + bg) = a\vartheta(f) + b\vartheta(g)$,
2. $\vartheta(fg) = \vartheta(f)g(p) + f(p)\vartheta(g)$, *(Leibniz rule)*

where $a, b \in \mathbb{R}$, $f, g \in \mathcal{F}(M)$.

Every point $p \in M$ let $T_p(M)$ be the set of all tangent vectors to M at p.

Then under the operations

$$\begin{aligned}(\vartheta+\omega)(f) &= \vartheta(f)+\omega(p)\\ (\alpha\vartheta)(f) &= \alpha\vartheta(f),\end{aligned} \tag{2.2}$$

the set $T_p(M)$ is made into a real vector space of dimension equal to that of M. A basis for this vector space is constructed as follows. Take a local chart (U,φ) of p, and let x_i $(i=1,...,n)$ be the ith component of φ (i.e. the result of the composition of $\varphi: U\to\mathbb{R}^n$ with the ith projection $u_i:\mathbb{R}^n\to\mathbb{R}$). Then the function

$$\frac{\partial}{\partial x_i}\Big|_p : \mathcal{F}(M)\to\mathbb{R} \tag{2.3}$$

sending each $f\in\mathcal{F}(M)$ to

$$\frac{\partial f}{\partial x_i}(p)=\frac{\partial(f\circ\varphi^{-1})}{\partial u_i}(\varphi(p)) \tag{2.4}$$

is a tangent vector to M at p, and the set of all such tangent vectors for $i=1,...,n$ is a basis for $T_p(M)$.

Using transformation properties of vectors under coordinate changes, one defines a tangent vector at $p\in M$ to be an equivalence class of triples (U,f,e), where $f:U\to\mathbb{R}$ is a chart and $e\in\mathbb{R}$, with two triples identified if they are related by the tangent of the corresponding overlap map evaluated at the point corresponding to $p\in M$.

A **curve** in a manifold M is a smooth map $\alpha: I\to\mathbb{R}$, where I is an open interval in $\mathbb{R}$. We abstract the idea that a tangent vector to a surface is the **velocity vector** $\alpha'(t)$ of a curve in the surface at t. A continuous map $\alpha: I\to\mathbb{R}$ is said to be smooth (or differentiable) at the endpoint p if there is a chart (U,f) at $\alpha(p)$ such that

$$\alpha'(t)f=\frac{d(f\circ\alpha)}{dt}(t)=\lim_{t\to p}\frac{(f\circ\alpha)(t)-(f\circ\alpha)(p)}{t-p}. \tag{2.5}$$

Definition 2.1.7 *Let $f : M \to N$ be smooth function. Then, for each $p \in M$, the differential of f is the function*

$$df_p = T_pM \to T_{f(p)}N \tag{2.6}$$

defined by

$$df_p(v)(g) = v(g \circ f) \tag{2.7}$$

for all $v \in T_pM$ and $g \in \mathcal{F}(N)$.

At each point $p \in M$, the differential df_p is a linear function between the tangent spaces.

If (V, g) is another chart at $\alpha(p)$ and let $v = (f \circ \alpha)(t) - (f \circ \alpha)(p)$, then in $U \cap V$ we have

$$\begin{aligned}(g \circ f^{-1})((f \circ \alpha)(t)) &- (g \circ f^{-1})((f \circ \alpha)(p)) \\ &= df_p((g \circ f^{-1}))((f \circ \alpha)(p)).v + o(||v||),\end{aligned} \tag{2.8}$$

whence

$$\frac{(f \circ \alpha)(t) - (f \circ \alpha)(p)}{t - p} = \frac{df_p((g \circ f^{-1}))((f \circ \alpha)(p)).v}{t - p} + \frac{o(||v||)}{t - p}. \tag{2.9}$$

Since

$$\lim_{t \to p} = \frac{v}{t - p} = (f \circ \alpha)'(p) \qquad \text{and} \qquad \lim_{t \to p} \frac{o(||v||)}{t - p} = 0 \tag{2.10}$$

it follows that

$$\lim_{t \to p} \frac{(f \circ \alpha)(t) - (f \circ \alpha)(p)}{t - p} = df_p((g \circ f^{-1}))((f \circ \alpha)(p)).(f \circ \alpha)'(p) \tag{2.11}$$

and therefore the map $\alpha : I \to \mathbb{R}$ is differentiable at p in the chart (U, f) if and only if it is differentiable at p in the chart (V, g).

2.2 Lie Group

Definition 2.2.1 *A **Lie group** is a group that is also a smooth manifold such that the multiplication map for all $g, h \in G$*

$$\begin{aligned} \mu &: G \times G \to G \\ \mu &: (g, h) \mapsto gh \end{aligned} \tag{2.12}$$

and the inversion map,

$$\begin{aligned} \iota &: G \to G \\ \iota &: g \mapsto g^{-1} \end{aligned} \tag{2.13}$$

are smooth. We remind that the group axioms for group elements $g, g_1, g_2, g_3 \in G$

$$\begin{aligned} eg &= ge = g && (identity) \\ g^{-1}g &= gg^{-1} = e && (inverses) \\ g_1(g_2g_3) &= (g_1g_2)g_3 = g_1g_2g_3 && (associativity), \end{aligned} \tag{2.14}$$

and the group operations (multiplication and inversion) satisfy the group axioms.For a Lie group these operations must be differentiable mappings.

Example 2.4 *Firstly consider ($\mathbb{R}^n$,+). This is commutative group.The binary operations are;*

$$\begin{aligned} \mu(u, v) &= u + v = (u_1, u_2, ..., u_n) + (v_1, u_2, ..., v_n) \\ \iota(u) &= -u = (-u_1, -u_2, ..., -u_n). \end{aligned} \tag{2.15}$$

where $u, v \in \mathbb{R}^n$. These operations are differentiable, and they satisfy the group axioms. The manifold of the group is the vector space $\mathbb{R}^n$.

Example 2.5 *Let us take the group of unit modulus complex numbers. Any of the group*

has the form $z = e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$*, where the real number* φ *lies between zero and* 2π*,* $(0 \leq \varphi < 2\pi)$*. The group operation is complex multiplication;*

$$\begin{aligned} z_1 z_2 &= e^{i\varphi_1} e^{i\varphi_2} \\ &= e^{i(\varphi_1+\varphi_2)} = \cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2) \end{aligned} \tag{2.16}$$

This operation is continuous since addition and multiplication are continuous. The inverse of element is the complex conjugate, and hence the identity element is 1. Again this group is commutative. The group manifold has the topology of a unit circle.

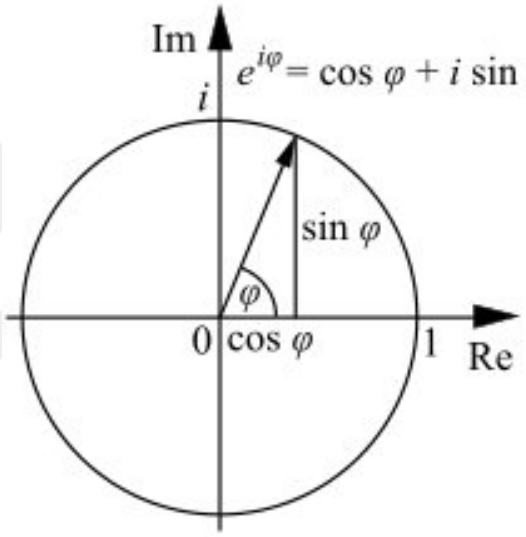


Figure 2.2 Unit circle

2.2.1 Matrix Lie Groups

We know that the group operations must be associative and have inverses, we are led to consider groups whose elements are square matrices. Certainly, matrix multiplication is associative, but not all square matrices have inverses. Now we restrict our attention to $n \times n$ matrices that are invertible, then we obtain Lie groups. Now we give list of matrix groups;

1. General Linear Group $\Rightarrow GL(n,\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : detA \neq 0\}$, $dimGL(n,\mathbb{R}) = n^2$.

2. Special Linear Group $\Rightarrow SL(n,\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : detA = 1\}$, $dimSL(n,\mathbb{R}) = n^2 - 1$.

3. General Linear Group $\Rightarrow GL(n,\mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) : detA \neq 0\}$, $dimGL(n,\mathbb{R}) = 2n^2$.

4. Special Linear Group $\Rightarrow SL(n,\mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) : detA = 1\}$, $dimSL(n,\mathbb{R}) = 2(n^2 - 1)$.

5. Orthogonal Group $\Rightarrow O(n) = \{A \in GL(n,\mathbb{R}) : AA^T = I\}$, $dimO(n) = \binom{n}{2}$.

6. Special Orthogonal Group $\Rightarrow SO(n) = \{A \in O(n) : detA = 1\}$, $dimSO(n) = \binom{n}{2}$.

7. Unitary Group $\Rightarrow U(n) = \{A \in GL(n,\mathbb{C}) : AA^* = I\}$, $dimU(n) = n^2$.

8. Special Unitary Group $\Rightarrow SU(n) = \{A \in U(n) : detA = 1\}$, $dimSU(n) = n^2 - 1$.

The above groups are matrix Lie groups. For example let's consider the general linear group, these group are usually denoted;

$$GL(n,\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : detA \neq 0\}, \tag{2.17}$$

with n the number of rows and columns in the matrices. The $\mathbb{R}$ refers to the field of scalars; that is, the matrix entries real numbers. When $n \geq 2$, $GL(n,\mathbb{R})$ is not commutative, since matrix multiplication is not in general commutative. Matrix multiplication is continuous though, since it is simply sequence of additions and multiplications. To specify an $n \times n$ matrix A, we must give its n^2 entries (a_{ij}). Hence, the group manifold of $GL(n,\mathbb{R})$ lies in $\mathbb{R}^{n^2}$, where the coordinates of $\mathbb{R}^{n^2}$ are given by the n^2 entries (a_{ij}). We conclude that the manifold of $GL(n,\mathbb{R})$ is an open set in $\mathbb{R}^{n^2}$ and has dimension n^2.

Of course several other matrix groups have been defined. But we are interested in $SO(2)$ and $SO(3)$. Because these groups will be important, since these are the rigid body rotations about a fixed center in two and three dimensions. The group $SO(2)$ consists of matrices of the form;

$$R(\varphi) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \tag{2.18}$$

Therefore we have a natural parametrization of $R(\varphi) \rightarrow SO(2)$. However, the parametrization is not injective because $R(\varphi) = R(\varphi + 2\pi i)$. We can have bijectivity by restricting the domain to the interval $[0, 2\pi)$. Thus there exists one to one correspondence between $SO(2)$ and unit modulus complex numbers. So the group manifold is a circle once more. The matrices of $SO(3)$ cannot be parameterised so neatly. Since,

$SO(3)$:Set of all rotations with axes through the origin.

Matrix Equation: $AA^T = I, detA = 1$.

System of equations in $\mathbb{R}^9$;

$$\begin{aligned} 1 &= a_{11}^2 + a_{21}^2 + a_{31}^2 \\ 1 &= a_{12}^2 + a_{22}^2 + a_{32}^2 \\ 1 &= a_{13}^2 + a_{23}^2 + a_{33}^2 \\ 0 &= a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} \\ 0 &= a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} \\ 0 &= a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} \end{aligned} \tag{2.19}$$

So we have 6 equations but we have 9 unknowns. How to solve these system of equations?

Theorem 2.2.2 *(Implicit Function Theorem)*
Given a smooth function $F : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$. *Suppose that* $F(x_0, y_0) = 0$ *and that the* $n \times n$ *matrix*

$$J = \frac{\partial F}{\partial y}_{|_{(x,y)=(x_0,y_0)}} = \left(\frac{\partial F_i}{\partial y_j}_{|_{(x,y)=(x_0,y_0)}} \right)_{i,j}. \tag{2.20}$$

is invertible $(detJ \neq 0)$. *Then there are open neighborhoods* $x_0 \in U \subset \mathbb{R}^k$, $y_0 \in V \subset \mathbb{R}^n$ *and a smooth function* $g : U \to V$ *with* $g(x_0) = y_0$ *such that* $F(x, y) = 0$ *holds if and only if* $y = g(x)$ *for* $(x, y) \in U \times V$.

Thus, we may solve the equations locally e.g. we find smooth functions $a_{11}(u, v, w)$, $a_{21}(u, v, w)$, $a_{31}(u, v, w)$, $a_{22}(u, v, w)$, $a_{32}(u, v, w)$, $a_{33}(u, v, w)$, $-0.5 < u, v, w < 0.5$ with $a_{ij}(0, 0, 0) = \delta_{ij}$ such that

$$A(u, v, w) = \begin{pmatrix} a_{11}(u, v, w) & u & v \\ a_{21}(u, v, w) & a_{22}(u, v, w) & w \\ a_{31}(u, v, w) & a_{32}(u, v, w) & a_{33}(u, v, w) \end{pmatrix} \tag{2.21}$$

The Jacobian of this matrix nonzero. And $-0.5 < u, v, w < 0.5$, are precisely the rotations "near 1" and for example $a_{33}(u, v, w) = \sqrt{1 - v^2 - w^2}$ from above system of equations for $a_{12}(u, v, w) = u$, $a_{13}(u, v, w) = v$, $a_{23}(u, v, w) = w$. Thus "locally" $SO(3)$ is smoothly equivalent to $\mathbb{R}^3$. There are another parametrization of $SO(3)$. Let $S_n := \{v \in \mathbb{R}^{n+1} : \|v\| = 1\}$ (n-sphere) and let $R(\varphi)$ be a rotation;

1. **Euler axis and angle**: Represent $R(\varphi)$ by a unit vector e in direction of the axis and angle φ.

2. **Euler angle**: Decompose $R(\varphi)$ into $R(\varphi) = Z_1 X Z_2$, where Z_1, Z_2 rotations around z-axis and X rotations x-axis.

3. Represent $R(\varphi)$ by a quaternion of length 1.

4. Other parameterizations...

But the parameter spaces are not open subset of $\mathbb{R}^k$. Hence we do not know what "differentiation" on S_n means and the parameterizations are not bijective. So we use always "local parameterizations" for differentiability in $SO(3)$. We will give detail of above parameterizations later.

2.2.2 Group Actions and Products

We may combine groups to obtain new groups. The simplest combination is direct product. Given two groups G and H, elements of their direct product, written $G \times H$, are pairs elements (g, h) where $g \in G$ and $h \in H$. The group operation is simply

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2). \tag{2.22}$$

Definition 2.2.3 *Consider a group G and a manifold X. We say that G acts on X if there is a differentiable map;*

$$\sigma : G \times X \to X, \tag{2.23}$$

that satisfies

1. $\sigma(e, x) = x, \qquad \forall x \in X.$

2. $\sigma(g_1, \sigma(g_2, x)) = \sigma(g_1 g_2, x), \qquad \forall x \in X, \forall g_1, g_2 \in G.$

So the σ is called group action.

Linear actions of groups on vector spaces are called **representations**, that is, actions which satisfy

$$\sigma(g, h_1 h_2) = g(h_1 + h_2) = g(h_1) + g(h_2), \qquad \forall g \in G, \forall h_1, h_2 \in H \tag{2.24}$$

Suppose we have a group G and a additive group H together a linear action of G on H. That is map $G \times H \to H$ given by $g(h)$ satisfying all three axioms given above. The **semi-direct product** of G and H written $G \ltimes H$,

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 + g_1(h_2)). \tag{2.25}$$

Since the second group is additive, we write the group operation here as "+";

$$(g, h)^{-1} = (g^{-1}, g^{-1}(h)). \tag{2.26}$$

Example 2.6 *Let's consider the matrix group $GL(n, \mathbb{R})$ an action of this group on $M_{n\times n}$ is given by*

$$S(g, M) = gMg^{-1} \tag{2.27}$$

A transformation with this shape is called a "similarity".

Example 2.7 *The second common action is given by*

$$C(g, M) = gMg^{-T} \tag{2.28}$$

A transformation with this shape is called a "congruence". If the original matrix symmetric ($M = M^T$), then transformed matrix also be symmetric.

2.2.3 The Proper (Special) Euclidean Group

This is the group of transformations of the vector space $\mathbb{R}^n$ that preserves the Euclidean metric. Transformations that preserves a metric are known as isometries, we could also called this group the isometry group of $\mathbb{R}^n$.

$$\| u - v \|^2 = \| u' - v' \|^2 \tag{2.29}$$

There are two possibilities: either we could transform vectors by adding a constant vector say t, to each or we could multiply the vectors by an orthogonal matrix.

More generally we may perform a combination of possibilities,

$$v' = Mv + t \tag{2.30}$$

where M is orthogonal matrix and t is a constant vector. (M, t), with $M \in O(n)$ and $t \in \mathbb{R}^n$. If $v' = M_1 v + t_1$ then

$$v'' = M_2 v' + t_1 = M_1 M_2 v + M_2 t_1 + t_2. \tag{2.31}$$

And so

$$(M_2, t_2)(M_1, t_1) = (M_1 M_2, M_2 t_1 + t_2). \tag{2.32}$$

The group of rigid body motions in $\mathbb{R}^n$ is thus the semi-product of the orthogonal group with $\mathbb{R}^n$ itself. We will denote it $E(n)$ for the Euclidean group

$$E(n) = O(n) \ltimes \mathbb{R}^n, \tag{2.33}$$

orthogonal matrices with determinant 1 corresponding to the rotations about the origin in $\mathbb{R}^n$,

$$SE(n) = SO(n) \ltimes \mathbb{R}^n, \tag{2.34}$$

There is a convenient (n+1)-dimensional representation of $SE(n)$, that is, an injective homomorphism $SE(n) \to GL(n+1)$ given by

$$(R, t) \to \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \tag{2.35}$$

where we have used a partitioned form for the matrix on the right. This is a homomorphism, since multiplying these matrices exactly replicates the product of the pairs

$$\begin{pmatrix} R_2 & t_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_1 & t_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_2 R_1 & R_2 t_1 + t_2 \\ 0 & 1 \end{pmatrix}. \tag{2.36}$$

The inverse of such a matrix is conveniently given by

$$\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R^T & -R^T t \\ 0 & 1 \end{pmatrix} \tag{2.37}$$

Since R is an orthogonal matrix.

CHAPTER THREE
LIE ALGEBRA AND THE EXPONENTIAL MAP

Lie groups are non-linear objects and their study requires quite a lot of effort. On the other hand, one of the simplest algebraic objects is that of a real vector space. In this chapter we will see that it is possible to associate to every point of a Lie group G a real vector space, which is the tangent space of the Lie group at that point, we will also see that it is enough to study the tangent space of a Lie group at its identity element e. The tangent space at that point is not only a vector space but, as we will see below, it is isomorphic to what is defined below to be the Lie algebra of the Lie group G.

3.1 Derivation and Vector Fields

Let M be a smooth manifold of dimension n and let p be a point in M. We write $C^\infty(p)$ for the set of smooth, real-valued functions on M whose domain of definition includes some open neighborhood of p. A map $X_p : C^\infty(p) \to \mathbb{R}$ is called a **derivation** if, for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(p)$, it satisfies,

1. $X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p g)$ (Linearity).
2. $X_p(fg) = X_p(f)g(p) + f(p)(X_p g)$ (Leibniz rule).

The set of all derivations $X_p : C^\infty(p) \to \mathbb{R}$ defines a vector space over the reals with the operations

$$\begin{aligned} (X_p + Y_p)f &= X_p f + Y_p f \\ (\alpha X_p)f &= \alpha(X_p f). \end{aligned} \tag{3.1}$$

The tangent space of M at a point p, denoted by T_pM, is the set of all derivations $X_p : C^\infty(p) \to \mathbb{R}$. Elements of the tangent space are called tangent vectors.

Let (U, ϕ) be a coordinate chart on M with local coordinates $(x_1, ..., x_n)$. Then, the set of derivations, $\{\frac{\partial}{\partial x_i}\}$ forms a basis for T_pM and hence we can write

$$X_p = X_1\frac{\partial}{\partial x_1} + X_2\frac{\partial}{\partial x_2} + ... + X_n\frac{\partial}{\partial x_n} \tag{3.2}$$

The vector $(X_1, ..., X_n) \in \mathbb{R}^n$ is a local coordinate representation of $X_p \in T_pM$.

Let $F : M \to N$ be a smooth map. We define the **tangent map** of F at p as the linear map $F_{*p} : T_pM \to T_{F(p)}N$ defined by

$$F_{*p}X_pf = X_p(f \circ F), \tag{3.3}$$

where $X_p \in T_pM$ and $f \in C^\infty(F(p))$. We also make use of the notation T_pF to denote the tangent map of F at p. The tangent map satisfies the following properties:

1. If $H = F \circ G$ is the composition of two smooth maps F and G, then $H_{*p} = F_{*G(p)} \circ G_{*p}$ or $T_pH = T_{G(p)}F \circ T_pG$.
2. If $F : M \to N$ is a diffeomorphism, then $F_{*p} : T_pM \to T_{F(p)}N$ is an isomorphism of tangent spaces with inverse $(F_{*p})^{-1} = (F^{-1})_{*F(p)}$.

If $c : M \to \mathbb{R}$ is a curve in M such that $c(0) = p$ and $\dot{c}(0) := c_{*t}(\frac{\partial}{\partial t}) \mid_{t=0} = X_p$, then $T_pF(X_p) = \frac{d}{dt}F(c(t)) \mid_{t=0}$.

The tangent bundle of M, denoted TM, is a manifold of dimension $2n$ defined by

$$TM = \bigsqcup_{p \in M} T_pM. \tag{3.4}$$

An element of TM will be written as (p, X_p) or simply X_p where $p \in M$ and $X_p \in T_pM$. There is a natural projection $\pi : TM \to M$ given by $\pi(X_p) = p$.

A smooth **vector field** X on a manifold M is defined as a smooth map $X : M \to TM$ satisfying $\pi \circ X = id$, where $\pi : TM \to M$ is the canonical projection and

$id : M \to M$ is the identity map on M. We let $\Psi(M)$ denote the set of all smooth vector fields on M. Relative to a coordinate chart (U, ϕ), a vector field is written as

$$X(x) = X_1(x)\frac{\partial}{\partial x_1} + X_2(x)\frac{\partial}{\partial x_2} + ... + X_n(x)\frac{\partial}{\partial x_n} \tag{3.5}$$

where each X_i is a smooth function defined on an open neighborhood of $x = \phi(p)$. It is customary to write a vector field as a column vector

$$X(x) = \begin{pmatrix} X_1(x) \\ . \\ . \\ . \\ X_n(x) \end{pmatrix} \tag{3.6}$$

Vector fields represent differential equations on manifolds. Let $c : (a, b) \to M$ be a curve on the manifold. The curve c is said to be an integral curve of the vector field X if

$$\dot{c}(t) = X(c(t)). \tag{3.7}$$

By the existence and uniqueness theorem for ordinary differential equations, the existence of integral curves for a given nonzero vector field is guaranteed locally. The vector field is said to be complete if the domain of definition of the integral curves can be chosen to be $(-\infty, \infty)$. In this case, the integral curves of a vector field define a one-parameter family of diffeomorphisms $\Phi_t(q) : M \to M$ with the understanding that $\Phi_t(q)$ is the point on the integral curve starting from initial condition q at $t = 0$.

This one parameter family of diffeomorphisms is referred to as the flow of the vector field X. Let X be a smooth vector field and $f \in C^\infty(p)$ a smooth function on M.

The Lie derivative of f with respect to X is a new function $Xf : M \to \mathbb{R}$ defined by

$$Xf(p) = X_p f. \tag{3.8}$$

In coordinate chart (U, ϕ), if we write $X = \sum_{i=1}^{n} X_i(x) \frac{\partial}{\partial x_i}$, then

$$Xf(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} X_i(x), \tag{3.9}$$

where all partial derivatives are evaluated at $x = \phi(p)$.

Let X and Y be two smooth vector fields. The Lie bracket of X and Y, denoted $[X, Y]$, is a new vector field defined by

$$[X, Y]f = X(Yf) - Y(Xf). \tag{3.10}$$

In local coordinates, if we write $X = \sum_{i=1}^{n} X_i(q) \frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^{n} Y_i(q) \frac{\partial}{\partial x_i}$ then the Lie bracket vector field $[X, Y]$ is given by

$$[X, Y] = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial Y_j}{\partial x_i} X_i - \frac{\partial X_j}{\partial x_i} Y_i \right) \frac{\partial}{\partial x_j} \tag{3.11}$$

A vector space V (over $\mathbb{R}$) is a **Lie algebra** if there exists a bilinear operator $V \times V \to V$, denoted $[.,.]$, satisfying

1. **Skew-symmetry**: $[v, w] = -[w, v]$ for all $v, w \in V$
2. **Jacobi identity**:

$$[[v, w], z] + [[z, v], w] + [[w, z], v] = 0 \tag{3.12}$$

for all $v, w, z \in V$

A subspace $W \subset V$ is called a **Lie subalgebra** if $[v, w] \in W$ for all $v, w \in W$. The vector space of all smooth vector fields on a manifold M is an infinite-dimensional Lie algebra under the Lie bracket operation on vector fields.

3.2 The Lie algebra associated with a Lie group

Let X be a vector field on G. X is left invariant if $(L_g)_* X = X$, that is

$$T_h L_g X(h) = X(gh) \qquad \text{for all } h \in G. \tag{3.13}$$

Let $\Psi_L(G)$ be the set of left invariant vector fields on G. Then, for $X, Y \in \Psi_L(G)$ we have

$$L_{g*}[X, Y] = [L_{g*}X, L_{g*}Y] = [X, Y]. \tag{3.14}$$

Thus, $\Psi_L(G)$ is a Lie subalgebra of the Lie algebra $\Psi(G)$ the set of all vector fields on G.

For each $\xi \in T_e G$, we define a vector field X_ξ on G by

$$X_\xi = T_e L_g \xi. \tag{3.15}$$

Since

$$\begin{aligned} X_\xi(gh) &= T_e L_{gh}\xi = T_e(L_g \circ L_h).\xi \\ &= T_h L_g(T_e L_h \xi) = T_h L_g(X_\xi(h)), \end{aligned} \tag{3.16}$$

X_ξ is left invariant. The linear maps $\rho_1 : \Psi_L(G) \to T_e G$ given by

$$\rho_1(X) = X_e \tag{3.17}$$

and $\rho_2 : T_e G \to \Psi_L(G)$ given by

$$\rho_2(\xi) = X_\xi \tag{3.18}$$

satisfy $\rho_1 \circ \rho_2 = id_{T_e G}$ and $\rho_2 \circ \rho_1 = id_{\Psi_L(G)}$. Hence $\Psi_L(G)$ and $T_e G$ are isomorphic.

Defining a Lie bracket in T_eG by

$$[\xi_1, \xi_2] = [X_{\xi_1}, X_{\xi_2}](e), \qquad \xi_1, \xi_2 \in T_eG \tag{3.19}$$

makes T_eG into a Lie algebra. The vector space T_eG with this Lie algebraic structure is called the Lie algebra of G and is denoted $\mathfrak{g}$. A Lie subalgebra of $\mathfrak{g}$ is a subspace$\mathfrak{h} \subset \mathfrak{g}$ such that $\xi, \eta \in \mathfrak{h}$ implies that$[\xi, \eta] \in \mathfrak{h}$. It can be shown that if H is a Lie subgroup of G with Lie algebra $\mathfrak{h}$ then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.

3.2.1 Examples of the Lie algebra

Example 3.1 ***The Lie algebra of (***$\mathbb{R}^n, +$***)***

For the group $\mathbb{R}^n$ we have $e = 0$, $T_0\mathbb{R}^n \cong \mathbb{R}^n$, and it is easy to see that the left invariant vector field defined by $v \in T_0\mathbb{R}^n$ is the constant vector field: $X_v(x) = v$ for all $x \in \mathbb{R}^n$. Therefore, the Lie algebra of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself, with the trivial Lie bracket $[v_1, v_2] = 0$ for all $v_1, v_2 \in \mathbb{R}^n$.

Example 3.2 ***The Lie algebra of (***$GL(n, \mathbb{R})$***)***

The Lie algebra of $GL(n, \mathbb{R})$ is the set of all $n \times n$ real matrices, denoted $\mathfrak{gl}(n, \mathbb{R})$, with the bracket structure

$$[A, B] = AB - BA \qquad A, B \in \mathfrak{gl}(n, \mathbb{R}) \tag{3.20}$$

To derive this, note that $GL(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$ and hence $T_eGL(n, \mathbb{R}) \cong \mathbb{R}^{n\times n}$. A vector field on $GL(n, \mathbb{R})$ can be written as

$$X(x) = \sum_{i,j} X_{ij}(x) \frac{\partial}{\partial x_{ij}} \tag{3.21}$$

where each x_{ij} represents a coordinate in $\mathbb{R}^{n\times n}$.

Example 3.3 ***The Lie algebra of (***$SO(3)$***)***

The Lie algebra of $SO(3)$, denoted $so(3)$,may be identified with the 3×3

skew-symmetric matrices of the form

$$\hat{w} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \tag{3.22}$$

with the bracket structure

$$[\hat{w}_1, \hat{w}_2] = \hat{w}_1\hat{w}_2 - \hat{w}_2\hat{w}_1, \qquad \hat{w}_1, \hat{w}_2 \in so(3) \tag{3.23}$$

We can identify $so(3)$ *with* $\mathbb{R}^3$ *using the mapping in above skew-symmetric matrix equation which maps a vector* $w \in \mathbb{R}^3$ *to a matrix* $\hat{w} \in so(3)$*. It is straight forward to show that*

$$[\hat{w}_1, \hat{w}_2] = (\hat{w}_1 \times \hat{w}_2) \qquad \hat{w}_1, \hat{w}_2 \in \mathbb{R}^3 \tag{3.24}$$

Thus $w \mapsto \hat{w}$ *is a Lie algebra isomorphism between the Lie algebra* $(\mathbb{R}^3, \times)$ *and the Lie algebra (*$so(3), [.,.]$*).*

3.3 The Exponential Map

For every $\xi \in T_eG$, let $\phi_\xi : \mathbb{R} \to G$ denote the integral curve of the left invariant vector field X_ξ passing through e at $t = 0$. That is $\phi_\xi(0) = e$ and $\frac{d}{dt}\phi_\xi(t) = X_\xi(\phi_\xi(t))$. It follows from its definition that

$$\phi_\xi(s + t) = \phi_\xi(s)\phi_\xi(t), \tag{3.25}$$

which means that $\phi_\xi(t)$ is a one-parameter subgroup of G. Indeed, as functions of t, both sides are equal at $t = 0$ and both satisfy the differential equation $\dot{\sigma} = X_\xi(\sigma(t))$ by left invariance of X_ξ, so they are equal by the uniqueness of integral curves. Using either left invariance or the fact that $\phi_\xi(s+t) = \phi_\xi(s)\phi_\xi(t)$ shows that $\phi_\xi(t)$ is defined for all $t \in \mathbb{R}$.

The function $\exp : T_eG \to G$ defined by $\exp(\xi) = \phi_\xi(1)$ is called the **exponential map** of the Lie algebra $\mathfrak{g}$ into G. The exponential map takes the line $\xi s \in \mathfrak{g}$, $s \in \mathbb{R}$, into a one-parameter subgroup of G, i.e.,

$$\exp(\xi s) = \phi_\xi(s) \tag{3.26}$$

To verify this formula, note that for fixed s, the curve $t \mapsto \phi_\xi(ts)$ which at $t = 0$ passes through e, satisfies the differential equation

$$\frac{d}{dt}\phi_\xi(ts) = sX_\xi(\phi_\xi(ts)) = X_{s\xi}(\phi_{s\xi}(t)). \tag{3.27}$$

Since $\phi_{s\xi}(t)$ and $\phi_\xi(ts)$ satisfy the same differential equation and both pass through e at $t = 0$, it follows that $\phi_{s\xi}(t) = \phi_\xi(ts)$. Putting $t = 1$ yields $\exp(\xi s) = \phi_\xi(s)$. In fact, all connected one-parameter subgroups of G are obtained in this way.

Differentiating the map $\exp(\xi s) = \phi_\xi(s)$ with respect to s at $s = 0$ shows that $T_e \exp = id_{T_eG}$. Therefore, by the inverse function theorem, $\exp : \mathfrak{g} \to G$ is a local diffeomorphism from a neighborhood of zero in $\mathfrak{g}$ onto a neighborhood of e in G. If G is compact it can be shown that the exponential map is surjective.

3.3.1 Examples of the Exponential Map

Example 3.4 ***The exponential map of*** $(\mathbb{R}^n, +)$

Consider $\mathbb{R}^n$ with addition as the group operation. The Lie algebra of $\mathbb{R}^n$ is $\mathbb{R}^n$ with the trivial bracket and the integral curve of a left invariant vector field $X_v(x) = v$ is given by $\phi_v(t) = vt$. Thus $\phi_v(t) \circ x = x + vt$ and $\exp : \mathbb{R}^n \to \mathbb{R}^n$ *is the identity.*

Example 3.5 ***The exponential map of*** $(GL(n, \mathbb{R}))$

Let $G = GL(n, \mathbb{R}) \subset \mathbb{R}^{n\times n}$ so that $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^{n\times n}$ with $[A, B] = AB - BA$.

For every $A \in \mathfrak{gl}(n, \mathbb{R})$*, the mapping* $\Phi_A : \mathbb{R} \to GL(n, \mathbb{R})$ *given by*

$$\Phi_A(t) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \tag{3.28}$$

is a one-parameter subgroup because $\Phi_A(0) = I$ *and*

$$\frac{d}{dt}\Phi_A(t) = \sum_{n=1}^{\infty} \frac{t^{n-1} A^n}{(n-1)!} = A\Phi_A(t), \tag{3.29}$$

which shows that Φ_A *is an integral curve of the left invariant vector field* X_A*. Therefore, the exponential map* $\exp : \mathfrak{gl}(n, \mathbb{R}) \to GL(n, \mathbb{R})$ *is given by*

$$\exp(A) = \Phi_A(1) = \sum_{n=0}^{\infty} \frac{A^n}{n!}. \tag{3.30}$$

CHAPTER FOUR
KINEMATIC COMPARISON OF EXPONENTIAL AND CAYLEY MAPS IN THE PLANAR MOTION GROUP

In this chapter, the exponential and the Cayley maps, from the Lie algebra $\mathfrak{se}(2)$ of the planar motion group $SE(2)$, to the group itself are investigated. The comparison between these maps on $SE(2)$ is given by means of the Rodrigues vector and as an application a three joint mechanism is discussed.

4.1 The Matrix Lie Group in the Plane Kinematics

Planar motion group $SE(2)$ is the matrix Lie group in the plane kinematics. It can be represented as,

$$S = \begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}.$$

where R is a rotation matrix and t is translation vector in Euclidean space $\mathbb{E}^2$. Here S defines a general rigid transformation, which is a rotation about a fixed point, the center of rotation and a translation along a vector $t = (t_x, t_y)^T$.

If $q = (q_x, q_y)^T$, is the center of rotation, then rotation about this point is given by the conjugation,

$$\begin{aligned} S &= \begin{pmatrix} I_2 & q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_2 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & (I_2 - R)q \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta & q_y\sin\theta + q_x(1-\cos\theta) \\ \sin\theta & \cos\theta & q_y(1-\cos\theta) - q_x\sin\theta \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

Thus we find,

$$(I_2 - R)q = t,$$

yields the fixed point or center of rotation q, where we avoid taking a pure translation, i.e. $R \neq I_2$(for details see .

4.2 The Exponential and The Cayley Maps

Let G be a matrix Lie group and $\mathfrak{g}$ be its Lie algebra. The exponential map, exp : $\mathfrak{g} \to G$, can be defined by the ordinary series;

$$e^{\Omega} = \sum_{k=0}^{\infty} \frac{\Omega^k}{k!}.$$

where Ω is an $n \times n$ matrix.

Consider the exponential mapping on rotation group $SO(2)$, which is the set of 2×2 real orthogonal matrices with determinant $+1$. Its Lie algebra $\mathfrak{so}(2)$ consists of 2×2 real skew-symmetric matrices. Let

$$\Omega = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix},$$

be an element of $\mathfrak{so}(2)$. We need to find inductive formula expressing the powers Ω^n. If

$$\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

then

$$\begin{aligned} \Omega^2 &= -\theta^2 I_2, \\ \Omega^3 &= -\theta^3 J_2, \\ \Omega^4 &= \theta^4 I_2, \end{aligned}$$

where

$$J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

Therefore,

$$\begin{aligned} \Omega^{4n} &= \theta^{4n} I_2, \\ \Omega^{4n+1} &= \theta^{4n+1} J_2, \\ \Omega^{4n+2} &= -\theta^{4n+2} I_2, \\ \Omega^{4n+3} &= -\theta^{4n+3} J_2. \end{aligned}$$

So one can obtain the Lie group element,

$$\begin{aligned} e^{\Omega} &= I_2 + \frac{\theta}{1!} J_2 - \frac{\theta^2}{2!} I_2 - \frac{\theta^3}{3!} J_2 + \frac{\theta^4}{4!} I_2 + \frac{\theta^5}{5!} J_2 - \frac{\theta^6}{6!} I_2 - \frac{\theta^7}{7!} J_2 .... \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...\right) I_2 + \left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...\right) J_2 \\ &= \cos\theta I_2 + \sin\theta J_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \end{aligned}$$

e^{Ω} is an orthogonal matrix of determinant $+1$, which defines rotation as it is expected.

To investigate the exponential map on $SE(2)$ we have to find the Lie algebra element close to the identity by differentiating the Lie group element S at $\theta = 0$. Hence

$$\frac{d}{d\theta} S\big|_{\theta=0} = K,$$

or

$$\begin{aligned} K = \frac{d}{d\theta} \begin{pmatrix} \cos\theta & -\sin\theta & q_y \sin\theta + q_x(1-\cos\theta) \\ \sin\theta & \cos\theta & q_y(1-\cos\theta) - q_x \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \Bigg|_{\theta=0} &= \begin{pmatrix} 0 & -1 & q_y \\ 1 & 0 & -q_x \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} J_2 & -J_2 q \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

where K satisfies the following cubic equation;

$$K^3 + K = 0.$$

A general element of the Lie algebra can be written as θK (see). Thus

$$\begin{aligned} e^{\theta K} &= I_3 + \frac{\theta}{1!}K + \frac{\theta^2}{2!}K^2 - \frac{\theta^3}{3!}K - \frac{\theta^4}{4!}K^2 + \frac{\theta^5}{5!}K + \frac{\theta^6}{6!}K^2 - \frac{\theta^7}{7!}K.... \\ &= I_3 + \left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...\right)K + \left(1 - 1 + \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - ...\right)K^2 \\ &= I_3 + \sin\theta K + (1 - \cos\theta)K^2, \end{aligned}$$

where I_3 is the 3×3 identity matrix and this formula obeys the familiar form of the Rodrigues formula in $SO(3)$. Hence we obtain,

$$e^{\theta K} = \begin{pmatrix} \cos\theta & -\sin\theta & q_y \sin\theta + q_x(1 - \cos\theta) \\ \sin\theta & \cos\theta & q_y(1 - \cos\theta) - q_x \sin\theta \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.1}$$

Similar to the exponential map, there exists a rational polynomial map from the skew-symmetric matrices to the orthogonal matrices, which is known as the Cayley map.

Definition 4.2.1 *Let $A \in \mathfrak{g}$ be an $n \times n$ skew-symmetric matrix then the map, $Cay : \mathfrak{g} \to G$, to the group of orthogonal matrices is called the Cayley map, where*

$$Cay(A) = (I_n + A)(I_n - A)^{-1},$$

and I_n is an $n \times n$ identity matrix.

Let we choose,

$$A = \begin{pmatrix} 0 & -a_{12} \\ a_{12} & 0 \end{pmatrix}, \tag{4.2}$$

so

$$Cay(A) = (I_2 + A)(I_2 - A)^{-1}, \tag{4.3}$$

and the inverse of $(I_2 - A)$ can be written as

$$(I_2 - A)^{-1} = I_2 + A + A^2 + A^3 + ..., \tag{4.4}$$

and (4.4) converges in some neighborhood of $A = 0$. Since A satisfies its charachteristic equation,

$$A^2 + a_{12}^2 I_2 = 0. \tag{4.5}$$

From (4.4) and (4.5) we have

$$(I_2 - A)^{-1} = \frac{1}{1 + a_{12}^2}(I_2 + A).$$

Therefore

$$\begin{aligned} Cay(A) &= (I_2 + A)(I_2 + A)\frac{1}{1 + a_{12}^2} \\ &= \left(\frac{1 - a_{12}^2}{1 + a_{12}^2}\right) I_2 + \left(\frac{2}{1 + a_{12}^2}\right) A. \end{aligned} \tag{4.6}$$

The result (5.30) in $SO(2)$ can be extended to the group of planar motion $SE(2)$. Let M be an element of $SE(2)$;

$$M = \begin{pmatrix} \Re & u \\ 0 & 1 \end{pmatrix},$$

where $\Re$ is a 2×2 rotation matrix and $u = (u_1, u_2)^T$ is a translation vector in $\mathbb{E}^2$. So the Lie algebra elements can be written as,

$$m = \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix},$$

where A is an 2×2 skew-symmetric matrix and $v = (v_1, v_2)^T$ is a velocity vector.

Cayley map is originally defined on skew-symmetric matrices. Hovewer using formula (5.36) one can extend Cayley map to $SE(2)$, by

$$Cay_3(m) = (I_3 + m)(I_3 - m)^{-1},$$

subscript 3 denotes the 2×2 representation of $SE(2)$. m satisfies its charachteritic equation,

$$m^3 + a_{12}^2 m = 0,$$

This leads to the formula,

$$\begin{aligned} Cay_3(m) &= (I_3 + m)(I_3 - m)^{-1} \\ &= I_3 + \frac{2}{1 + a_{12}^2}\left(m + m^2\right) \\ &= \begin{pmatrix} 1 - \frac{2a_{12}^2}{1+a_{12}^2} & \frac{2a_{12}}{1+a_{12}^2} & u_1 \\ -\frac{2a_{12}}{1+a_{12}^2} & 1 - \frac{2a_{12}^2}{1+a_{12}^2} & u_2 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{4.7}$$

4.3 Comparison Between The Exponential and The Cayley Maps

The equivalance of the range spaces of Cayley and exponential maps motivates us to compare them. Therefore $e^{\theta K} = Cay_3(m)$ implies

$$\begin{pmatrix} \cos\theta & -\sin\theta & q_y \sin\theta + q_x(1 - \cos\theta) \\ \sin\theta & \cos\theta & q_y(1 - \cos\theta) - q_x \sin\theta \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{2a_{12}^2}{1+a_{12}^2} & \frac{2a_{12}}{1+a_{12}^2} & u_1 \\ -\frac{2a_{12}}{1+a_{12}^2} & 1 - \frac{2a_{12}^2}{1+a_{12}^2} & u_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence from the rotation parts of the matrices;

$$\begin{aligned} 1 - \frac{2a_{12}^2}{1 + a_{12}^2} &= \cos\theta \\ \frac{2a_{12}}{1 + a_{12}^2} &= \sin\theta. \end{aligned} \tag{4.8}$$

Thus,

$$a_{12} = \tan(\frac{\theta}{2}). \tag{4.9}$$

As a result we obtain the relation between the magnitude of the eigenvalue of a skew symmetric matrix A (which is used in the Cayley map), and the tangent of the half of the relation angle θ.

From the translation parts;

$$\begin{aligned} u_1 &= q_y \sin\theta + q_x(1 - \cos\theta) \\ u_2 &= q_y(1 - \cos\theta) - q_x \sin\theta. \end{aligned} \tag{4.10}$$

At this moment we would like to discuss the kinematic meaning of a_{12}. If we take a rigid body which moves on a plane (like a mobile robot motion) then the motion of this rigid body only relates with $SE(2)$. So any point with position vector $\vec{r_1}$ on the rigid body rotates to $\vec{r_2}$ around the rotation axis by the angle θ, since $\vec{r_1'}$ rotates $\vec{r_2'}$ onto the plane which is perpendicular to rotation axis with the same angle θ, where $\vec{r_1'}$ and $\vec{r_2'}$ are projections of $\vec{r_1}$ and $\vec{r_2}$ respectively.

Now we import a vector $\vec{b}$, which is parallel to the rotation axis (z-axis here) and well known by the Rodrigues vector, to our computations . From fig 4.1 it follows that

$$\tan\left(\frac{\theta}{2}\right) = \frac{\frac{|\vec{r_2} - \vec{r_1}|}{2}}{\frac{|\vec{r_1} + \vec{r_2}|}{2}} = \frac{|\vec{r_2} - \vec{r_1}|}{|\vec{r_1} + \vec{r_2}|}. \tag{4.11}$$

Since the diagonals of the rhombus are perpendicular to each other

$$(\vec{r_1'} - \vec{r_2'})^T.(\vec{r_1'} + \vec{r_2'}) = 0, \tag{4.12}$$

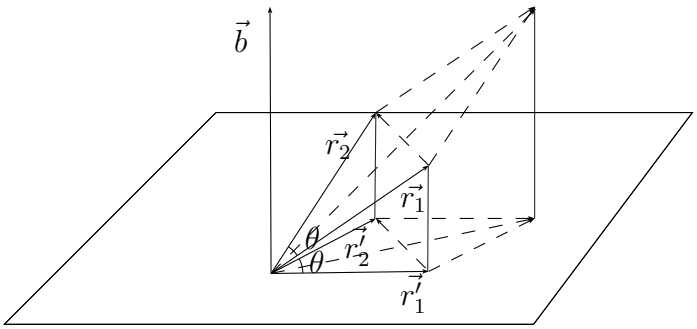


Figure 4.1 The instantaneous rotation axis for a rigid body and the rhombuses drawn by $\vec{r_1}$, $\vec{r_2}$ and $\vec{r_1'}$, $\vec{r_2'}$.

where $\vec{r_2'} = R\vec{r_1'}$. So

$$\vec{r_2'} - \vec{r_1'} = R\vec{r_1'} - \vec{r_1'} = (R-I)\vec{r_1'},$$

$$\vec{r_2'} + \vec{r_1'} = (R+I)\vec{r_1'},$$

$$\vec{r_1'} = (R+I)^{-1}(\vec{r_2'} + \vec{r_1'}),$$

$$(\vec{r_2'} - \vec{r_1'}) = (R-I)(R+I)^{-1}(\vec{r_2'} + \vec{r_1'}). \tag{4.13}$$

The matrix $A = (R-I)(R+I)^{-1}$ in (4.13) has the property that for a general vector $\vec{n}$ (one can specify $\vec{r_1'}$ and $\vec{r_2'}$ to obtain $\vec{r_1'} + \vec{r_2'} = \vec{n}$), $A.\vec{n}$ is perpendicular to $\vec{n}$ by (4.12).

Hence

$$\vec{n}^T A\vec{n} = \sum_{i,j}(a_{ij} + a_{ji})n_i n_j = 0, \qquad \forall \vec{n} \in \mathbb{E}^2, \tag{4.14}$$

where a_{ij} are the entries of A. So $a_{ij} = -a_{ji}$, that is A is skew symmetric and can be taken as (4.2). If we extend A and $\vec{n}$ to $\mathbb{E}^3$ by

$$A' = \begin{pmatrix} 0 & -a_{12} & 0 \\ a_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, n' = \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} \tag{4.15}$$

then

$$A'.\vec{n}' = \vec{b} \times \vec{n}', \text{where } \vec{b} = (0, 0, a_{12}), \tag{4.16}$$

is the Rodrigues vector.

Finally from (4.9), (4.11) and (4.16)

$$|\vec{b}| = \tan\left(\frac{\theta}{2}\right) = |a_{12}|. \tag{4.17}$$

(4.17) is the relation which combines Rodrigues vector, the exponential map and the Cayley map.

On the other hand

$$\begin{aligned}
\frac{\vec{r_2'} - \vec{r_1'}}{|\vec{r_2'} - \vec{r_1'}|} &= \vec{s} \times \frac{\vec{r_2'} + \vec{r_1'}}{|\vec{r_2'} + \vec{r_1'}|} \\
\vec{r_2'} - \vec{r_1'} &= \frac{|\vec{r_2'} - \vec{r_1'}|}{|\vec{r_2'} + \vec{r_1'}|}\vec{s} \times (\vec{r_2'} + \vec{r_1'}) \\
&= |\vec{b}|\vec{s} \times (\vec{r_2'} + \vec{r_1'}) \\
&= (\tan\left(\frac{\theta}{2}\right)\vec{s}) \times (\vec{r_1'} + \vec{r_2'}) \\
&= |a_{12}|\vec{s} \times (\vec{r_1'} + \vec{r_2'}),
\end{aligned} \tag{4.18}$$

where $\vec{s}$ is the unit vector along $\vec{b}$ and the equation (4.18) is the Rodrigues formula for a planar displacement (for the dual form see also).

4.3.1 Application (A Three-Link Mechanism)

Consider the three-link mechanism shown in Fig.2. This planar mechanism has three degrees of freedom to specify the position of a link in a plane. We can use one point $(x, y) \in \mathbb{E}^2$ and angle θ to represent the position and orientation of a link. The joint variables are $\theta_1, \theta_2, \theta_3$, and length of the links are just q_1, q_2, q_3. By taking $\theta_1 = \theta_2 = \theta_3 = 0$ we define a convenient home configuration. For a final position of manipulator the following rotations are taken;

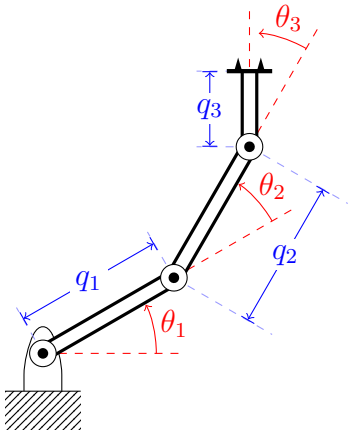


Figure 4.2 A three-link planar manipulator

First rotate joint 3 by θ_3 with center of rotation $((q_1 + q_2), 0)$;

$$A(\theta_3) = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & (q_1+q_2)(1-\cos\theta_3) \\ \sin\theta_3 & \cos\theta_3 & -(q_1+q_2)\sin\theta_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Rotate joint 2 by θ_2 with center of rotation $(q_1, 0)$;

$$A(\theta_2) = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & q_1(1-\cos\theta_2) \\ \sin\theta_2 & \cos\theta_2 & -q_1\sin\theta_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally rotate joint 1 by θ_1 with center of rotation $(0, 0)$;

$$A(\theta_1) = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the effect of such a movement is given by the product of the three matrices;

$$A(\theta_1)A(\theta_2)A(\theta_3) = \begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & l_x \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & l_y \\ 0 & 0 & 1 \end{pmatrix}.$$

where the unknown variables l_x and l_y are given by;

$$\begin{aligned} l_x &= q_1 \cos\theta_1 + q_2 \cos(\theta_1 + \theta_2) - (q_1 + q_2) \cos(\theta_1 + \theta_2 + \theta_3) \\ l_y &= q_1 \sin\theta_1 + q_2 \sin(\theta_1 + \theta_2) - (q_1 + q_2) \sin(\theta_1 + \theta_2 + \theta_3). \end{aligned}$$

The forward kinematics of the mechanism can be derived by the product of $A(\theta_1)A(\theta_2)A(\theta_3)$,

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1 + \theta_2 + \theta_3) & -\sin(\theta_1 + \theta_2 + \theta_3) & l_x \\ \sin(\theta_1 + \theta_2 + \theta_3) & \cos(\theta_1 + \theta_2 + \theta_3) & l_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 + q_2 + q_3 \\ 0 \\ 1 \end{pmatrix}.$$

Hence we find the kinematic equations;

$$\begin{aligned} x &= q_1 \cos(\theta_1) + q_2 \cos(\theta_1 + \theta_2) + q_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ y &= q_1 \sin(\theta_1) + q_2 \sin(\theta_1 + \theta_2) + q_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ \theta &= \theta_1 + \theta_2 + \theta_3. \end{aligned}$$

Since the matrix multiplication $A(\theta_1)A(\theta_2)A(\theta_3)$ is the exponential matrix as the equation (4.1), we can compare the matrix $A(\theta_1)A(\theta_2)A(\theta_3)$ and the Cayley matrix (5.10). Then

$$a_{12} = \tan\left(\frac{(\theta_1 + \theta_2 + \theta_3)}{2}\right),$$

as in (5.19), and

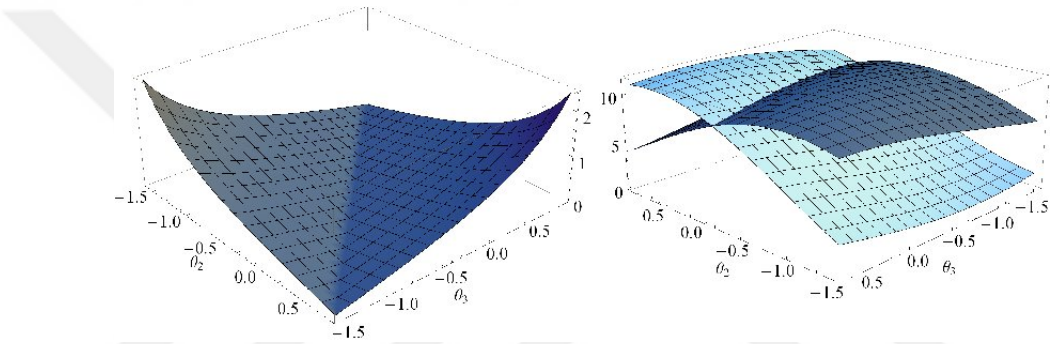
$$\begin{aligned} u_1 &= q_1 \cos\theta_1 + q_2 \cos(\theta_1 + \theta_2) - (q_1 + q_2) \cos(\theta_1 + \theta_2 + \theta_3) \\ u_2 &= q_1 \sin\theta_1 + q_2 \sin(\theta_1 + \theta_2) - (q_1 + q_2) \sin(\theta_1 + \theta_2 + \theta_3), \end{aligned} \tag{4.19}$$

as in (4.10). The equation (4.17) gives the length of the Rodrigues vector and the equation (4.19) yields the position of joints in the plane.

For a 3D-graph of length of the Rodrigues vector and the joint positions of three-link

planar mechanism we need two free variables. Taking θ_1 constant we have θ_2, θ_3 free variables. Let we choose $\theta_1 = \frac{\pi}{4}$, $q_1 = 8$,$q_2 = 4$, $q_3 = 2$, $\theta_2, \theta_3 \in \left[-\frac{\pi}{2}, \frac{\pi}{4}\right]$. Hence we get

$$\begin{aligned} x &= 4\sqrt{2} + 4\cos(\frac{\pi}{4} + \theta_2) + 2\cos(\frac{\pi}{4} + \theta_2 + \theta_3) \\ y &= 4\sqrt{2} + 4\sin(\frac{\pi}{4} + \theta_2) + 2\sin(\frac{\pi}{4} + \theta_2 + \theta_3) \\ \theta &= \frac{\pi}{4} + \theta_2 + \theta_3. \end{aligned} \tag{4.20}$$



(a) Length of Rodrigues vector (b) Joint positions

Figure 4.3 The length of Rodrigues vector and the location of joint positions with $-\frac{\pi}{2} \leq \theta_2 \leq \frac{\pi}{4}$ (dark) and $-\frac{\pi}{2} \leq \theta_3 \leq \frac{\pi}{4}$ (light).

Figure 3(a) represents the length of the Rodrigues vector $\vec{b}$ depending on θ_2 and θ_3, and also figure 3(b) is the joint positions with the same variables, θ_2 and θ_3. These two figures have information about the eigenvalue of the Lie algebra element a_{12} and the position of the center of rotation, u_1, u_2, in the plane respectively.

CHAPTER FIVE
CONTINUOUS DISPLACEMENTS WITH EXPONENTIAL MAPPING IN THE PLANAR MOTION GROUP

This chapter deals with the planar motion of a rigid body relative to a reference system with exponential mapping. As a body moves, all of its points trace trajectories in the reference system. These trajectories are studied in terms of their instantaneous differential properties. When time is used as the motion parameter, the study is referred to as time-dependent kinematics. Although it is most common to think of motions as being functions of time, the concept of time is not necessary for defining many important properties that characterize the trajectories produced by a motion. Eliminating time from the development allows for much simpler derivations and clearer understanding of the geometric properties of the trajectory. The term geometric kinematics is used to describe the study of time-independent kinematic properties. When studying the instantaneous differential properties of a motion, it is possible to take advantage of the intrinsic properties of the motion in order to choose a coordinate system that yields the simplest analytical description. This system is called the canonical coordinate system. The parameters that define a specific motion in such a system are called the instantaneous invariants of the motion. This chapter shows that how to relate instanteneous invariants and the exponential mapping in the palnar motion.

5.1 Active and Passive Transformation

To define rigid body displacement, we can use active coordinate transformation and passive coordinate transformation. In the active coordinate transformation, we have a fixed coordinate system and the transformation matrix change the points. In the passive coordinate transformation is a change in the position of the coordinate system or change of basis. These two coordinate transformations are equal. The passive coordinate transformation is the inverse of active coordinate transformation.

We take an arbitrary point P in the fixed plane Σ_f and this point has position vector

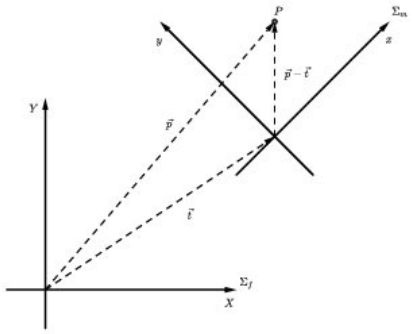


Figure 5.1 Change of Coordinate Frames

$\vec{p'} = \vec{p} - \vec{t}$ in the moving frame Σ_m see figure 1. If we choose basis vectors of the moving frame e_1^m, e_2^m then we find

$$e_1^m = Re_1^f, \qquad e_2^m = Re_2^f \tag{5.1}$$

where R is a rotation matrix and e_1^f, e_2^f is a basis vector of a fixed frame. In the fixed frame, the position vector of an arbitrary point $\vec{p}$, is given by

$$\vec{p} = Xe_1^f + Ye_2^f \tag{5.2}$$

In the moving frame, the position vector of an arbitrary point $\vec{p'}$, is given by

$$\vec{p'} = xe_1^m + ye_2^m \tag{5.3}$$

where $x = \vec{p'}.e_1^m = (\vec{p} - \vec{t})^T.Re_1^f$ an so forth. Thus the moving coordinates can be written in terms of the fixed as

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = R^T \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} - R^T\vec{t}. \tag{5.4}$$

Therefore we find,

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}. \tag{5.5}$$

Suppose now that M is a matrix representing a rigid motion referred to the moving frame. Now we want to transform this matrix to the fixed frame. Assume that M sends point $\vec{p_1'}$ to $\vec{p_2'}$ so that

$$M \begin{pmatrix} \vec{p_1'} \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{p_2'} \\ 1 \end{pmatrix} \tag{5.6}$$

In the fixed frame coordinates the corresponding points are $\vec{p_1}$ and $\vec{p_2}$, using the above result, we can find

$$\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} M \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \vec{p_1} \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{p_2} \\ 1 \end{pmatrix}. \tag{5.7}$$

This transformation is called conjugation transformation in the group theory.

Planar motion group $SE(2)$ is the matrix Lie group in the plane kinematics. It can be represented as,

$$S = \begin{pmatrix} \cos\theta & -\sin\theta & f \\ \sin\theta & \cos\theta & g \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}.$$

where R is a rotation matrix and t is translation vector in Euclidean space $\mathbb{E}^2$. Here S defines a general rigid transformation, which is a rotation about a fixed point and a translation along a vector $t = (f, g)^T$.

If $p = (x, y)^T$ is the center of rotation (or pole point), then rotation about this point

is given by the conjugation,

$$\begin{aligned} S &= \begin{pmatrix} I_2 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_2 & -p \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} R & (I_2 - R)p \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta & y\sin\theta + x(1-\cos\theta) \\ \sin\theta & \cos\theta & y(1-\cos\theta) - x\sin\theta \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where I_2 is the 2×2 identity matrix. Thus,

$$(I_2 - R)p = t,$$

yields the pole point or center of rotation p, where we avoid taking a pure translation, i.e. $R \neq I_2$. So we can find pole point coordinates, using conjugation transformation.

$$e^{\theta K} = \begin{pmatrix} \cos\theta & -\sin\theta & y\sin\theta + x(1-\cos\theta) \\ \sin\theta & \cos\theta & y(1-\cos\theta) - x\sin\theta \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.8}$$

The matrix (5.8) is the exponential representation of the matrix S. Therefore the coordinate transformation of pole points in the planar motion group $SE(2)$ is given by with exponential matrix as the following,

$$\begin{pmatrix} X_f \\ Y_f \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & f \\ \sin\theta & \cos\theta & g \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_m \\ y_m \\ 1 \end{pmatrix} = e^{\theta K} \begin{pmatrix} x_m \\ y_m \\ 1 \end{pmatrix}. \tag{5.9}$$

which gives the (X_f, Y_f) position, in the fixed plane Σ_f, of the point (x_m, y_m) in the moving plane Σ_m. There exist three degrees of freedom of displacement which are θ, f, g. In the continuous displacement, θ, f, g are given functions of a parameter t. If t is the time (5.9) defines a motion of Σ_m with respect to Σ_f. Any motion in the plane

can be described by the three functions $\theta(t), f(t), g(t)$. They find out position, velocity, acceleration and so on of any point of Σ_m.

If θ is constant then any displacement of the set is a pure translation. We shall exclude this special case and suppose that θ is not constant. Then we may take it as the motion's parameter, having a geometrical meaning. In this case we have,

$$\begin{pmatrix} X_f \\ Y_f \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & f(\theta) \\ \sin\theta & \cos\theta & g(\theta) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_m \\ y_m \\ 1 \end{pmatrix} = e^{\theta K} \begin{pmatrix} x_m \\ y_m \\ 1 \end{pmatrix}. \tag{5.10}$$

and the motion is defined by the $f(\theta), g(\theta)$. All properties depend only on these two functions. We will denoted by prime of the differentiation with respect to θ. Now if we take derivative of (5.10) with respect to θ, we get

$$\begin{pmatrix} X'_f \\ Y'_f \\ 0 \end{pmatrix} = Ke^{\theta K} \begin{pmatrix} x_m \\ y_m \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.11}$$

This derivatives are called geometric velocity of the point x, y. In particular,

$$\begin{aligned} f'(\theta) &= x\sin\theta + y\cos\theta \\ g'(\theta) &= -x\cos\theta + y\sin\theta. \end{aligned}$$

These two linear equations for x, y is at cusp of its path or the pole point. The pole point coordinates in Σ_m;

$$\begin{pmatrix} x_p \\ y_p \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f'(\theta) \\ g'(\theta) \\ 1 \end{pmatrix}. \tag{5.12}$$

The pole point coordinates in Σ_f;

$$\begin{pmatrix} X_p \\ Y_p \\ 1 \end{pmatrix} = e^{\theta K} \begin{pmatrix} x_p \\ y_p \\ 1 \end{pmatrix} = \begin{pmatrix} f(\theta) - g'(\theta) \\ f'(\theta) + g(\theta) \\ 1 \end{pmatrix}. \tag{5.13}$$

Therefore the matrices (5.12) and (5.13) can be obtained for any value of θ. These representations define by two curves in the plane, called centrodes.The matrix (5.12) is the parametric representation of the moving centrode p_m in Σ_m and the matrix (5.13) represents the fixed centrode p_f in Σ_f. The fixed centrode is the locus of the rotation pole in the fixed plane and the moving centrode is the locus of the rotation pole in the moving plane. So at any moment they have p as a common point. The geometric velocity of the point along p_m can be by differentiating (5.12),

$$\begin{pmatrix} x'_p \\ y'_p \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f'(\theta) - g''(\theta) \\ f''(\theta) + g'(\theta) \\ 0 \end{pmatrix}. \tag{5.14}$$

Thus we find the geometric rate at which the contact point p between p_m and p_f moves along p_m. The velocity with respect to Σ_f is given by

$$\begin{pmatrix} X'_p \\ Y'_p \\ 0 \end{pmatrix} = Ke^{\theta K} \begin{pmatrix} x_p \\ y_p \\ 1 \end{pmatrix} + e^{\theta K} \begin{pmatrix} x'_p \\ y'_p \\ 0 \end{pmatrix} = \begin{pmatrix} f'(\theta) - g''(\theta) \\ f''(\theta) + g'(\theta) \\ 0 \end{pmatrix}. \tag{5.15}$$

This time, we find the geometric rate at which the contact point p between p_m and p_f moves along p_f. Thus the rate of change of contact along p_m is the same as its change along p_f. This means that, at any moment p_m and p_f are tangent at p. Therefore the moving centrode rolls without sliding along the fixed centrode.

5.2 Instantaneous Geometric Invariants in the Planar Motion Group

Consider a moving frame Σ_m is continuous displacement to the fixed frame Σ_f. The point P is fixed in the moving frame Σ_m.

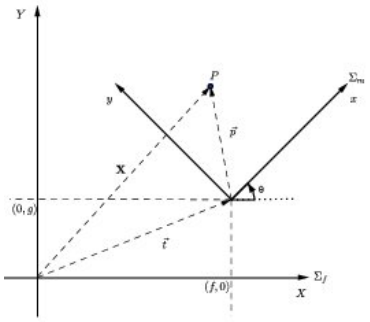


Figure 5.2 Change of Coordinate Frames

So its position is represented by the constant vector $\vec{p}$ in Σ_m. If $\vec{t}$ is the position vector of the origin of Σ_m respect to Σ_f, then the position vector of P respect to Σ_f is defined by

$$\mathbf{X}(\theta) = e^{\theta K}\vec{p} = R(\theta)\vec{p} + \vec{t}(\theta) \tag{5.16}$$

where,

$$\mathbf{X}(\theta) = \begin{pmatrix} X_f(\theta) \\ Y_f(\theta) \\ 1 \end{pmatrix}, \qquad \vec{p} = \begin{pmatrix} x_m \\ y_m \\ 1 \end{pmatrix}, \qquad \vec{t}(\theta) = \begin{pmatrix} f(\theta) \\ g(\theta) \\ 1 \end{pmatrix},$$

and $R(\theta)$ is a 2×2 rotation matrix. To find instantaneous invariants, we use Taylor series approximation. So we need to find $\mathbf{X_n}(\theta)$,

$$\mathbf{X_n}(\theta) = \frac{d^n}{d\theta^n} e^{\theta K}\big|_{\theta=0}\vec{p} = K^n\vec{p} = R_n(\theta)\vec{p} + \vec{t}_n(\theta),$$

where t_n are instantaneous invariants.

Thus we can write Taylor series approximation, evaluated at $\theta = 0$ then,

$$\mathbf{X}(\theta) = \sum_{j=0}^{\infty} \frac{\theta^j}{j!}(R_j\vec{p} + \vec{t_j}) \tag{5.17}$$

To find instantaneous geometric invariants are the best way to understand analytically continuous motion in the planar motion group. We described time-independent planar motion as (5.10) and we found Taylor series approximation in the equation as (5.30). This equation consists the instantaneous invariants. If the instantaneous invariants write explicitly then we get,

$$f(\theta) = \sum_{j=0}^{\infty} f_j \left(\frac{\theta^j}{j!}\right), \qquad g(\theta) = \sum_{j=0}^{\infty} g_j \left(\frac{\theta^j}{j!}\right). \tag{5.18}$$

With these relation we have elements to construct the canonical system for the planar motion group $SE(2)$. Firstly we choose the two frames coincide in the zero position. So we have from (5.10) and (5.18), $f_0 = g_0 = 0$. If we place the common origin of the frames at the pole at moment $\theta = 0$, we have from (5.12) and (5.13), $f_1 = g_1 = 0$. Moreover the axes O_X and o_x are chosen along the common tangent at the pole of the centrodes. From (5.14) or (5.15) it follows $f_2 = 0$ and assuming that $g_2 \neq 0$ and also we choose positive direction $g_2 > 0$. Due to these choice we have geometrical meaning for the pole point and the pole-tangent at $\theta = 0$.

where the X_n and Y_n are again the nth derivative of X and Y with respect to θ and $f_3, f_4, ..., f_n, g_2, g_3, ...g_n$ are the instantaneous invariants of the motions. So if we use canonical coordinates and instantaneous invariants for description of the planar motion then we have $2(n-2)+1$ parameters.

For $n = 2$ we have one parameter g_2 where $g_2 > 0$. Since our canonical coordinate system, the contact point of centrodes move in the direction of the negative X-axis. To describe to curvature trajectory of the circular path, we take to equation of circle,

$$(X-a)^2 + (Y-b)^2 = \rho^2, \tag{5.19}$$

where X, Y are the Cartesian coordinates of a moving point and (a, b) is the center of a circle of radius ρ. If we take first and second derivative of equation (5.19), we get

$$2(X-a)X' + 2(Y-b)Y' = 0, \tag{5.20}$$

$$X'^2 + Y'^2 + (X-a)X'' + (Y-b)Y'' = 0, \tag{5.21}$$

If we solve equations (5.20) and (5.21) for $(X-a)$ and $(Y-b)$ then we yields,

$$(X-a) = \frac{Y'(X'^2+Y'^2)}{X'Y''-Y'X''} \quad \text{and} \quad (Y-b) = \frac{X'(X'^2+Y'^2)}{X'Y''-Y'X''} \tag{5.22}$$

Substituting in equation (5.19), we get well-known formula for the radius of curvature,

$$\rho = \frac{(X'^2+Y'^2)^{3/2}}{X'Y''-Y'X''} \tag{5.23}$$

If we use definition of curvature $\kappa = \frac{1}{\rho}$ then we get,

$$\kappa = \frac{X'Y''-Y'X''}{(X'^2+Y'^2)^{3/2}} \tag{5.24}$$

We substitute our canonical coordinate system;

Thus we yield,

$$\kappa = \frac{-x^2-y^2+g_2y}{(x^2+y^2)^{3/2}}, \tag{5.25}$$

which is valid for any point different from the pole. If we consider $\kappa = 0$ then we get,

$$x^2 + y^2 - g_2y = 0 \tag{5.26}$$

This means that the path is an inflection curve with center $(0, g_2/2)$. Therefore g_2 is the diameter of the inflection circle.

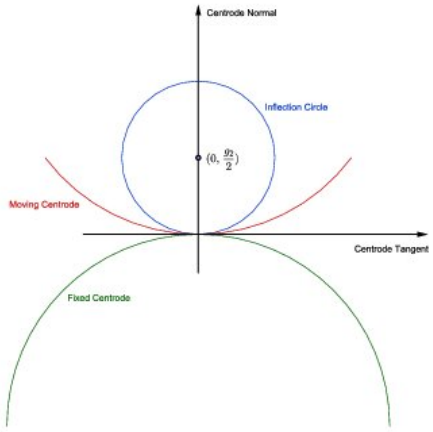


Figure 5.3 Inflection Circle

If we change polar coordinates (r, ψ), $x = r\sin(\psi)$, $y = r\cos(\psi)$ then we obtain

$$\kappa = \frac{r - g_2 \sin(\psi)}{r^2} \qquad 0 \leq \psi < \pi, \quad -\infty < r < \infty, \quad r \neq 0. \tag{5.27}$$

Now if we consider the radius curvature at the zero position as $\rho_0 = r - r_0$ then we

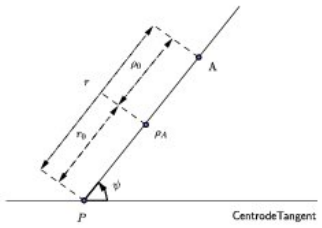


Figure 5.4 Euler-Savary Equation

get,

$$\left(\frac{1}{r} - \frac{1}{r_0}\right)\sin(\psi) = \frac{1}{g_2} \tag{5.28}$$

where r is the distance from the origin to the moving point, and r_0 is the distance from the origin to the center of curvature.Therefore the difference between inverses values of point and curvature center from the pole multiplied by sinus of angle between point normal and centrode tangent is constant for all the points of the body in given position. The equation (5.28) is called the Euler-Savary equation with invariant.

5.3 Time Dependent Motion in the Planar Motion Group

We characterized time independent kinematics in the planar motion group. Now we suppose that the motions are such that three parameters (θ, f, g) are functions of time. This time the coordinate transformations depend on time.

$$\mathbf{X}(\theta(\mathbf{t})) = e^{\theta(t)K}\vec{p} = R(\theta(t))\vec{p} + \vec{t}(\theta(t)) \tag{5.29}$$

Of course we can write,

$$\mathbf{X}(\theta(\mathrm{t})) = S(t)\vec{p} \tag{5.30}$$

where

$$S(t) = \begin{pmatrix} \cos(\theta(t)) & -\sin(\theta(t)) & f(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) & g(\theta(t)) \\ 0 & 0 & 1 \end{pmatrix} \tag{5.31}$$

If we take time derivatives of equation (5.30) then we get,

$$\frac{d}{dt}\mathbf{X}(\theta(\mathrm{t})) = \frac{d}{dt}S(t)\vec{p} \tag{5.32}$$

where

$$\begin{aligned} \frac{d}{dt}S(t)\big|_{t=0} &= \dot{\theta}\begin{pmatrix} -\sin(\theta(t)) & -\cos(\theta(t)) & f'(\theta(t)) \\ \cos(\theta(t)) & -\sin(\theta(t)) & g'(\theta(t)) \\ 0 & 0 & 0 \end{pmatrix}\Big|_{t=0} \\ &= \dot{\theta}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \dot{\theta}J_3, \end{aligned} \tag{5.33}$$

where

$$J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Therefore we find velocity distribution such as

$$\frac{d}{dt}\mathbf{X}(\theta(\mathrm{t}))\big|_{t=0} = \dot{\theta} J_2 \vec{p}, \qquad \dot{\theta} \neq 0. \tag{5.34}$$

Similarly we find second time derivatives,

$$\frac{d^2}{dt^2}\mathbf{X}(\theta(\mathrm{t})) = \frac{d^2}{dt^2} S(t)\vec{p} \tag{5.35}$$

where

$$\begin{aligned} \frac{d^2}{dt^2}S(t)\big|_{t=0} &= \ddot{\theta} \begin{pmatrix} -\sin(\theta(t)) & -\cos(\theta(t)) & f'(\theta(t)) \\ \cos(\theta(t)) & -\sin(\theta(t)) & g'(\theta(t)) \\ 0 & 0 & 0 \end{pmatrix}\Big|_{t=0} + \\ &+ \dot{\theta}^2 \begin{pmatrix} -\cos(\theta(t)) & \sin(\theta(t)) & f''(\theta(t)) \\ -\sin(\theta(t)) & -\cos(\theta(t)) & g''(\theta(t)) \\ 0 & 0 & 0 \end{pmatrix}\Big|_{t=0} \\ &= \ddot{\theta} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \dot{\theta}^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & g_2 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence we obtain

$$\frac{d^2}{dt^2}\mathbf{X}(\theta(\mathrm{t}))\big|_{t=0} = \ddot{\theta} J_3 \vec{p} - \dot{\theta}^2(\vec{p} - \vec{m}), \tag{5.36}$$

where $\vec{m} = (0, g_2)^T$. So the equation (5.36) shows that the acceleration of the velocity pole of Σ_f is always $\dot{\theta}^2 g_2$.

CHAPTER SIX
CONCLUSION

In the present thesis, we studied planar motion group $SE(2)$ in the classical kinematics. We have presented a starting point for developing matrix Lie groups and plane kinematics. We have shown that how to relate exponential and Cayley map in the planar motion group $SE(2)$. It is important result for classical kinematics.

Another contribution of this thesis, we have studied instantaneous invariants for planar motion group. Some important connection between instanteneous invariants and Lie group is given in the canonical coordinate systems.

In the future work we are thinking next problems; The mobile robot motion application in the canonical coordinate systems with instanteneous invariants and finding geodesics in the canonical coordinate systems with instantaneous invariants.

REFERENCES

Bottema, O., & Roth, B. (1990). *Theoretical kinematics*. North holland publishing company, Dover publication.

Duffy, J. (1991). Introduction to theoretical kinematics. *SIAM Review*, *33*(2), 302.

Ebetiuc, S., & Haraldw, S. (2005). Applying differential geometry to kinematic modeling in mobile robotics. In *2005 WSEAS International Conference on Dynamical Systems and Control*, (2–4).

Gallier, J. (2011). *Geometric methods and applications: for computer science and engineering*. U.S.A, Springer Science & Business Media.

Müller, H. R. (1962). *Sphärische Kinematik*. Deutscher Verlag der Wissenschaften.

Murray, R. M., Li, Z., Sastry, S. S., & Sastry, S. S. (1994). *A mathematical introduction to robotic manipulation*. Boca Raton CRC press.

Ouerfelli, M., Kumar, V., & Harwin, W. S. (2000). Methods for kinematic modeling of biological and robotic systems. *Medical Engineering & Physics*, *22*(7), 509–520.

Paraskevopoulos, E., & Natsiavas, S. (2013). A new look into the kinematics and dynamics of finite rigid body rotations using lie group theory. *International Journal of Solids and Structures*, *50*(1), 57–72.

Roth, B. (2015). On the advantages of instantaneous invariants and geometric kinematics. *Mechanism and Machine Theory*, *89*, 5–13.

Selig, J. (2004). Lie groups and lie algebras in robotics. In *Computational Noncommutative Algebra and Applications* (101–125). Springer.

Selig, J. (2007). Cayley maps for se (3). In *the international federation of the theory of machines and mechanisms 12th World Congress, Besançon*.

Selig, J. (2011). On the geometry of point-plane constraints on rigid-body displacements. *Acta Applicandae Mathematicae*, *116*(2), 133–155.

Selig, J. M. (2005). *Geometric fundamentals of robotics*. Holland, Springer Science & Business Media.

Selig, J. M. (2006). Active versus passive transformations in robotics. *IEEE Robotics & Automation Magazine*, *13*(1), 79–84.

Veldkamp, G. (1967). Canonical systems and instantaneous invariants in spatial kinematics. *Journal of Mechanisms*, *2*(3), 329–388.